Future Challenges in Non-Leptonic B Decays:
Theory and Experiment

Extraction of the angle γ from charmless 3-body *B* decays

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CKM parameters and charmless B meson decays

Measure CKM parameters:

- SM: V_{CKM} is unitary.
- SM + NP: V_{CKM} may not be unitary.
- Need to test unitarity and self-consistency.

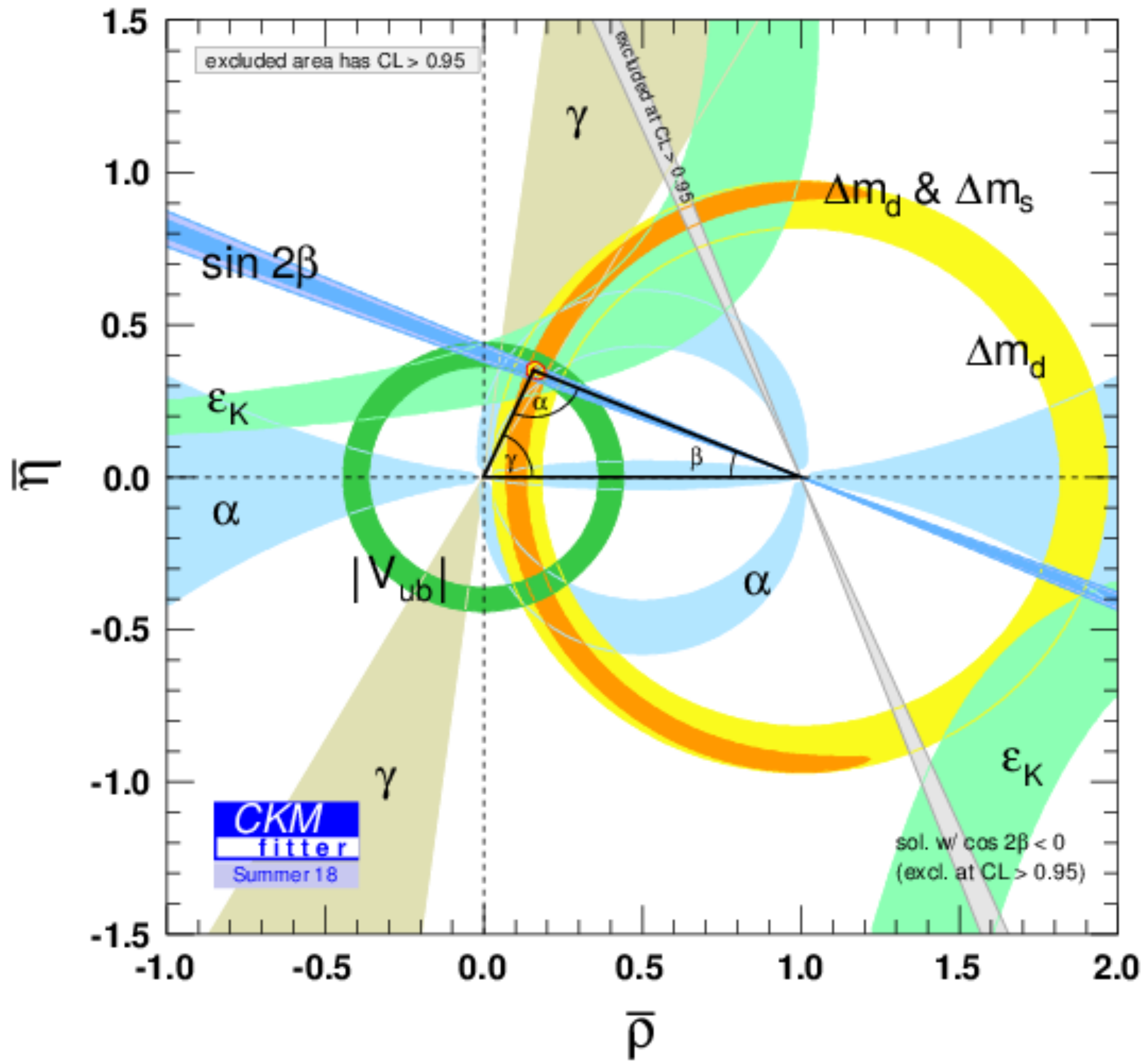
→ over-constrain the Unitarity Triangle.

