Executice Summary of the MITP Topical Workshop: "The Mathematics of Linear Relations between Feynman Integrals"

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Goals of the Topical Workshop

The precise theoretical prediction of observables at modern particle colliders relies to a large extent on our ability to compute large numbers of intricate Feynman integrals. The systematic use of linear relations between Feynman integrals has become indispensable in such computations. These relations allow for efficient reductions of millions of integrals, they are crucial in the derivation of differential and difference equations and in certain cases they can even provide direct ways to express the integrals in terms of known special functions. Since their first and most famous incarnation, the classical integration by parts (IBP) method, linear relations between Feynman integrals have seen various new representations and strategies, powerful implementations on the computer and countless applications over the past forty years.

In recent publications, mathematical concepts such as finite field techniques, syzygies, D-modules, Gröbner bases and intersection numbers have proven to be useful to arrive at a better understanding of the systematics behind these Feynman integral relations. Today, making progress in this direction is more urgent than ever, as efficient IBP reductions and the derivation of appropriate differential equations are crucial bottlenecks in highly demanding computations for physics at the LHC. The use of computer algebra and abstract mathematics in these computations has reached a level, at which the knowledge of specialists in these fields is needed to make further progress.

Our workshop brought together mathematicians with relevant expertise in the mentioned fields, physicists with experience in Feynman integral reductions and specialists on computer algebra tools. It was the main goal of the workshop to enable the direct dialogue between specialists from different communities and to provide an overview of the state of the art on linear relations between Feynman integrals and related methods and applications. The event was also intended to further the exchange of new ideas on how to tackle some long-lasting questions of the field, such as: How do we derive the number of master integrals of a reduction?

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How are master integrals of top level and sub-topologies related? Does the IBP method generate all linear relations? Are there competitive alternatives to Laporta's reduction algorithm? While some of the above questions naturally had to remain open, the workshop showed exciting progress in some of these directions.

Scientific Highlights

The content of the presentations can be structured into three categories:

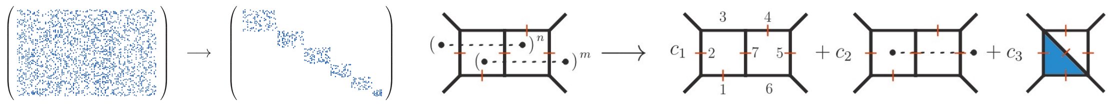
- New developments in the software implementations of reduction programs and other specialized computing tools
- Mathematical concepts and methods related to integral relations
- New benchmarks in physics applications of Feynman integral relations

While most speakers covered more than one of these fields, the topical focus differed strongly between the presentations. Thanks to the very diverse expertise brought together at this event, the workshop was able to cover a wide topical range from impressive physics results such as the four-loop quark-antiquark-photon form factor to latest developments of powerful computer algebra systems and intricacies of abstract D-module theory. The individual presentations can be summarized as follows:

Johann Usovitsch from Trinity College Dublin presented the powerful program Kira for reductions to master integrals based on Laporta's algorithm. The talk showed features of the new version 1.2, including an enormous speed-up in the equation generator, improved parallelization and the algebraic reconstruction of reductions from runs with fixed variables for multi-scale problems. The impressive speed of this new version was demonstrated with some highly involved benchmark examples.

Alexander Smirnov from Moscow State University showed the new developments in his famous program FIRE, which for many years now serves the community in countless research projects. Its latest version, FIRE6, includes runs of reductions with fixed values of variables and algebraic reconstruction, involved techniques for parallelization and an incorporation of Roman Lee's program LiteRed. In particular the talk explained the main reduction strategy behind the program and illuminated aspects of the algebraic reconstruction.

Kasper Larsen from the University of Southampton presented the powerful method of constructing and solving IBP relations by use of spanning sets of cut integrals and corresponding syzygy equations, obtained from Baikov's representation of the integrals. The method is implemented in the program Azurite which serves for the construction of bases of master integrals. The strength of this approach and the program was impressively demonstrated in several benchmark examples, including the fully analytic reduction of the highly involved non-planar hexagon box with numerator insertions of degree four.



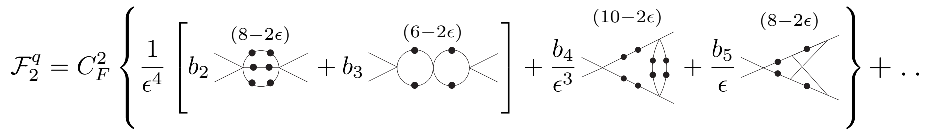
Andreas von Manteuffel from Michigan State University discussed the use of his famous reduction program Reduze2 in the resolution of singularities by use of the systematic construction of a finite integral basis. The talk furthermore presented the new program FINRED for the fast integral reduction by use of the finite field approach. In ongoing work, this program is extended by a customized syzygy finder, such that a strategy based on syzygies of parametric differential operators is combined with the Laporta based strategy in one program.

