Computer algebra and ring theory help Feynman integrals

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The old good Weyl algebra

The 1st polynomial Weyl algebra

$$A_1(\mathbb{K}) \ = \ D_1 \ = \ \mathbb{K}\langle x, \partial \mid \partial x = x\partial + 1 \rangle$$

or, stressing that we work over the ring of *polynomial coefficients*

$$\mathbb{K}[x]\langle \partial \mid \partial x = x\partial + 1 \rangle \ = \ \mathbb{K}[x][\partial; 1, \tfrac{\partial}{\partial x}]$$

where the latter is the formulation via **Ore extension**.

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The 1st rational Weyl algebra

$$B_1(\mathbb{K}) = \mathbb{K}(x)\langle \partial \mid \partial x = x\partial + 1\rangle \;=\; \mathbb{K}(x)[\partial; 1, \tfrac{\partial}{\partial x}]$$

It is the **Ore localization** of A_1 at the Ore set $S = K[x] \setminus \{0\}$, and thus $B_1 \cong S^{-1}A_1$.

Dimension of the space of holomorphic solutions

Theorem (Cauchy-Kowalewska-Kashiwara)

Let $\mathbb{K} = \mathbb{C}$, $D = A_n(\mathbb{C})$ the n-th Weyl algebra, $I \subset D$ an ideal such that D/I is a holonomic D-module (i. e. $\operatorname{GKdim} D/I = n$*). Moreover, let* $\operatorname{Sing}(I)$ *be the singular locus of I and U a simply connected domain in* $\mathbb{C}^n \setminus \operatorname{Sing}(I)$*. Consider the system of differential equations $\{\mathfrak{o} \bullet f = 0 : \mathfrak{o} \in I\}$ for holomorphic functions f on U. Then the dimension of the complex vector space of solutions to this system is equal to the holonomic rank of D/I.*

... where the **holonomic rank** of D/I (or of a fin. pres. D-module) is nothing else but

$$\dim_{\mathbb{K}(x)} S^{-1}D/S^{-1}I = \dim_{\mathbb{K}(x)} B_n/B_nI$$

for $S = \mathbb{K}[x] \setminus \{0\}$. It is computable as well as $\operatorname{Sing}(I)$.

The old good shift algebra

Let g be a sequence in a discrete argument k and $\mathbf{s}$ (or σ) is the shift operator $\mathbf{s}(g(k)) = g(k+1)$ which is multiplicative per def.

Thus $\mathbf{s}(kg(k)) = (k+1)g(k+1) = (k+1)\mathbf{s}(g(k))$ holds.

The 1st polynomial shift algebra

$$S_1 \; = \; K\langle k, s \mid sk = (k+1)s = ks + s \rangle.$$

or, stressing that we work over the ring of *polynomial coefficients*

$$\mathbb{K}[k]\langle s \mid sk = \mathbf{s}(k)s = (k+1)s \rangle \; = \; \mathbb{K}[k][s; \mathbf{s}(), 0]$$

where the latter is the formulation via Ore extension.

Dimension of the space of meromorphic solutions

Theorem (Norlund 1924)

For $\mathbb{K} \subseteq \mathbb{C}$, $d \in \mathbb{N}$ *and a matrix* $A \in \mathrm{GL}_d(\mathbb{K}(x))$, *the set of meromorphic solutions of the system*

$$Y(x+1) = A(x) \cdot Y(x)$$

is a vector space of dimension d *over the field of* 1*-periodic meromorphic functions.*

Via the companion matrix this extends to a recurrence operator of order d with coefficients in $\mathbb{K}(x)$.

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Question: is there a computable dimension for modules !?

Answer: Yes! It is the Gel'fand–Kirillov dimension, well-known in the Ring Theory.

Higher dimensional versions of algebras

... of Weyl and shift algebras are obtained by tensoring univariate versions over $\mathbb{K}$, resulting in component-wise commutativity.