Measure γ:

- from tree decays (eg. $B \to DK$).
- from loop decays [charmless].

Charmless *B* meson decays:

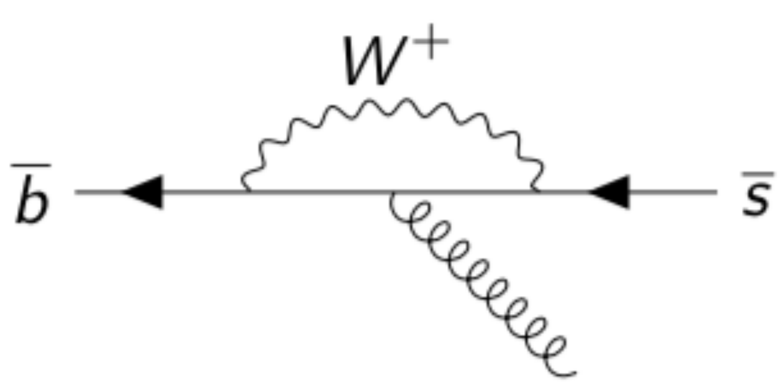
- Tree and penguin contributions can have similar size.
- CPV
- NP searches

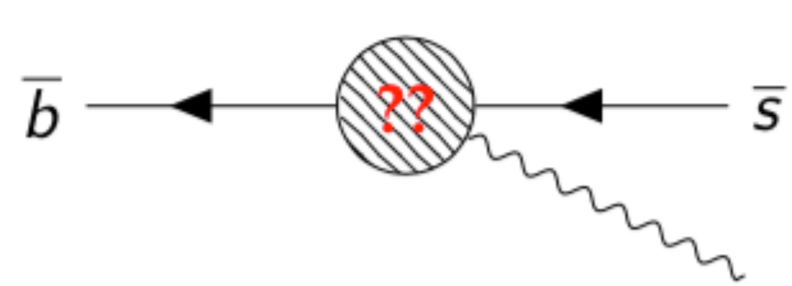


$$\alpha = 86.4^{\circ}{}^{+4.5^{\circ}}_{-4.3^{\circ}}$$

$$\beta = 22.14^{\circ}{}^{+0.69^{\circ}}_{-0.67^{\circ}}$$

$$\gamma = 72.1^{\circ}{}^{+5.4^{\circ}}_{-5.7^{\circ}}$$





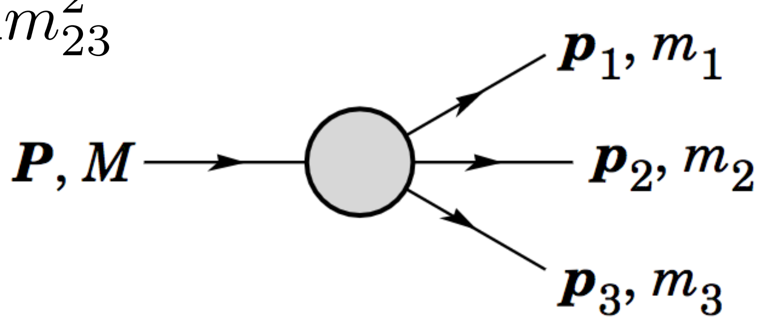
3-body decays

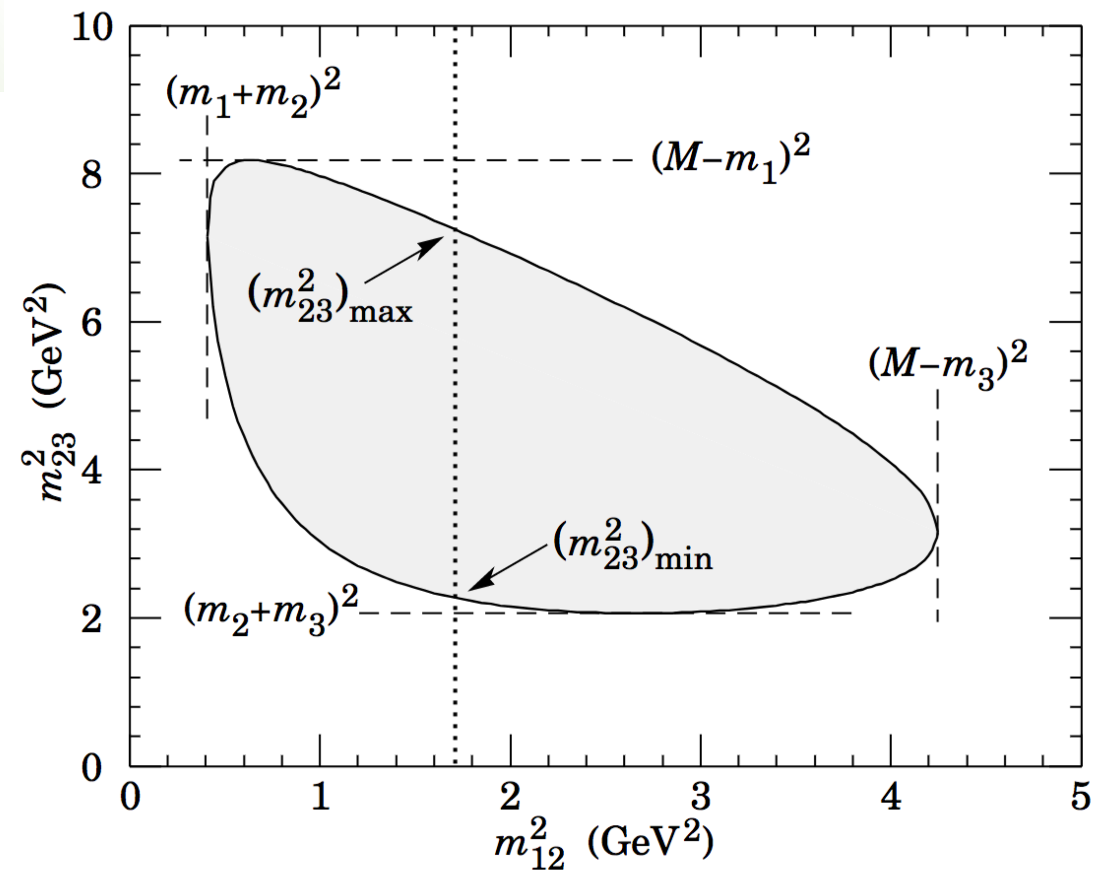
Dalitz plot

- Information on the resonant structure.
- Direct access to phases
- Branching ratios, direct and indirect CP asymmetries.

- Partial width of the decay:

$$d\Gamma = \frac{1}{(2\pi^3)}\frac{1}{32M^2}|\overline{A}|^2 dm_{12}^2 dm_{23}^2$$





Experimental parametrisation of the DP: Isobar Model

The total amplitude of a 3-body decay is described as a coherent sum of partial amplitudes:

$$A(m_{12}^2, m_{23}^2) = \sum_{j=1}^{N} c_j e^{i\phi_j} F_j(m_{12}^2, m_{23}^2)$$

Isobar parameters: weak dynamics

Lineshape: strong dynamics

Method overview

Phys. Lett. B728 (2014) 206-209

- Method to extract the CKM angle γ from charmless loop processes (NP sensitive) developed by Bhubanjyoti Bhattacharya, Maxime Imbeault and David London.