Stefano Laporta from the University of Padova presented a new approach to finding linear relations between Feynman integrals based on the PSLQ algorithm. Using Baikov's representation, precise numerical evaluations of cut integrals at suitable kinematic points serve as the input for the search for linear relations by the PSLQ algorithm. The presentation illuminated the role of closed paths and self-intersections in the zero set of Baikov polynomials for this search and presented several impressive examples for which the new method provides the desired reduction relations.

Pierpaolo Mastrolia from the University of Padova presented a new method for a direct decomposition of Feynman integrals in terms of a basis of master integrals, based on the use of intersection numbers. In this approach, the Feynman integral is considered as a pairing of a cycle, given by the integration domain, with a co-cycle, given by the integrand, and as each of these constituents admit decompositions into bases, one obtains a master formula for the direct decomposition of the integral. The talk showed the power of this method for many highly involved examples, and discussed implications of the new method for a variety of other techniques, including dimension shifts and the derivation of systems of differential equations for Feynman integrals.

Hui Luo from Graduate School of China Academy of Engineering Physics, presented a novel integrand reduction method based on IBP relations, unitarity cuts and polarization decomposition. This method extracts the gauge-invariant expression for gauge boson and fermion scattering amplitudes and then reduces the amplitude to master integral coefficients by IBP relations on unitarity cuts. The breakthrough on the 3-loop 4-point gauge amplitude computation is obtained by Luo's method.

Robert Schabinger from Michigan State University gave an introduction to the concept of uniformly-finite Feynman integrals with a special focus on QCD form factors and their singularity structure. The talk emphasised convincingly that a decomposition in terms of uniformly-finite integrals unites the advantages of integrals with uniform weight and of finite integrals. A main point of this presentation was the intriguing conjecture that in virtual corrections, the origin of the deeper poles in contributions of lower loop-numbers could be manifest, if the results are expressed in terms of the proposed bases of uniformly-finite integrals. Supporting results for three-loop form factors were discussed in detail.

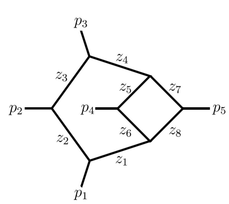


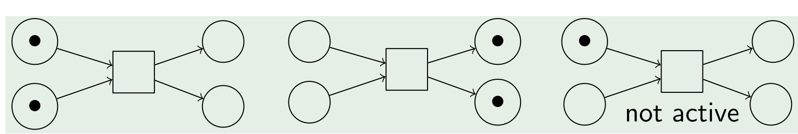
Toshinori Oaku from Tokyo Woman's Christian University discussed his frontier research on evaluating Feynman integrals by modern D-module theory. His method identifies the differential operator annihilator of each Feynman integral, then the integration ideal provides the Feynman integral evaluation up to a constant. Feynman integral examples like bubble, triangle and sunset Feynman integral are evaluated by his powerful method.

Viktor Levandovskyy form RWTH Aachen University discussed the conceptual and computational aspects of modern D-module theory. He emphasized the polynomial annihilator computation and its application in finding linear relations between Feynman integrals. The most up-to-date computer implementations of D-module computations are announced in his talk.

Roman Lee from Budker Institute of Nuclear Physics, Novosibirsk discussed the application of IBP relations, dimension shift identities and differential equations to compute L-loop watermelon and sunset graphs. Using the novel auxiliary function approach, he obtained the compact form of differential equations for such graphs. Then these integrals are expressed analytically as Lauricella functions.

Janko Böhm from Technische Universität Kaiserslautern, in companion with Kasper Larsen's talk, discussed the technical aspects of generating syzygies for a simplified IBP system, as well as the new linear reduction algorithm. He demonstrated the computation of syzygies and linear reduction, with the software Singular, powered by current computational algebraic geometry developments. He also announced the highly-efficient parallel computing scheme, the so-called Petri nets, for large-scale IBP reduction problems in the future.



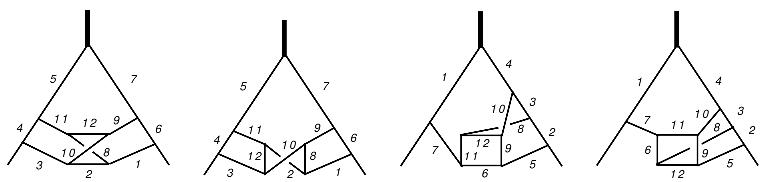


Oleg Tarasov from JINR, Dubna, Russia presented functional relations of Feynman integrals. Such functional relations can be easily obtained by algebraic relations of the propagators. These relations can be used to reduce Feynman integrals to integrals with fewer propagators or fewer kinematic scales, and thereby for the evaluation of Feynman integrals.

David Kosower from Institut de Physique Théorique, CEA-Saclay presented the novel method of direct reduction of IBP relations. In the algorithm, the IBP relations with arbitrary numerators are derived and then solved by the difference equation method. This method will be a very promising tool for reduction of integrals with arbitrarily high numerator degrees for theoretical research.