The *n*th polynomial Weyl algebra

$$D_n = \mathbb{K}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \mid \partial_j\partial_i = \partial_i\partial_j, x_jx_i = x_ix_j, \partial_jx_i = x_i\partial_j + \delta_{ij} \rangle$$

The *n*th polynomial shift algebra

$$S_n = \mathbb{K}[k_1, \ldots, k_n]\langle s_1, \ldots, s_n \mid s_js_i = s_is_j, k_jk_i = k_ik_j, s_jk_i = k_is_j + \delta_{ij}s_j \rangle$$

We can also tensor algebras betweem themselves, like $A_1 \otimes_{\mathbb{K}} S_1$.

A unifying framework: G-algebras

Definition

For $n \in \mathbb{N}$ and $1 \leq i < j \leq n$ consider the units $c_{ij} \in \mathbb{K}^*$ and polynomials $d_{ij} \in \mathbb{K}[x_1, \ldots, x_n]$. Suppose, that there exists a monomial total well-ordering $\prec$ on $\mathbb{K}[x_1, \ldots, x_n]$, such that for any $1 \leq i < j \leq n$ either $d_{ij} = 0$ or the leading monomial of d_{ij} is smaller than $x_i x_j$ with respect to $\prec$. The $\mathbb{K}$-algebra

$$A := \mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij} : 1 \leq i < j \leq n\}\rangle$$

is called a G-**algebra**, if $\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \alpha_i \in \mathbb{N}_0\}$ is a $\mathbb{K}$-basis of A.

Remark

- *"G-algebra" is also known as "an algebra of solvable type" and "a Poincaré–Birkhoff–Witt (PBW) algebra",*
- *A is a Noetherian domain with nice Gröbner bases theory.*

Examples for G-Algebras

- Weyl algebras ($\mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \forall i : \partial_i x_i = x_i \partial_i + 1\rangle$)
- Shift algebras ($\mathbb{K}\langle x_1, \ldots, x_n, s_1, \ldots, s_n \mid \forall i : s_i x_i = (x_i + 1) s_i\rangle$)
- q-Weyl algebras ($\mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \forall i \exists q_i \in \mathbb{K}^* : \partial_i x_i = q_i x_i \partial_i + 1\rangle$)
- q-Shift algebras ($\mathbb{K}\langle x_1, \ldots, x_n, s_1, \ldots, s_n \mid \forall i \exists q_i \in \mathbb{K}^* : s_i x_i = q_i x_i s_i\rangle$)
- integration algebras ($\mathbb{K}\langle x_1, \ldots, x_n, I_1, \ldots, I_n \mid \forall i : I_i x_i = x_i I_i + I_i^2\rangle$)
- Universal enveloping algebras of fin.-dim. Lie algebras
- Many quantum groups
- ...

Moreover, a GR-**algebra** is a factor of a G-algebra by a two-sided ideal (these include Clifford (and exterior) algebras and yet more quantum groups).

There are more GR-algebras than you think

For $f \in \mathbb{K}[x]$, let $t := f^{-1}$. Then the localization of $A_n(\mathbb{K})$ at an Ore set $S := \{f^i : i \in \mathbb{N}_0\}$ is **finitely presented** as an algebra and, moreover, can be realized as a GR-algebra, being the factor of

$$\mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, t \mid \partial_j x_i = x_i \partial_j + \delta_{ij}, x_j x_i = x_i x_j, \partial_j \partial_i = \partial_i \partial_j,$$

$$t x_j = x_j t, \quad \partial x_i \cdot t = t \cdot \partial x_i - \frac{\partial f}{\partial x_i} \cdot t^2 \rangle,$$

(note, that an ordering $\partial x_i \gg x_j, t$ makes any such a G-algebra) modulo the two-sided ideal $\langle tf - 1 \rangle$.

Hence one can **work with modules over this algebra** in SINGULAR:PLURAL: a subsystem of freely available SINGULAR, devoted to GR-algebras since 2002.

Gröbner Technology = Gröbner Trinity + Gröbner Basics

With essentially the same algorithm, we can compute the (left) **Gröbner Trinity**:

1. GB (left) Gröbner basis G of a module M
2. SYZ (left) Gröbner basis of the 1st (left) syzygy module of M
3. LIFT the transformation matrix between two bases G and M

Besides, here's the url: `http://www.singular.uni-kl.de` .