- Combine information from 5 charmless 3-body decays of B mesons

$$B^0 \to K_S K_S K_S \quad B^0 \to K^+ \pi^0 \pi^- \quad B^+ \to K^+ \pi^+ \pi^-$$
$$B^0 \to K_S K^+ K^- \quad B^0 \to K_S \pi^+ \pi^-$$

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- Combine information from 5 charmless 3-body decays of B mesons

1) $B^0 \to K_S K_S K_S$ 2) $B^0 \to K^+ \pi^0 \pi^-$ 3) $B^+ \to K^+ \pi^+ \pi^-$
4) $B^0 \to K_S K^+ K^-$ 5) $B^0 \to K_S \pi^+ \pi^-$

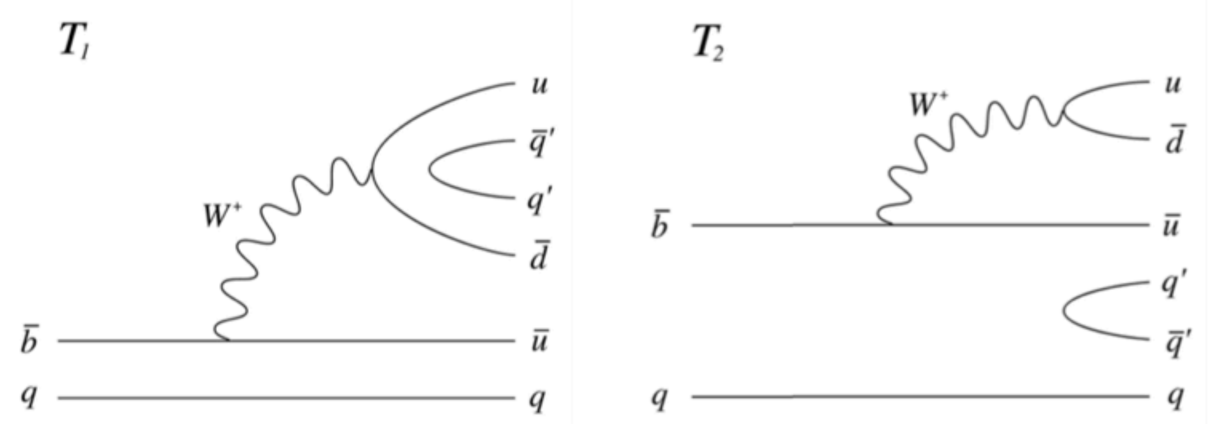
- Use BABAR models (resonant content, lineshapes) and analysis results (isobar parameters, correlation matrices) to reconstruct the amplitudes of the different decay modes over the DP and extract γ with its uncertainty.

1) Phys. Rev. D85 (2012) 054023 2) Phys. Rev. D83 (2011) 112010 3) Phys. Rev. D78 (2009) 112004
4) Phys. Rev. D78 (2012) 112010 5) Phys. Rev. D80 (2009) 112001

Flavour SU(3) diagrammatics

Phys. Rev. D.84.034040

- Different topologies for b→ s transitions: T'$_1$, T'$_2$, C'$_1$, C'$_2$, P'$_1$, P'$_2$...



- γ from 2-body decays: $N_{obs} > N_{param}$
- γ from 3-body decays: $N_{obs} < N_{param}$

Need to make assumptions to reduce the number of parameters

Flavour SU(3) limit: tree and penguin diagrams are proportional

$P_{EW}^{(C)} = \kappa T^{(C)}$ with $\kappa \equiv -\frac{3}{2}\frac{|\lambda_t^{(s)}|c_9 + c_{10}}{|\lambda_u^{(s)}|c_1 + c_2}$ $\left\{ \begin{array}{l} \lambda_p^{(s)} = V_{pb}^* V_{ps} \\ c_i \text{ :Wilson coefficients} \end{array} \right.$

This relation holds only for **fully symmetric amplitudes**:

$$A_{\text{fs}}(s_{12}, s_{13}) = \frac{1}{\sqrt{6}}\left(A(s_{12}, s_{13}) + A(s_{12}, s_{23}) + A(s_{13}, s_{23}) + A(s_{13}, s_{12}) + A(s_{23}, s_{12}) + A(s_{23}, s_{13})\right)$$

Theoretical expressions for the amplitudes

Theoretical amplitudes for each mode can be expressed in terms of:

- 5 effective diagrams
- 1 weak phase
- 1 parameter related to flavour SU(3) breaking

$$
\begin{aligned}
2A_{\rm fs}(B^0 \to K^+\pi^0\pi^-) &= Be^{i\gamma} - \kappa C \\
\sqrt{2}A_{\rm fs}(B^0 \to K^0\pi^+\pi^-) &= -De^{i\gamma} - \tilde{P}'_{\rm uc}e^{i\gamma} - A + \kappa D \\
A_{\rm fs}(B^0 \to K^0K^0\overline{K}^0) &= \alpha_{\rm SU(3)}(\tilde{P}'_{\rm uc}e^{i\gamma} + A) \\
\sqrt{2}A_{\rm fs}(B^0 \to K^+K^0K^-) &= \alpha_{\rm SU(3)}(-Ce^{i\gamma} - \tilde{P}'_{\rm uc}e^{i\gamma} - A + \kappa B)
\end{aligned}
$$

Theoretical expressions for the amplitudes

Theoretical amplitudes for each mode can be expressed in terms of:

- 5 effective diagrams: A, B, C, D and P.
- 1 weak phase
- 1 parameter related to flavour SU(3) breaking

$$2A_{\rm fs}(B^0 \to K^+\pi^0\pi^-) = Be^{i\gamma} - \kappa C$$

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$$A_{\rm fs}(B^0 \to K^0K^0\overline{K}^0) = \alpha_{\rm SU(3)}(\tilde{P}'_{\rm uc}e^{i\gamma} + A)$$

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Theoretical expressions for the amplitudes

Theoretical amplitudes for each mode can be expressed in terms of:

- 5 effective diagrams: *A*, *B*, *C*, *D* and *P*.
- 1 weak phase: γ.
- 1 parameter related to flavour SU(3) breaking

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- 1 weak phase: γ.
- 1 parameter related to flavour SU(3) breaking: $\alpha_{SU(3)}$.

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Parameter counting for 4 modes (5 modes)
10 (11) theoretical parameters

Observables

From the extracted amplitudes of the 4 (5) modes, we construct **momentum dependent observables**

$$X(s_{13}, s_{23}) = |A_{fs}(s_{13}, s_{23})|^2 + |\overline{A}_{fs}(s_{13}, s_{23})|^2$$

X: branching ratio [available for 4 (5) modes]

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Y: direct ACP [available for 4 (5) modes]

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Parameter counting for 4 modes (5 modes)
11 (13) observables