Yanqing Ma from Peking University discussed the brand-new IBP reduction method by the eta expansion. The multi-loop Feynman integrals after expansion for the large value of eta, present simple linear relations which can be solved in $\mathcal{O}(N)$ time complexity. This new method is very useful for the next-to-next-to-leading-order QCD precision calculations.

Vladimir Smirnov from Moscow State University presented the computation of the quark-antiquark-photon form factor with massless quarks and quartic fundamental colour factor to four loops. The result of this elaborate computation, which included a demanding IBP reduction with the program FIRE and the evaluation of hundreds of master integrals by use of the differential equations method, served for the extraction of the cusp and the collinear anomalous dimension. In the context of this workshop, this computation served as an impressive demonstration of state of the art techniques, as the very latest improvements of integral reduction methods were required to make the full decomposition to master integrals possible.



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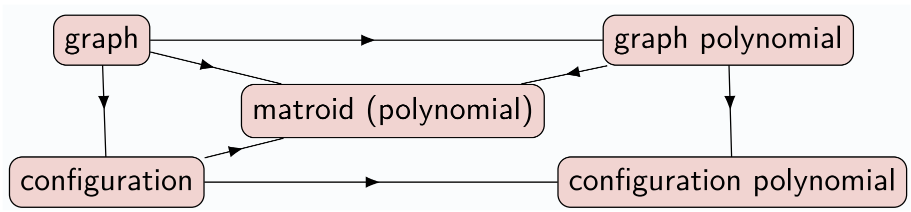
Thomas Bitoun from University of Toronto gave an introduction to the theory of Loeser-Sabbah on the algebraic Mellin transform, the approach to systems of difference equations in the language of D-modules. A key concept is the class of holonomic D-modules, and the main theorem is that the number of master integrals can be computed explicitly as the topological Euler characteristic of a toric hypersurface.

Subhajit Mazumdar from Hebrew University of Jerusalem presented the "Symmetries of Feynman integrals" (SFI) method to reduce a Feynman integral to a line integral over simpler Feynman diagrams. The integration-by-parts identities are the starting point to define a group of symmetries. It was illustrated explicitly for the kite diagram, where it is maximally effective with 6-dimensional orbits.

Mikhail Kalmykov, affiliated with DESY Hamburg and JINR Dubna, explained the hypergeometric take on Feynman integrals. Using Mellin-Barnes reprentations, algorithms using Gröbner bases can determine contiguous relations, which provide relations between Feynman integrals. Illustrative examples include the number of master integrals for all sunrise integrals, and an intriguing relation of 2-loop integrals found in 1999, which is not directly clear from momentum space IBP.

Mathias Schulze from Technische Universität Kaiserslautern explained that the hypersurfaces defined by Symanzik polynomials of (Feynman) graphs are maximally singular: the codimension of their singular locus is 3. He explained how this observation holds more generally in the context of configuration polynomials, and explained the techniques from matroid theory that enter the proof.



Najam ul Basat from Institute of High Energy Physics CAS, China, presented the "direct reduction" method in the example of the $u\bar{d} \to W^+$ process at two loops in QCD. The expansion around infinite mass (parameter η) leads to massive tadpole master integrals and completely avoids an explicit tensor reduction. This makes the approach very effective.

Open problems

While the talks gave answers, the two plenary discussions of our workshop were intended to determine the most relevant open questions of the field. The resulting list of open problems attempts to capture the state of the art by embracing what we do not know yet and intends to inspire future directions of research in the field. The following list of problems was extracted from the discussion sessions:

1. Do all linear relations between Feynman integrals (except for symmetry relations) come from momentum space IBP relations?

2. What is the correspondence between the top level and the sub-topologies with respect to existing relations and the counting of master integrals?

3. How much information on the integral relations can be derived from syzygies?

4. What do we learn from IBP relations about the actual computation of the (master) integrals?

5. How do we arrive at a 'good' choice of master integrals?

6. What do we know about non-linear relations between Feynman integrals?

7. Is there a graphical interpretation of the "alphabet" of the iterated integrals, assuming they can be expressed in terms of such functions?

8. Which physical information can be used to obtain more effective IBP reductions?

9. For a reduction $I_{n+1} = \sum_{i=1}^{n} c_i I_i$ with a set of master integrals $\{I_1, \ldots, I_n\}$, can we impose a projection (inner product?) such that $c_i \propto \langle I_{n+1} | I_i \rangle$?

Conclusions

We are very grateful to all participants of the workshop for exciting contributions and lively discussions throughout the whole week. Thanks to the excellent expertise, the commitment and open-mindedness of the participants, and the conducive infrastructure, the event made a successful contribution in furthering the scientific progress in this field and also in establishing new common ground between different research communities. Indeed, we were able to bring together a very diverse group of scientists from various origins, many meeting each other for the first time. We are convinced that the interdisciplinary dialogue between experts from particle physics, computer algebra and pure mathematics will continue on this exciting subject and we are curious to see the future answers to the questions listed above.

Last but not least, we thank the MITP for hosting and financing this workshop, and for providing perfect administrative support in a most welcoming environment.