Computation of holonomic systems for Feynman amplitudes associated with some simple diagrams

Toshinori Oaku
Department of Mathematics, Tokyo Woman's Christian University

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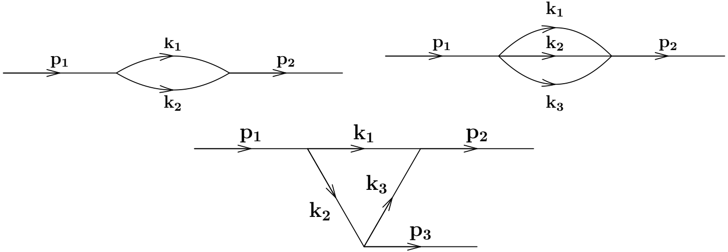
Holonomic systems are a class of systems of linear (partial or ordinary) differential equations. One of the most fundamental properties of a holonomic system is that its solution space is finite-dimensional.

A Feynman amplitude is the integral of a rational function, or more generally, the product of complex powers of polynomials. Hence we can, in principle, apply the following two facts in (computational) D-module theory:

1. For (multi-variate) polynomials $f_1, \ldots, f_d$ and complex numbers $\lambda_1, \ldots, \lambda_d$, the multi-valued analytic function $f_1^{\lambda_1} \cdots f_d^{\lambda_d}$ satisfies a holonomic system, which can be computed algorithmically. Moreover, if we ragard λ's as parameters, then it satisfies a difference-differential system which corresponds to a holonomic system by Mellin transform.
2. If a funciton satisfies a holonomic system, its integral with respect to some of its variables also satisfies a holonomic system, which can be computed algorithmically.

In the integration, it would also be natural to ragard the integrand as a local cohomology class associated with $f_1, \dots, f_d$, which corresponds to the Cutkosky-type phase space integral in case of Feynman amplitude.

However, actual computation is hard in general because of the complexity. I have only succeeded in the compuation for Feynman amplitudes associated with simple diagrams as below mostly in the two-dimensinal space-time.

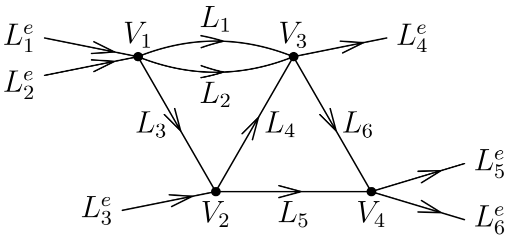


Feynman diagrams and Feynman integrals

Let G be a connected Feynman graph (diagram), i.e., G consists of

- vertices $V_1, \cdots, V_{n'}$,
- oriented line segments $L_1, \ldots, L_N$ called internal lines,
- oriented half-lines $L_1^e, \ldots, L_n^e$ called external lines.

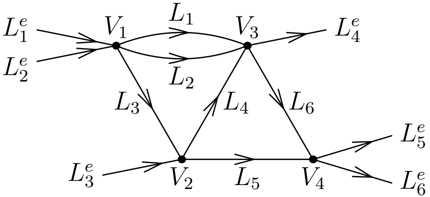
The end-points of each internal line L_l are two distinct vertices, and each external line has only one end-point, which coincides with one of the vertices.



• We associate ν-dimensional vector $\mathbf{p}_r$ to each external line L_r^e $(1 \leq r \leq n')$,
and ν-dimensional vector $\mathbf{k}_l$ and a real number (mass) $m_l \geq 0$ to each internal line L_l $(1 \leq l \leq N)$.

• For a vertex V_j and an internal or external line L_l, the incidence number $[j : l]$ is defined as follows:

$$[j : l] = 1 \text{ if } L_l \text{ ends at } V_j,$$
$$[j : l] = -1 \text{ if } L_l \text{ starts from } V_j,$$
$$[j : l] = 0 \text{ otherwise.}$$



The Feynman integral associated with G is defined to be

$$F_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n'} \delta\left(\sum_{r=1}^{n}[j:r]\mathbf{p}_r + \sum_{l=1}^{N}[j:l]\mathbf{k}_l\right)}{\prod_{l=1}^{N}(\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)} \prod_{l=1}^{N} d^\nu \mathbf{k}_l.$$

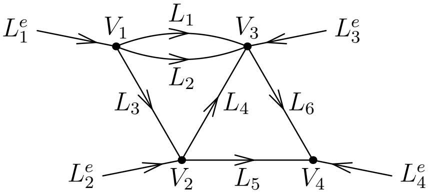
Here δ denotes the ν-dimensional delta function,

$$\mathbf{k}_l^2 := k_{l0}^2 - k_{l1}^2 - \cdots - k_{l\nu}^2,$$

$d^\nu \mathbf{k}_l$ is the ν-dimensional volume element,
and $(\cdots + \sqrt{-1}\,0)$ means the limit $(\cdots + \sqrt{-1}\,\varepsilon)$ as $\varepsilon \to +0$.
The integrand is well-defined as a generalized function at least if all $m_l > 0$.