Gröbner basics (as coined by Buchberger, Sturmfels et al)

... are the most important and fundamental applications of Gröbner Bases.

Universal Gröbner Basics

- Ideal (resp. module) membership problem (NF, REDUCE)
- Intersection with subrings (elimination of variables) (ELIMINATE)
- Intersection of ideals (resp. submodules) (INTERSECT)
- Quotient and saturation of ideals (QUOT)
- Kernel of a module homomorphism (MODULO)
- Kernel of a ring homomorphism (NCPREIMAGE.LIB)
- Algebraic relations between pairwise commuting polynomials
- Hilbert and Hilbert-Samuel polynomials of modules (NCHILBERT.LIB)

The needs for localizations

Recall: adjoining the inverse of any $f \in \mathbb{K}[x]$ in D_n,
what is the same as the localization of D_n at $S = f^{\mathbb{N}_0}$,
results in a fin. pres. algebra, which can be directly addressed by a computer algebra system.

Infinite presentation happens as well

Suppose we'd like to add an inverse k^{-1} to the shift algebra

$$S_1 \ = \ K\langle k, s \mid sk = (k+1)s = ks + s\rangle.$$

However, $s \cdot \frac{1}{k} = \frac{1}{k+1}s$, so one needs to add inverses to $\{k + \mathbb{Z}\}$...

Rational localizations $\mathbb{K}(\underline{x})\langle\underline{\mathfrak{o}}\rangle$: S was $\mathbb{K}[\underline{x}] \setminus \{0\}$

Good news: there exist a monomial (block) ordering $\prec$, such that a Gröbner basis with respect to $\prec$ over $\mathbb{K}[\underline{x}]\langle\underline{\mathfrak{o}}\rangle$ is a (non-reduced, non-monic) Gröbner basis over $\mathbb{K}(\underline{x})\langle\underline{\mathfrak{o}}\rangle$.

$K(x \cdot \partial)\langle x, \partial\rangle$: still not clear, how to handle properly.

Dimension function

Let A be a Noetherian algebra. A dimension function δ assigns a value $\delta(M)$ to each finitely generated A-module M and satisfies the following properties:

(i) $\delta(0) = -\infty$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is exact sequence, then $\delta(M) \geq \sup\{\delta(M'), \delta(M'')\}$ with equality if the sequence is split.

(iii)* If M is a torsion module, then $\delta(M) \leq \delta(A) - 1$.

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- a dimension function is **exact**, if $\delta(M) = \sup\{\delta(M'), \delta(M'')\}$ for all modules
- generalized Krull dimension is an exact dimension function
- Gel'fand-Kirillov dimension is a dimension function, which is not always exact

Filtrations on algebras and modules

Let A be an associative $\mathbb{K}$-algebra, generated by $x_1, \ldots, x_m$.

Degree filtration

Let $V = \mathbb{K}x_1 \oplus \ldots \oplus \mathbb{K}x_m \subset A$ be a vector space.

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Set $V_0 = \mathbb{K}$, $V_1 = \mathbb{K} \oplus V$ and $V_{k+1} = V_k \oplus V^{k+1}$. If

$$V_i \subseteq V_{i+k}, \quad V_i \cdot V_j \subseteq V_{i+j}, \quad A = \bigcup_{k=0}^{\infty} V_k,$$

then $\{V_k : k \in \mathbb{N}\}$ is the **standard (ascending) filtration** of A.

Gel'fand-Kirillov dimension and its properties

Let M be a left A-module and let $M_0 \subset M$ be a finite $\mathbb{K}$-vector space, spanned by the generators of M. That is $\dim_{\mathbb{K}} M_0 < \infty$ and $AM_0 = M$.

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The **Gel'fand-Kirillov dimension** of M is defined as follows

$$\mathrm{GKdim}(M) \;=\; \limsup_{d\to\infty} (\log_d(\dim_{\mathbb{K}} H_d))$$

In the standard construction one puts $\deg x_i := 1$ and defines $V_d := \{f \mid \deg f \leq d\}$.