Fit principle

10 (11) parameters

11 (13) observables

observables as functions of the parameters

γ extracted with a fit

Fit principle

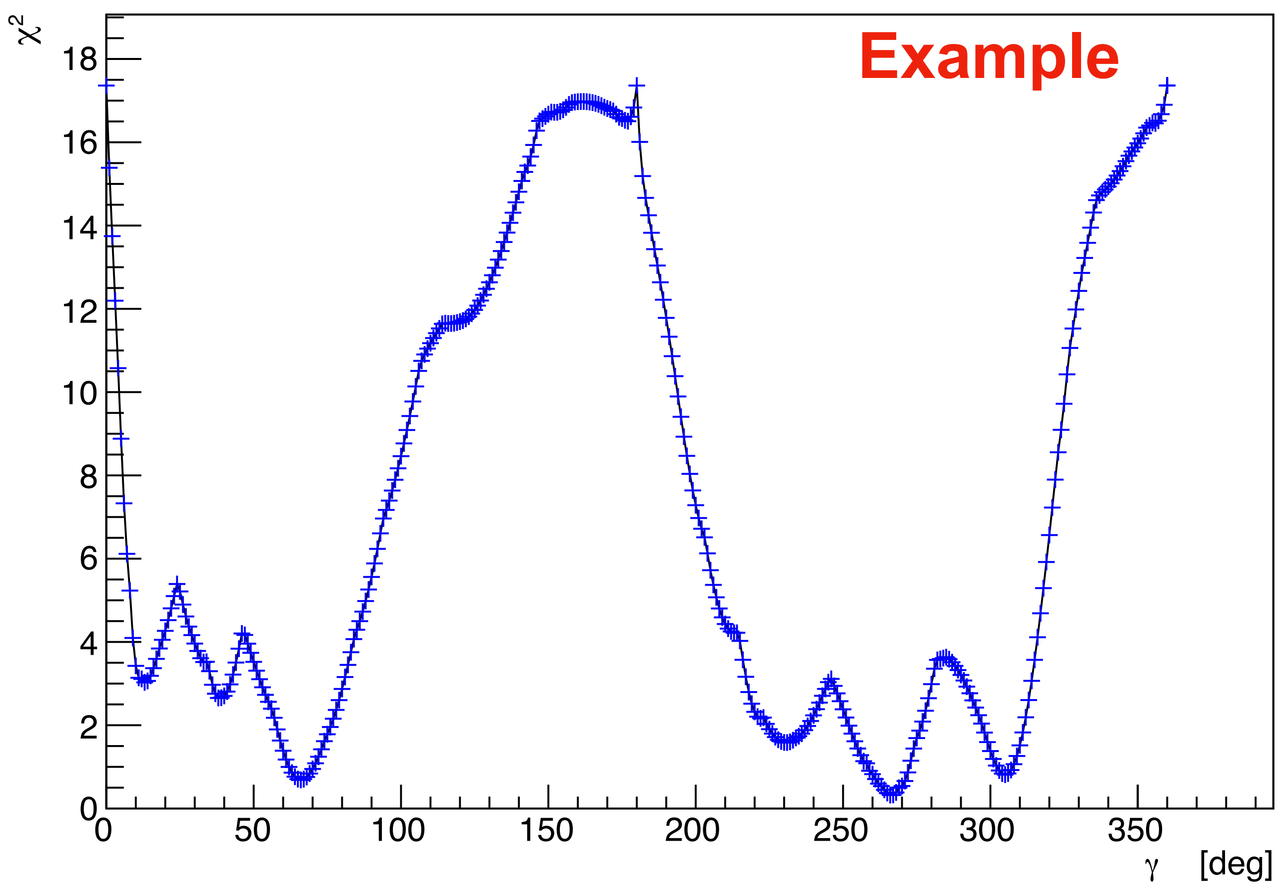
10 (11) parameters → **observables as functions of the parameters** → **γ extracted with a fit**

11 (13) observables → **observables as functions of the parameters**

Extraction of γ at one point (s_{13}, s_{23}) on the DP:

- Compute observables: X (s_{13}, s_{23}), Y (s_{13}, s_{23}), Z(s_{13}, s_{23}).
- Compute the covariance matrix including the correlations.
- Scan on γ: fix γ to consecutive values and evaluate the other parameters minimising a χ^2 function.

Cov matrix: 11x11 (13x13)