In what follows, we assume that G is external, i.e., for each vertex V_j, there exists a unique external line, which we may assume to be L^e_j, that ends at V_j and that no external line starts from V_j. Then $n = n'$ holds and the Feynman integral is

$$F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n} \delta\Big(\mathbf{p}_j + \sum_{l=1}^{N} [j:l]\mathbf{k}_l\Big)}{\prod_{l=1}^{N}(\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)} \prod_{l=1}^{N} d^\nu \mathbf{k}_l$$



Rewriting the Feynman integral

The delta factors of the integrand of the Feynman integral correspond to the linear equations (momentum preservation)

$$p_j + \sum_{l=1}^{N} [j : l] k_l = 0 \qquad (1 \leq j \leq n)$$

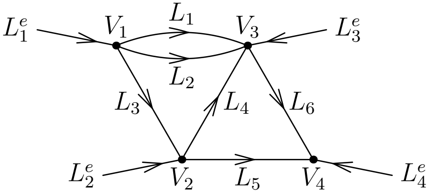
for indeterminates p_j and k_l which correspond to the vectors $\mathbf{p}_j$ and $\mathbf{k}_l$. These equations define an N-dimensional linear subspace of $\mathbb{R}^{n+N}$, which is contained in the hyperplane $p_1 + \cdots + p_n = 0$ since $\sum_{j=1}^{n} [j : l] = 0$.

Lemma

Let A be the $n \times N$ matrix whose (j, l)-element is $[j : l]$. Then the rank of A is $n - 1$.

For the example below, the matrix A is given by

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



In view of the lemma above, we can choose a set of indices

$$J = \{l_1, \dots, l_{N-n+1}\} \subset \{1 \dots, N\}$$

and integers a_{lr} and b_{lj} so that the system

$$p_j + \sum_{l=1}^{N} [j : l] k_l = 0 \qquad (1 \le j \le n)$$

of linear equations is equivalent to

$$\sum_{j=1}^{n} p_j = 0, \quad k_l - \psi_l(p_1, \dots, p_{n-1}, k_{l_1}, \dots, k_{l_{N-n+1}}) = 0 \quad (l \in J^c).$$

Then the Feynman integral is written in the form

$$
\begin{aligned}
F_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathbb{R}^{N\nu}} & \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \\
& \times \prod_{l \in J^c} \delta(\mathbf{k}_l - \psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})) \\
& \times \prod_{l=1}^{N} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \prod_{l=1}^{N} d\mathbf{k}_l \\
= \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) & \tilde{F}_G(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})
\end{aligned}
$$

with the amplitude function

$$
\begin{aligned}
\tilde{F}_G(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) &= \int_{\mathbb{R}^{(N-n+1)\nu}} \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \\
&\times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \prod_{l \in J} d\mathbf{k}_l.
\end{aligned}
$$

D-modules

In general, let D_n be the ring of differential operators in the variables $x = (x_1, \dots, x_n)$ with polynomial coefficients (the n-th Weyl algebra). D_n is the $\mathbb{C}$-algebra generated by $x_1, \dots, x_n$ and $\partial_{x_1}, \dots, \partial_{x_n}$ with commutation relations

$$\partial_{x_i} x_j - x_j \partial_{x_i} = \delta_{ij}, \quad x_i x_j = x_j x_i, \quad \partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}.$$

A system

$$\sum_{j=1}^{l} P_{ij} u_j = 0 \quad (i = 1, \dots, m)$$

of linear differential equations with $P_{ij} \in D_n$ for unknown functions $u_1, \dots, u_l$ corresponds to the left D_n-module

$$M = D_n^l / (D_n \vec{P}_1 + \cdots + D_n \vec{P}_m), \quad \vec{P}_i = (P_{i1}, \dots, P_{il}).$$

In fact, the solution space of this system in a function space $\mathcal{F}$ is given by the left D_n-module homomorphisms

$$\mathrm{Hom}_{D_n}(M, \mathcal{F}) \simeq \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_l \end{pmatrix} \in \mathcal{F}^l \;\middle|\; \sum_{j=1}^{l} P_{ij} u_j = 0 \quad (i = 1, \ldots, m) \right\}.$$

This follows from the exact sequence

$$D_n^m \xrightarrow{\psi} D_n^l \xrightarrow{\varphi} M \longrightarrow 0$$

of left D_n-modules. Here φ and ψ are D_n-homomorphisms defined by

$$\varphi((P_1, \ldots, P_l)) = [(P_1, \ldots, P_l)] \quad \text{(the residue class in } M\text{)},$$

$$\psi((Q_1, \ldots, Q_m)) = \begin{pmatrix} Q_1 & \cdots & Q_m \end{pmatrix} \begin{pmatrix} P_{11} & \cdots & P_{1l} \\ \vdots & & \vdots \\ P_{m1} & \cdots & P_{ml} \end{pmatrix}.$$

Characteristic variety and holonomic systems

For the sake of simplicity, let us consider a system

$$\sum_{j=1}^{m} P_j u = 0 \quad (P_j \in D_n)$$

for a single unknown function u, i.e., the D_n-module

$$M = D_n/I, \qquad I := D_n P_1 + \cdots + D_n P_m.$$

In general, a nonzero $P \in D_n$ is written in the form

$$P = \sum_{|\alpha| \leq l} a_\alpha(x) \partial_x^\alpha, \quad a_\alpha(x) \in \mathbb{C}[x]$$

with multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\mathbb{N} = \{0, 1, 2, \dots\}$. The principal symbol of P is defined to be

$$\sigma(P) = \sigma(P)(x, \xi) = \sum_{|\alpha| = l} a_\alpha(x) \xi^\alpha \in \mathbb{C}[x, \xi]$$

with commutative indeterminates $\xi = (\xi_1, \dots, \xi_n)$.