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Conventions: $\mathrm{GKdim}(0) = -\infty$. $\quad\mathrm{GKdim}_{\mathbb{Q}}(\mathbb{Q}) = 0$.

Lemma

Let A be a $\mathbb{K}$-algebra and a domain. If the standard filtration on A is compatible with the PBW Basis $\{x^\alpha \mid \alpha \in \mathbb{N}_0^m\}$, then $\mathrm{GKdim}_{\mathbb{K}}(A) = m$.

$$\dim V_d = \binom{d+m-1}{m-1}, \; \dim V^d = \binom{d+m}{m}.$$

Thus $\binom{d+m}{m} = \frac{(d+m)\ldots(d+1)}{m!} = \frac{d^m}{m!} + \ldots$ and

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Hence for any G-algebra A in n variables has $\mathrm{GKdim}_{\mathbb{K}}(A) = n$.

Consider the free associative algebra $T = \mathbb{K}\langle x_1, \ldots, x_n \rangle, n \geq 2$. Then $\mathrm{GKdim}(T) = \infty$.

$\dim V_d = n^d, \dim V^d = \frac{n^{d+1}-1}{n-1}$. Note, that $\frac{n^{d+1}-1}{n-1} > n^d$. Since $\log_d n^d = d \log_d n = \frac{d}{\log_n d} \to \infty, d \to \infty$, it follows that $\mathrm{GKdim}(T) = \infty$.

Gel'fand-Kirillov dimension: examples and properties

Properties (similarities and differences)

- $\operatorname{GKdim} M = \sup\{\operatorname{GKdim}(N) : N \in A - mod,\ N \subseteq M\}$,

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Hence, if $|K| = \infty$, then $\text{GKdim}(\mathbb{K}[[x_1, \ldots, x_n]]) = \infty$ for $n \geq 1$.

Lemma (R is commutative, note the difference to Krull)

(i) *Let R be a commutative affine $\mathbb{K}$-algebra. Then (by Noether normalization) $\exists S = \mathbb{K}[x_1, \ldots, x_t] \subseteq R$ and R is finitely generated S-module. Then* $\operatorname{GKdim}_{\mathbb{K}} R = \operatorname{Kr.dim} S = t$.

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Proposition (Exactness, J. Gomez-Torrecillas et al.)

Let R be a G-algebra. Then GKdim is **exact** on short exact sequences of fin. gen. left R-modules. That is,

$$0 \to L \to M \to N \to 0 \;\Rightarrow\; \operatorname{GKdim} M = \sup\{\operatorname{GKdim} L, \operatorname{GKdim} N\}$$

Elimination and GK-dimension

Lemma

Let $L \subset A$ be a left ideal and $S \subset A$ be a subalgebra. Then

- $L \cap S = 0$ *implies* $\operatorname{GKdim} A/L \geq \operatorname{GKdim} S$,

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Bernstein's inequality and holonomy

Over any field $\mathbb{K}$ one has $\operatorname{GKdim}_{\mathbb{K}}(D_n(K)) = 2n$.

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Over any field $\mathbb{K}$ one has $\text{GKdim}_{\mathbb{K}}(D_n(K)) = 2n$.

$\text{char}\, \mathbb{K} = 0$ implies, that the **holonomic number** of $D_n(\mathbb{K})$ is $\min\{\text{GKdim}_{\mathbb{K}}(M) : 0 \neq M \in D - mod\} = n$.

GK-dimension on our continuous service

Usually, GKdim of localized algebras is expected to increase.
But (all the proofs are suppressed for brevity)

$$2n = \mathrm{GKdim}\, \mathbb{K}[\underline{x}]\langle\underline{\partial}\rangle = \mathrm{GKdim}\, \mathbb{K}[\underline{x}, \tfrac{1}{f}]\langle\underline{\partial}\rangle = \mathrm{GKdim}\, \mathbb{K}(\underline{x})\langle\underline{\partial}\rangle$$