Definition

The characteristic variety of $M = D_n/I$ is the algebraic set

$$\mathrm{Char}(M) = \{(x,\xi) \in \mathbb{C}^{2n} \mid \sigma(P)(x,\xi) = 0 \quad (\forall P \in I \setminus \{0\})\}$$

of the cotangent bundle $T^*\mathbb{C}^n = \mathbb{C}^{2n}$.

Theorem (Sato-Kawai-Kashiwara, J. Bernstein)

The dimension of $\mathrm{Char}(M)$ *is not less than* n *if* $M \neq 0$.

Definition

The left D_n-module $M = D_n/I$ is holonomic if $\mathrm{Char}(M)$ is n-dimensional or else $M = 0$.

Theorem (Kashiwara)

The solution space of a holonomic system is finite dimensional.

Local cohomology

In general, let $f_1, \ldots, f_d$ be polynomials in the variables $x = (x_1, \ldots, x_n)$ with complex coefficients such that the variety

$$Y = \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_d(x) = 0\}$$

is d-codimensional, i.e., $f_1, \cdots, f_d$ are of complete intersection. Then the (algebraic) d-th local cohomology group associated with $f_1, \ldots, f_d$ is defined to be the quotient space

$$H^d_Y(\mathbb{C}[x]) := \mathbb{C}[x, f^{-1}] / \sum_{k=1}^{n} \mathbb{C}[x, (f/f_k)^{-1}]$$

with $f = f_1 \cdots f_d$. It consists of the cohomology classes $[g/f^\nu]$ with $\nu = 1, 2, 3, \ldots$ and $g \in \mathbb{C}[x]$.

• $H^d_Y(\mathbb{C}[x])$ has a natural structure of left D_n-module and is holonomic as such.

• The simplest example of the local cohomology group is

$$H^1_{\{0\}}(\mathbb{C}[x]) = \mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$$

with $x = x_1$ (one variable), which is spanned by the classes $[x^{-k}]$ with $k = 1, 2, 3, \ldots$ as a $\mathbb{C}$-vector space. As a left D_1-module, it is generated only by $[x^{-1}]$ since $\partial_x^k [x^{-1}] = (-1)^k k! [x^{-k-1}]$.

• There are algorithms (U. Walther, Oaku-Takayama) to compute the local cohomology group as a D-module, in particular, the annihilator (the holonomic system) for each cohomology class.

Integrands of Feynman amplitudes

Let $D_{\nu N}$ be the ring of differential operators with polynomial coefficients in $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-1}}$. We regard the integrand

$$\prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}\,0)^{-1} \\ \times \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}\,0)^{-1}$$

of the Feynman amplitude as the local cohomology class

$$\left[\prod_{l \in J} (\mathbf{k}_l^2 - m_l^2)^{-1} \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-1} \right]$$

associated with polynomials $\mathbf{k}_l^2 - m_l^2$ with $l \in J$ and $\psi_l - m_l^2$ with $l \in J^c$.

The holonomic system for the local cohomology class

$$\left[\prod_{l\in J}(\mathbf{k}_l^2 - m_l^2)^{-1}\prod_{l\in J^c}(\psi_l(\mathbf{p}_1,\ldots,\mathbf{p}_{n-1},\mathbf{k}_{l_1},\ldots,\mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-1}\right]$$

can be computed by general algorithms; in case $m_l > 0$, it is generated by first order operators, which are much easier to compute.

Integrals of D-modules

In general, let M be a finitely generated left module over D_{n+d}, the ring of differential operators in the variables $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_d)$.
Then the integral of M w.r.t. y is the left D_n-module

$$\int M\, dy := M/(\partial_{y_1} M + \cdots + \partial_{y_d} M).$$

Theorem (J. Bernstein)

If M is a holonomic D_{n+d}-module, then $\int M\, dy$ is a holonomic D_n-module.

• Let $\mathcal{F}$ be a $\mathbb{C}$-vector space of (generalized) functions in (x, y) such that $f \in \mathcal{F}$ implies $Pf \in \mathcal{F}$ for any $P \in D_{n+d}$.

• Let $\mathcal{F}'$ be a $\mathbb{C}$-vector space of (generalized) functions in x such that $f \in \mathcal{F}'$ implies $Pf \in \mathcal{F}'$ for any $P \in D_n$.