However, even for $n = 1$

$$S = \{(x\partial)^i : i \in \mathbb{N}_0\} =: [x\partial], \widehat{S} = [x\partial - j : j \in \mathbb{Z}] \text{ and } \bar{S} = [x, \partial]$$

lead to the same localization and

$$\mathrm{GKdim}\, \widehat{S}^{-1}\mathbb{K}[x]\langle\partial\rangle = 3.$$

Moreover, this holds multivariate;
also for $S = \mathbb{K}[x_1\partial_1, \ldots, x_n\partial_n] \setminus \{0\}$ we also have

$$\mathrm{GKdim}\, S^{-1}\mathbb{K}[\underline{x}]\langle\underline{\partial}\rangle = 3n.$$

GK-dimension on our discrete service

A natural localization of the shift algebra is at the powers of the (forward) shift operators, which results in an algebra with both forward and backward shifts, called the **full shift algebra**:
for $S = [s_1, \ldots, s_n]$ we have

$$2n = \mathrm{GKdim}\, \mathbb{K}[\underline{k}]\langle \underline{s} \rangle = \mathrm{GKdim}\, \mathbb{K}[\underline{k}]\langle \underline{s}^{\pm 1} \rangle$$

but

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What about homomorphisms between Weyl and shift algebras?

Fourier transform

automorphism(s) $\tau_{\pm}$ of $A_1 = \mathbb{K}\langle x, \partial \mid \partial x = x\partial + 1 \rangle$,
$x \mapsto \pm\partial, \partial \mapsto \mp x$.

Algebraic Mellin transform: Blackboard!

The $D_n[s]$-module $\mathbb{K}\left[x, s, \frac{1}{f}\right] \bullet f^s$

- Let $D_n[s] := D_n \otimes_{\mathbb{K}} \mathbb{K}[s_1, \ldots, s_m]$.
- Let $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ be non-zero polynomials.
- Let us denote $f := f_1 \cdot \ldots \cdot f_m$ and $f^s := f_1^{s_1} \cdot \ldots \cdot f_m^{s_m}$
- By $\mathbb{K}[x, s, \frac{1}{f}]$ we denote the localization of $\mathbb{K}[x, s]$ at $S = \{f^i : i \in \mathbb{N}_0\}$,
- and by $\mathbb{K}[x, s, \frac{1}{f}] \bullet f^s$ the free $\mathbb{K}[x, s, \frac{1}{f}]$-module, generated by the formal symbol f^s.

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$$\partial_i \bullet f^s = \Big(\sum_{j=1}^{m} s_j \frac{\partial f_j}{\partial x_i} \frac{1}{f_j}\Big) \cdot f^s \quad \in \quad \mathbb{K}\Big[x, s, \frac{1}{f}\Big] \cdot f^s$$

One classically has

$$D_n[s] \bullet f^s \cong D_n[s] / \operatorname{Ann}_{D[s]}(f^s).$$

f^s as a special function

Recall, that $\text{Ann}_{D[s]}(f^s) = \{Q(s) \in D[s] \mid Q(s) \bullet f^s = 0\} \subset D[s]$ is a left ideal, **the annihilator of f^s in $D_n[s]$**.

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- $\text{Ann}_{D[s]}(f^s) \cap \mathbb{K}[x, s] = 0$.
- $\text{Ann}_{D[s]}(f^s)$ is **left saturated** with respect to $\mathbb{K}[x, s]$, that is $h(x, s)r(x, s, \partial) \in \text{Ann}_{D[s]}(f^s) \Leftrightarrow r(x, s, \partial) \in \text{Ann}_{D[s]}(f^s)$.

Bernstein-Sato ideals

For a polynomial mapping $F : \mathbb{K}^n \to \mathbb{K}^m$,
$F = (f_1, \ldots, f_m), f_i \in \mathbb{K}[x_1, \ldots, x_n]$ consider the ideal
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$b(s) \in B_F(s) \Leftrightarrow$ there exist a linear partial differential operator $P(x, s, \partial) \in D_n[s]$ with polynomial coefficients in $\mathbb{K}[x_1, \ldots, x_n, s_1, \ldots, s_m]$, such that

$$P(s) \bullet f^{s+1} = P(s)f \bullet f^s = b(s) \cdot f^s.$$

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Theorem (J. Bernstein ($m = 1$), C. Sabbah, R. Bahloul)

The Bernstein-Sato ideal of $F = (f_1, \ldots, f_m), f_i \in \mathbb{K}[x_1, \ldots, x_n]$ is a nonzero proper ideal in $\mathbb{K}[s]$.