Suppose that for $u(x, y) \in \mathcal{F}$, the integral $\int u(x, y)\, dy$ is well-defined and belongs to $\mathcal{F}'$ so that

$$P \int u(x, y)\, dy = \int Pu(x, y)\, dy \quad (\forall P \in D_n)$$
$$\int \partial_{y_j} u(x, y)\, dy = 0 \quad (j = 1, \dots, d)$$

for any $u \in \mathcal{F}$. Then there is a natural $\mathbb{C}$-linear map

$$\mathrm{Hom}_{D_{n+d}}(M, \mathcal{F}) \longrightarrow \mathrm{Hom}_{D_n}(\int M\, dy,\, \mathcal{F}').$$

More concretely, let $M = D_{n+d}/I$ with a left ideal I of D_{n+d}. Then

$$\int M\,dy = D_{n+d}/(I + \partial_{y_1} D_{n+d} + \cdots + \partial_{y_d} D_{n+d})$$

as left D_n-module. If M is holonomic, there exists $m \in \mathbb{N}$ such that the left D_n-module $\int M\,dy$ is generated by the residue classes of y^α with $\alpha \in \mathbb{N}^d$ such that

$$|\alpha| = \alpha_1 + \cdots + \alpha_d \leq m.$$

Then the homomorphism

$$\mathrm{Hom}_{D_{n+d}}(M, \mathcal{F}) \longrightarrow \mathrm{Hom}_{D_n}(\int M\,dy,\, \mathcal{F}')$$

is given by the map, for $u(x,y)$ such that $Pu = 0$ for any $P \in I$,

$$u(x,y) \longmapsto \Big(\int y^\alpha u(x,y)\,dy \mid \alpha \in \mathbb{N}^d,\ |\alpha| \leq m \Big)$$

and for $|\beta| > m$, there exist $P_{\beta,\alpha} \in D_n$ such that

$$\int y^\beta u(x,y)\,dy = \sum_{|\alpha|\le m} P_{\beta,\alpha} \int y^\alpha u(x,y)\,dy.$$

This relation amounts to

$$y^\beta - \sum_{|\alpha|\le m} P_{\beta,\alpha} y^\alpha \;\in\; I + \partial_{y_1} D_{n+d} + \cdots + \partial_{y_d} D_{n+d}.$$

In particular, $\int u(x,y)\,dy$ is annihilated by the ideal

$$J = D_n \cap (I + \partial_{y_1} D_{n+d} + \cdots + \partial_{y_d} D_{n+d})$$

of D_n, which is called the integration ideal of I.

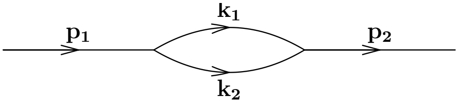
There is an algorithm to compute $\int M\,dy$ as well as J precisely (Oaku-Takayama). In the algorithm, the bound m is given as the largest integer root of what is called the b-function (an analogue of the indicial polynomial for ODE).

There are also heuristic algorithms to compute an ‘approximation’ of J, which might be smaller than J, such as ‘creative telescoping’ method.

I used a computer algebra system Risa/Asir developed by M. Noro for computing the following examples:

Example 1 (as a toy case)

Let us study the Feynman diagram G below:



Then the Feynman integral is written in the form

$$F_G(\mathbf{p}_1, \mathbf{p}_2) = \int_{\mathbb{R}^4} \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2)$$

$$\times (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}\,0)^{-1}\,d\mathbf{k}_1 d\mathbf{k}_2$$

$$= \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1)$$

with the amplitude

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}\,0)^{-1}\,d\mathbf{k}_1.$$

In view of the invariance under Lorentz transformations, let us set $\mathbf{p}_1 = (x, 0, \ldots, 0)$.

In case $\nu = 2$, the integration ideal of the annihilator of the local cohomology class

$$[(k_{10}^2 - k_{11}^2 - m_1^2)^{-1}((x - k_{10})^2 - k_{11}^2 - m_2^2)^{-1}]$$

is generated by

$$(x-m_1-m_2)(x-m_1+m_2)(x+m_1-m_2)(x+m_1+m_2)\partial_x+2x(x^2-m_1^2-m_2^2).$$

The solutions (the kernel) of this operator are constant multiples of

$$(x-m_1+m_2)^{-1/2}(x+m_1-m_2)^{-1/2}(x+m_1+m_2)^{-1/2}(x-m_1-m_2)^{-1/2}.$$

In case of $\nu = 4$, by using the 3-dimensional polar coordinates, we have

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}\,0)^{-1}\, d\mathbf{k}_1$$
$$= 2\pi \int_{\mathbb{R}^2} (k_{10}^2 - r^2 - m_1^2 + \sqrt{-1}\,0)^{-1}$$
$$\times ((x - k_{10})^2 - r^2 - m_2^2 + \sqrt{-1}\,0)^{-1} r_+^2\, dk_{10} dr.$$

The integration ideal of the annihilator of the cohomology class

$$[r^2(k_{10}^2 - r^2 - m_1^2)^{-1}((x - k_{10})^2 - r^2 - m_2^2)^{-1}]$$

is generated by

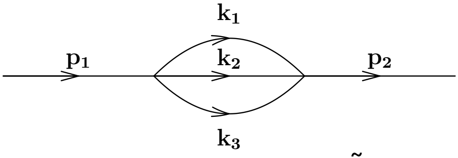
$$x(x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x$$
$$- 2((m_1^2 + m_2^2)x^2 - m_1^4 + 2m_2^2 m_1^2 - m_2^4).$$

The solutions (the kernel) of this operator are constant multiples of

$$x^{-2}(x - m_1 + m_2)^{1/2}(x + m_1 - m_2)^{1/2}(x + m_1 + m_2)^{1/2}(x - m_1 - m_2)^{1/2}.$$

Example 2 (Cf. Adams-Bogner-Weinzierl 2015)

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1)$$

with

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^4} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}\,0)^{-1} \times ((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2 + \sqrt{-1}\,0)^{-1}\, d\mathbf{k}_1 d\mathbf{k}_2.$$

We work in the 2-dimensional space-time ($\nu = 2$) and compute holonomic systems for $\tilde{F}_G((x, 0))$ by assigning some special values to m_1, m_2, m_3 since the computation for general m_1, m_2, m_3 (as parameters) is intractable.

First let us set $m_1 = 1$, $m_2 = 2$, $m_3 = 4$ so that $(-m_1 + m_2 + m_3)^2$, $(m_1 - m_2 + m_3)^2$, $(m_1 + m_2 - m_3)^2$ are distinct. Then $\tilde{F}_G((x, 0))$ is annihilated by the differential operator

$$
\begin{aligned}
&30x(x-1)(x+1)(x-3)(x+3)(x-5)(x+5)(x-7)(x+7)\underline{\partial_x^3} \\
&\quad + (-2x^{12} + 191x^{10} - 5340x^8 + 35954x^6 + 273082x^4 \\
&\qquad\qquad - 2071305x^2 + 661500)\underline{\partial_x^2} \\
&\quad + (-10x^{11} + 675x^9 - 12108x^7 + 15454x^5 + 936462x^3 \\
&\qquad\qquad - 2692665x)\underline{\partial_x} \\
&\quad - 8x^{10} + 372x^8 - 3300x^6 - 36028x^4 + 457932x^2 - 356760.
\end{aligned}
$$

The singular points $x = 0, \pm 1, \pm 3, \pm 5, \pm 7$ are all regular and the indicial equations are all $s^2(s-1)$.

Next set $m_1 = m_2 = m_3 = 1$. Then $\tilde{F}_G((x,0))$ is annihilated by

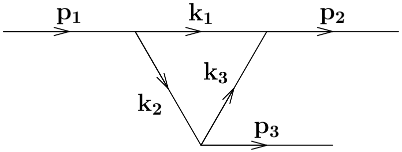
$$x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x.$$

The points $0, \pm 1, \pm 3$ are regular singular and the indicial equations at these points are all s^2.

See Adams-Bogner-Weinzierl (2015) for complete computation with arbitrary m_1, m_2, m_3 by a different (and more efficient) method.

Example 3

The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2) = \int_{\mathbb{R}^\nu} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}\,0)^{-1}$$
$$\times ((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}\,0)^{-1}((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2 + \sqrt{-1}\,0)^{-1}\, d\mathbf{k}_1.$$

We work in the 2-dimensional space-time ($\nu = 2$). In view of the invariance under Lorentz transformation, we set $\mathbf{p}_1 = (x, 0)$, and $\mathbf{p}_2 = (y, z)$.

First let us set $m_1 = m_2 = m_3 = 1$. We compute the integration ideal I of the annihilator of the integrand regarded as a local cohomology class.
Since I is too complicated (with tens of generators), we will present only the characteristic variety of $M := D_3/I$.

The characteristic variety of $M := D_3/I$ is given by

$$\mathrm{Char}(M) = T^*_{\{f=0\}}\mathbb{C}^3 \cup T^*_{\{x=0\}}\mathbb{C}^3 \cup T^*_{\{x=f_0=0\}}\mathbb{C}^3 \cup T^*_{\{x=y^2-z^2-4=0\}}\mathbb{C}^3 \\ \cup T^*_{\{x=y+z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y-z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y=z=0\}}\mathbb{C}^3$$

with

$$\begin{aligned} f(x,y,z) &= (y-z)(y+z)x^2 - 2(y-z)(y+z)yx \\ &\qquad + (y-z)^2(y+z)^2 + 4z^2, \\ f_0(y,z) &= f(0,y,z), \end{aligned}$$

where we denote by $T^*_Z\mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of $\mathbb{C}^3$.

Only the first component $T^*_{\{f=0\}}\mathbb{C}^3$ should be physically significant. Especially, it roughly means that the Feynman amplitude is analytic outside the surface $f = 0$, which has rather complicated singularities.

Definition of the conormal bundle

In general, if a complex analytic submanifold Z is defined by

$$f_1(x, y, z) = \cdots = f_d(x, y, z) = 0$$

with holomorphic functions $f_1, \ldots, f_d$ such that $df_1 \wedge \cdots \wedge df_d \neq 0$, then the conormal bundle of Z is defined to be the submanifold

$$T_Z^* \mathbb{C}^3 = \{(x, y, z; c_1 df_1 + \cdots + c_d df_d) \mid c_1, c_2, c_3 \in \mathbb{C}\}$$

of the contangent bundle

$$T^* \mathbb{C}^3 = \{(x, y, z; \xi dx + \eta dy + \zeta dz) \mid \xi, \eta, \zeta \in \mathbb{C}\}.$$

In fact, $T_Z^* \mathbb{C}^3$ is a Lagrangean submanifold of $T^* \mathbb{C}^3$.