Lemma

Consider the rational Weyl algebra, tensored with $\mathbb{K}[s_1, \dots, s_m]$*:*

$$D_{rat}[s] = \mathbb{K}(X)\langle \partial_1, \dots, \partial_n \mid [\partial_j, h(x)] = \frac{\partial h(x)}{\partial x_j} \rangle \otimes_{\mathbb{K}} \mathbb{K}[s].$$

Let $f^s = f_1^{s_1} \cdot \ldots \cdot f_m^{s_m}$ *for* $f_i \in \mathbb{K}[X] \setminus \mathbb{K}$*, then*

$$\mathrm{Ann}_{D_{rat}[s]}(f^s) = \langle \{ \partial_i - \sum_{k=1}^{m} f_k^{-1} \frac{\partial f_k}{\partial x_i} s_k : 1 \leq i \leq n \} \rangle.$$

Therefore, $\mathrm{GKdim}(D_{rat}[s] / \mathrm{Ann}_{D_{rat}[s]}(f^s)) = n + m$*.*

Dimension theory and annihilator

Stemming from Ring Theory, the Gel'fand-Kirillov dimension GKdim is very suitable for computations with various $D[s]$-modules.

Theorem (L.)

Consider $\operatorname{Ann} f^s \subset D_n[s_1, \ldots, s_m]$. *The latter is of* GKdim *equal to* $2n+m$ *with the holonomic number* n. *Then*

- GKdim $D_n[s_1, \ldots, s_m]/\operatorname{Ann} f^s = n+m$
- $D_n[s_1, \ldots, s_m]/\operatorname{Ann} f^s$ *has no torsion with respect to* $\mathbb{K}[x, s]$
- $D_n[s_1, \ldots, s_m]/\operatorname{Ann} f^s$ *is equidimensional (pure), i.e. any nonzero submodule has the same dimension* $n+m$
- $D_n[s_1, \ldots, s_m]/\operatorname{Ann} f^s$ *is uniform, i.e. it contains no direct sum of two submodules*

Purity filtration with $\delta = \mathrm{GKdim}$

Let $\mathfrak{O}$ be a Noetherian domain, being Auslander-regular and Cohen-Macaulay algebra (like a G-algebra) with $\mathrm{GKdim}\,\mathfrak{O} = n$ and the holonomic number h.

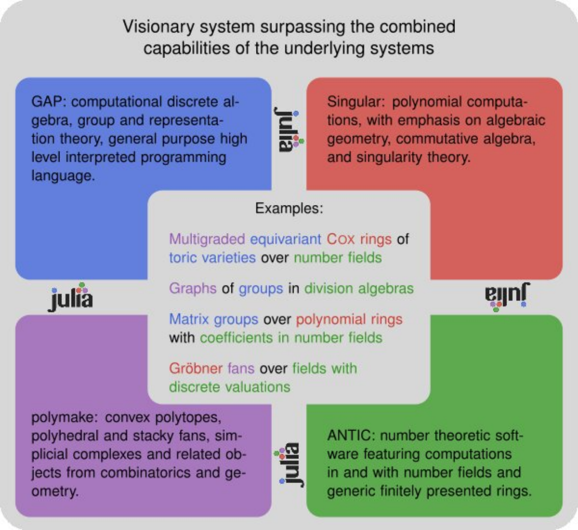
Given a fin. pres. $\mathfrak{O}$-module M of dimension $n > d \geq 0$, then the purity filtration of M is the sequence

$$M = M_d \supset M_{d-1} \ldots \supset M_{h+1} \supset M_h.$$

where $\mathrm{GKdim}\, M_j = j$. Moreover, M_{j+1}/M_j is either 0 or pure of dimension $j + 1$.

Moreover, such a filtration is algorithmically computable.

… and there is more: OSCAR!



Thank you for your attention!