Singularities of the surface $f = 0$

Let us investigate the singularities of the complex surface

$$Z = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\},$$
$$f = (y - z)(y + z)x^2 - 2(y - z)(y + z)yx + (y - z)^2(y + z)^2 + 4z^2.$$

Following N. Honda and T. Kawai, we rewrite f as

$$f = yzx^2 - yz(y + z)x + y^2z^2 + (y - z)^2$$

by change of coordinates $(y + z, y - z) \to (y, z)$.
Then the singular locus (the set of the singular points) of Z is the union of two complex lines $\{x = y = z\}$ and $\{y = z = 0\}$.
The projection $Z \ni (x, y, z) \mapsto (y, z)$ defines a doube covering on $\{(x, y) \mid xy \neq 0\}$ branched along the union of curves $y - z = 0$ and $yz - 4 = 0$.

The stratification of Z with respect to the (local) b-function $b_{f,p}(s)$ of f at a point p is

strata	$b_{f,p}(s)$
$\{(0,0,0)\}$	$(s+1)^3(2s+3)$
$\{(2,0,0),(-2,0,0),(2,2,2),(-2,-2,-2)\}$	$(s+1)^2(2s+3)$
$\{x=y=z\}\cup\{y=z=0\}$ $\setminus\{(0,0,0),(\pm 2,0,0),\pm(2,2,2)\}$	$(s+1)^2$
$\{f=0\}\setminus(\{x=y=z\}\cup\{y=z=0\})$	$s+1$

In comparison, that of $g := x^2 - y^2 z$ (Whitney umbrella) is

strata	$b_{g,p}(s)$
$\{(0,0,0)\}$	$(s+1)^2(2s+3)$
$\{x=y=0\}\setminus\{(0,0,0)\}$	$(s+1)^2$
$\{g=0\}\setminus\{x=y=0\}$	$s+1$

Local b-function (Bernstein-Sato polynomial)

In general, let $f(x) = f(x_1, \dots, x_n)$ be a holomorphic function defined on a neighborhood of $p \in \mathbb{C}^n$. Let $\mathcal{D}_p$ be the ring of differential operators whose coefficients are holomorphic on a neighborhood of p. Then the local b-function (Bernstein-Sato polynomial) of f at p is the monic polynomial $b_{f,p}(s) \in \mathbb{C}[s]$ of the least degree such that

$$P(s)f^{s+1} = b_{f,p}(s)f^s \qquad (\exists P(s) \in \mathcal{D}_p[s]).$$

By the definition, $b_{f,p}(s)$ is a complex analytic invariant of f at p.

In case $m_1 = 1$, $m_2 = 2$, $m_3 = 3$

In case $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, we succeeded in computation of the integration ideal I of the integrand as a local cohomology class. The characteristic variety of $M := D_3/I$ is given by

$$\mathrm{Char}(M) = T^*_{\{g=0\}}\mathbb{C}^3 \cup T^*_{\{x=y-z=0\}}\mathbb{C}^3,$$

with

$$g(x,y,z) = (y-z)(y+z)x^4 - 2y(y^2-z^2-8)x^3$$
$$+(y^4+(-2z^2-22)y^2+z^4+26z^2+64)x^2+6y(y^2-z^2-8)x+9(y^2-z^2).$$

The decomposition of $\mathrm{Char}(M)$ was done by using a library file `noro_pd.rr` of Risa/Asir for prime and primary decomposition of polynomial ideals developed by M. Noro.

The singularities of the surface $g = 0$

Again by change of coordinates $(y+z, y-z) \to (y, z)$, we rewirte g as

$$g = yzx^4 - (y+z)(yz-8)x^3 + ((z^2+1)y^2 - 24zy + z^2 + 64)x^2 + 3(y+z)(yz-8)x + 9yz.$$

The set of the singular points of the surface $g = 0$ is given by the curve

$$zx^2 - (z^2-8)x - 3z = y - z = 0.$$

The local b-function is $(s+1)^2(2s+3)$ at the 8 points

$$\pm(1,2,2), \quad \pm(1,-4,-4), \quad \pm(3,-2,-2), \quad \pm(3,4,4);$$

$(s+1)^2$ on the curve $zx^2 - (z^2-8)x - 3z = y - z = 0$ other than the 8 points above.

The local b-function of g at the 8 points is the same as that of the Whitney umbrella at the origin.

Microlocal point of view

We work in the d-dimensional Euclidean space $\mathbb{R}^n$ together with its complexification $\mathbb{C}^n$. We use complex coordinates $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with $z_k = x_k + \sqrt{-1}\, y_k$.
We mean by an open cone Γ in $\mathbb{R}^n$ an open set of $\mathbb{R}^n$ such that $y \in \Gamma$ implies $cy \in \Gamma$ for any $c > 0$. For two open cones Γ and Γ', we denote $\Gamma \Supset \Gamma'$ if $\Gamma \cup \{0\} \supset \overline{\Gamma'}$ holds.
Let U be an open set of $\mathbb{R}^n$, and Γ an open cone in $\mathbb{R}^n$. We say that a function $F(z)$ is holomorphic on $U + \sqrt{-1}\,\Gamma 0$ and denote $F \in \mathcal{O}(U + \sqrt{-1}\,\Gamma 0)$ if, for any compact subset K of U and any open cone Γ' such that $\Gamma' \Subset \Gamma$, there exists $\varepsilon > 0$ so that $F(z)$ is holomorphic (complex analytic) on the set

$$\{x + \sqrt{-1}\, y \mid x \in K,\, y \in \Gamma',\, |y| < \varepsilon\}.$$

Hyperfunctions

A hyperfunction (of Mikio Sato) $u(x)$ on $U \subset \mathbb{R}^n$ is defined as the sum of 'the boundary values'

$$u(x) = \sum_{k=1}^{N} F_k(x + \sqrt{-1}\,\Gamma_k 0)$$

with open cones Γ_k and $F_k \in \mathcal{O}(U + \sqrt{-1}\,\Gamma_k 0)$. Here $F_k(x + \sqrt{-1}\,\Gamma_k 0)$ means the formal (cohomological) limit of $F_k(x + \sqrt{-1}\,y)$ as $y \in \Gamma_k$ tends to zero. This limit often makes sense as a distribution of L. Schwartz.

Abstractly, the space of the hyperfunctions on $\mathbb{R}^n$ is defined to be the local cohomology group $H^n_{\mathbb{R}^n}(\mathcal{O}^n_{\mathbb{C}})$, where $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf on $\mathbb{C}^n$ of holomorphic functions.

Micro-analyticity

The dual (or polar) cone Γ° of an open cone Γ is defined to be the closed cone

$$\Gamma^\circ = \{\xi \in \mathbb{R}^n \mid \langle \xi, y\rangle \geq 0 \quad (\forall y \in \Gamma)\},$$

where $\langle \xi, y\rangle$ is the Euclidean inner product. Let

$$\sqrt{-1}\, T^*\mathbb{R}^n = \{(x, \sqrt{-1}\,\langle \xi, dx\rangle) \mid x, \xi \in \mathbb{R}^n\} \simeq \mathbb{R}^n$$

be the purely imaginary cotangent bundle of $\mathbb{R}^n$. We identify $(x, \sqrt{-1}\,\langle \xi, dx\rangle)$ with $(x, \sqrt{-1}\,\xi)$.

Definition

A hyperfunction $u(x)$ is said to be micro-analytic at $(p, \sqrt{-1}\,\xi_0)$ if it is written, on a neighborhood U of p, in the form

$$u(x) = \sum_{k=1}^{N} F_k(x + \sqrt{-1}\,\Gamma_k 0), \quad F_k \in \mathcal{O}(U + \sqrt{-1}\,\Gamma_k 0)$$

with open convex cones Γ_k such that $\xi_0 \notin \Gamma_k^\circ$.

Definition

The analytic wave-front set $WA(u)$ of a hyperfunction $u(x)$ on U is defined to be the closed suset

$$WA(u) = \{(p, \sqrt{-1}\,\xi_0) \mid u \text{ is not micro-analytic at } (p, \sqrt{-1}\,\xi_0)\}$$

of $\sqrt{-1}\,T^*U$.

Microfunctions

Definition

A germ of microfunction at $(p, \sqrt{-1}\,\xi_0 0)$ is a hyperfunction $u(x)$ defined on a neighborhood of p modulo hyperfunctions which are micro-analytic at $(p, \sqrt{-1}\,\xi_0 0)$. This defines the sheaf $\mathcal{C}$ of microfunctions on $\sqrt{-1}\,T^*\mathbb{R}^n$.

Let us denote by $\mathcal{A}$ the sheaf on $\mathbb{R}^n$ of real analytic functions, by $\mathcal{B}$ that of hyperfunctions, and by $\pi : \sqrt{-1}\,T^*\mathbb{R}. \to \mathbb{R}^n$ the projection. Then there exists an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \pi_*\mathcal{C} \longrightarrow 0.$$

This means that the sheaf $\mathcal{C}$ describes the cotangential decompositon of the analytic singularity $\mathcal{B}/\mathcal{A}$ of hyperfunctions.

Feynman amplitudes as microfunctions

As was pointed out by Sato-Kawai-Kashiwara in the 1970's, the Feynman amplitude $\tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p})$ associated with an external diagram G with positive masses is well-defined as a microfunction on the set

$$\sqrt{-1}\, T^*\mathbb{R}^{\nu(n-1)} \setminus \varpi\big(\Lambda(G) \setminus \Lambda_+(G)\big)$$

and its support (analytic wave-front set) is contained in $\varpi(\Lambda_+(G))$. These sets are called Landau (-Nakanishi) varieties and defined as follows:

We set

$$\begin{aligned}\Lambda(G) = \{(&\mathbf{p}_1,\dots,\mathbf{p}_{n-1},\mathbf{k}_{l_1},\dots,\mathbf{k}_{l_{N-n+1}};\mathbf{u}_1,\dots,\mathbf{u}_{n-1};\alpha_1,\dots,\alpha_N)\\ &\in \mathbb{R}^{\nu N}\times\mathbb{R}^{\nu(n-1)}\times\mathbb{R}^N \mid\\ &\alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1\le j\le N-n+1),\\ &\alpha_l(\psi_l^2 - m_l^2) = 0 \ (l\in J^c),\\ &\alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l\in J^c}\alpha_l b_{lj}\psi_l = 0 \ (1\le j\le N-n+1),\\ &\mathbf{u}_r = \sum_{l\in J^c}\alpha_l a_{lr}\psi_l \ (1\le r\le n-1),\\ &\alpha_l \ge 0 \ (1\le l\le N)\}\end{aligned}$$

with

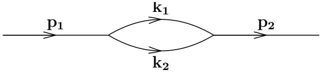
$$\psi_l = \sum_{r=1}^{n-1} a_{lr}\mathbf{p}_r + \sum_{j=1}^{N-n+1} b_{lj}\mathbf{k}_{l_j},$$

$$
\begin{aligned}
\Lambda_+(G) = \{ & (\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}; \mathbf{u}_1, \dots, \mathbf{u}_{n-1}; \alpha_1, \dots, \alpha_N) \\
& \in \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu(n-1)} \times \mathbb{R}^N \mid \\
& \alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1 \le j \le N - n + 1), \\
& \alpha_l(\psi_l^2 - m_l^2) = 0 \ (l \in J^c), \\
& \alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj} \psi_l = 0 \ (1 \le j \le N - n + 1), \\
& \mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr} \psi_l \ (1 \le r \le n - 1), \\
& \alpha_l > 0 \ (1 \le l \le N) \},
\end{aligned}
$$

and ϖ is the projection of $\Lambda(G)$ to

$$
\begin{aligned}
& \sqrt{-1}\, T^*\mathbb{R}^{\nu(n-1)} \\
& \quad = \{ (\mathbf{p}_1, \dots, \mathbf{p}_{n-1}; \sqrt{-1}\,(\langle \mathbf{u}_1, d\mathbf{p}_1 \rangle + \cdots + \langle \mathbf{u}_{n-1}, d\mathbf{p}_{n-1} \rangle)) \}.
\end{aligned}
$$

For example, for the graph G below



$$\Lambda(G) = \{(\mathbf{p}_1, \mathbf{k}_1, \mathbf{u}_1, \alpha_1, \alpha_2) \mid \alpha_1(\mathbf{k}_1^2 - m_1^2) = \alpha_2((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) = 0$$
$$\alpha_1 \mathbf{k}_1 - \alpha_2(\mathbf{p}_1 - \mathbf{k}_1) = \mathbf{0}, \quad \mathbf{u}_1 = \alpha_2 \mathbf{p}_1, \quad \alpha_1, \alpha_2 \geq 0\}$$
$$\Lambda_+(G) = \{(\mathbf{p}_1, \mathbf{k}_1, \mathbf{u}_1, \alpha_1, \alpha_2) \mid \alpha_1(\mathbf{k}_1^2 - m_1^2) = \alpha_2((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2) = 0$$
$$\alpha_1 \mathbf{k}_1 - \alpha_2(\mathbf{p}_1 - \mathbf{k}_1) = \mathbf{0}, \quad \mathbf{u}_1 = \alpha_2 \mathbf{p}_1, \quad \alpha_1, \alpha_2 > 0\},$$

from which, we can confirm that

$$\varpi(\Lambda(G) \setminus \Lambda_+(G)) = \{(\mathbf{p}_1, \sqrt{-1}\,\langle \mathbf{u}_1, d\mathbf{p}_1 \rangle) \mid \mathbf{u}_1 = \mathbf{0}\},$$
$$\varpi(\Lambda_+(G)) = \sqrt{-1}\; T^*_{\{\mathbf{p}_1^2 - (m_1+m_2)^2 = 0\}} \mathbb{R}^2 \cup \sqrt{-1}\; T^*_{\{\mathbf{p}_1^2 - (m_1-m_2)^2 = 0\}} \mathbb{R}^2.$$

This implies that the Feynman amplitude $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as an element of $\mathcal{B}(\mathbb{R}^2)/\mathcal{A}(\mathbb{R}^2)$.

Theorem

Let $\tilde{F}_G(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ be the Feynman amplitude associated with an external diagram with positive masses. Let I be the annihilator of the integrand of the Feynman amplitude as a local cohomology class. Then as a microfunction $\tilde{F}_G(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ is annihilated by the integration ideal

$$D_{\nu(n-1)} \cap \left(I + \sum_{l=1}^{N-n+1} \sum_{j=1}^{\nu} \partial_{k_{lj}} D_{\nu N} \right)$$

as a microfunction on the set

$$\sqrt{-1}\, T^*\mathbb{R}^{\nu(n-1)} \setminus \varpi\big(\Lambda(G) \setminus \Lambda_+(G)\big).$$