Fit principle

10 (11) parameters

11 (13) observables

→ **observables as functions of the parameters** → **γ extracted with a fit**

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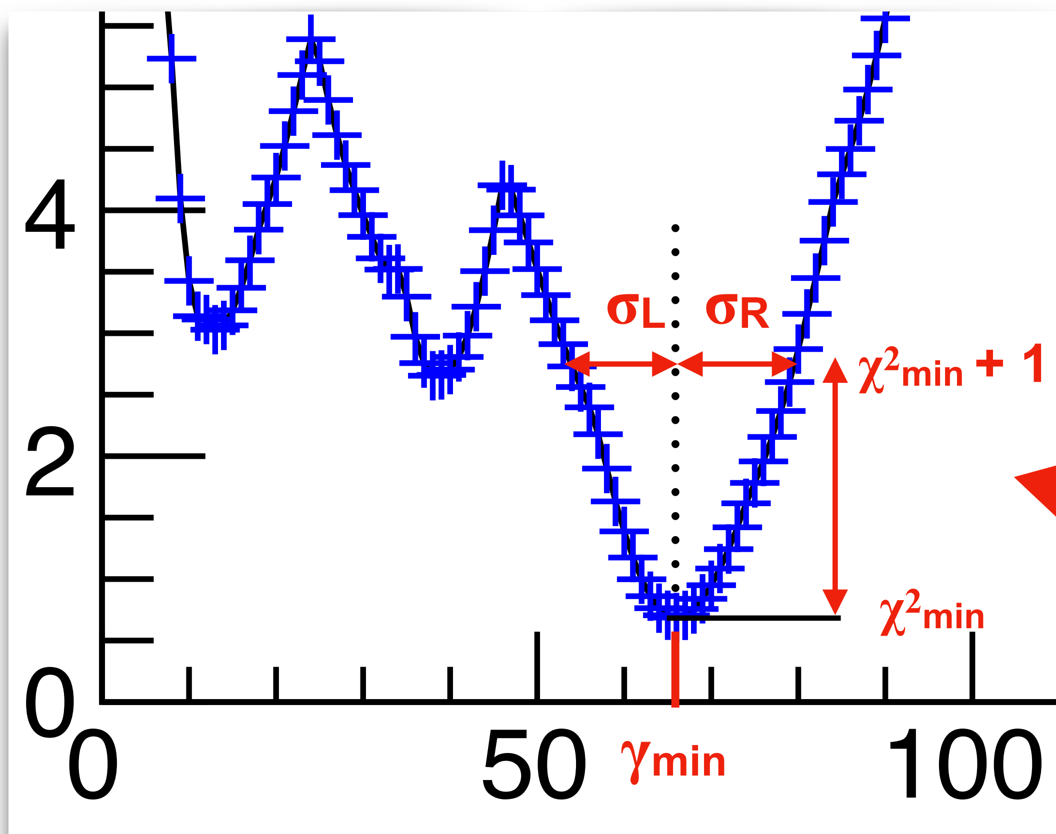
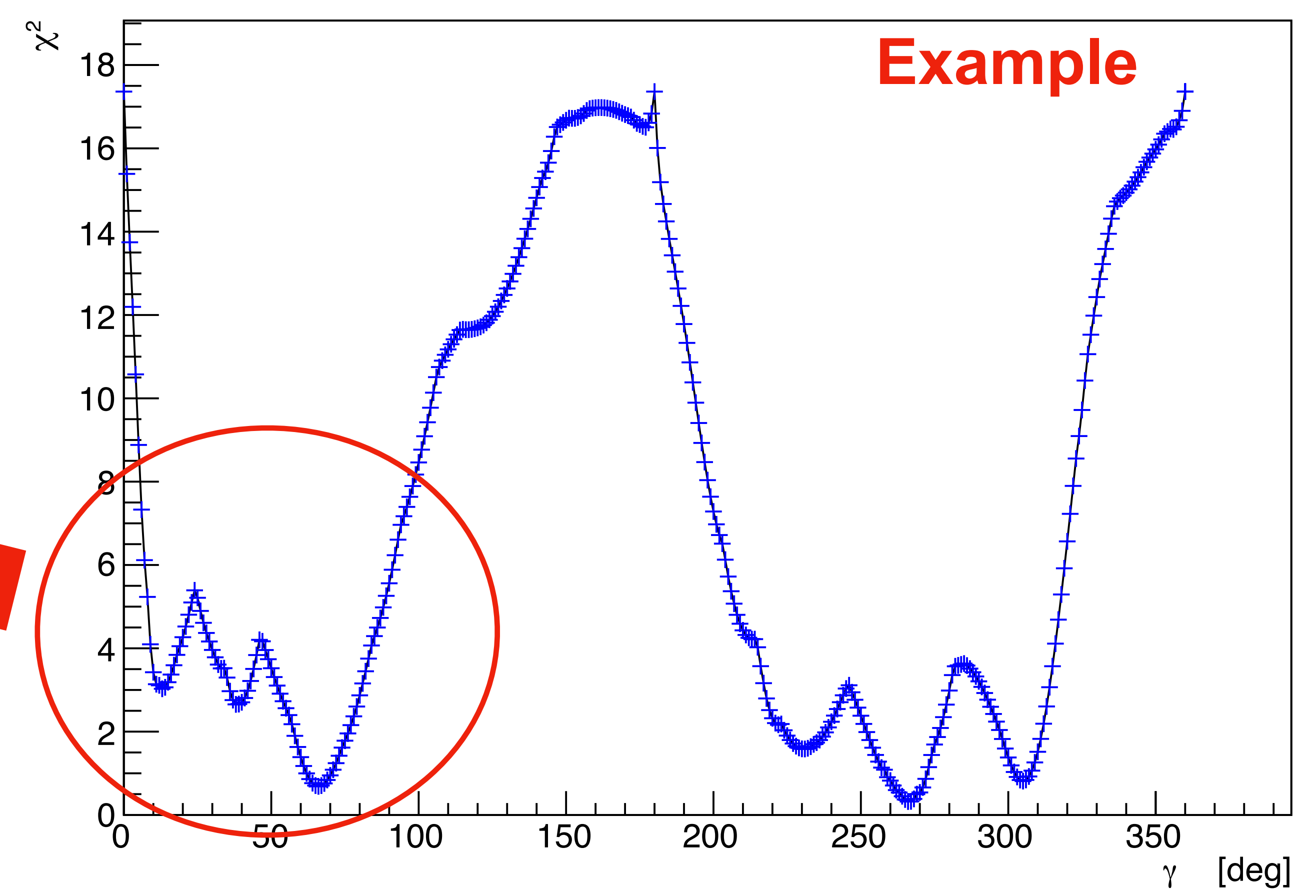
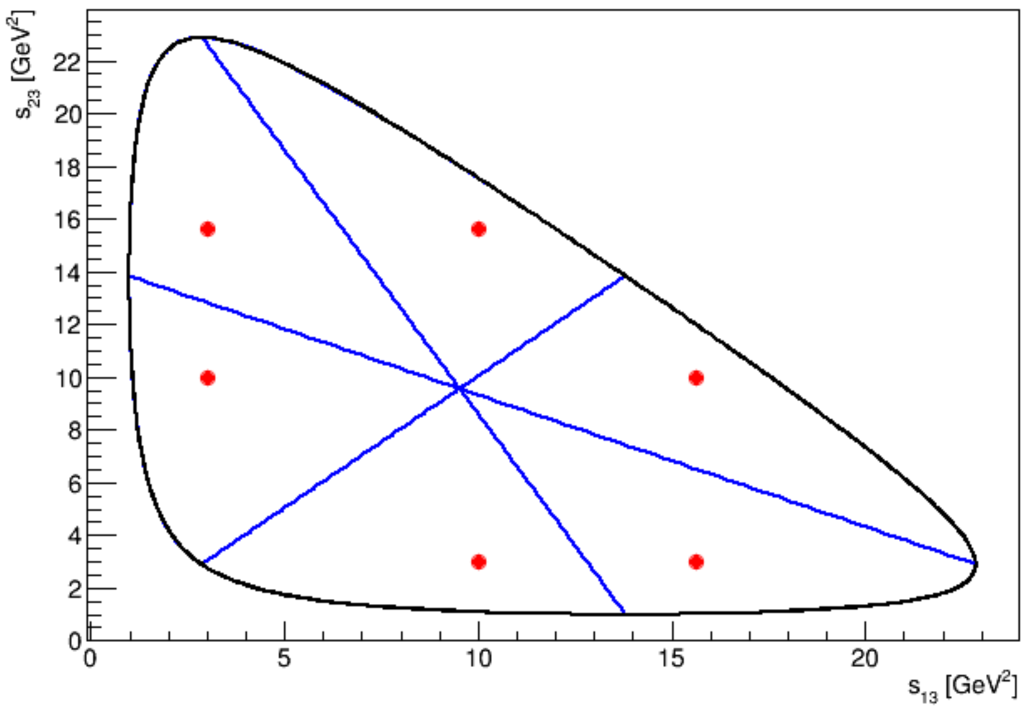




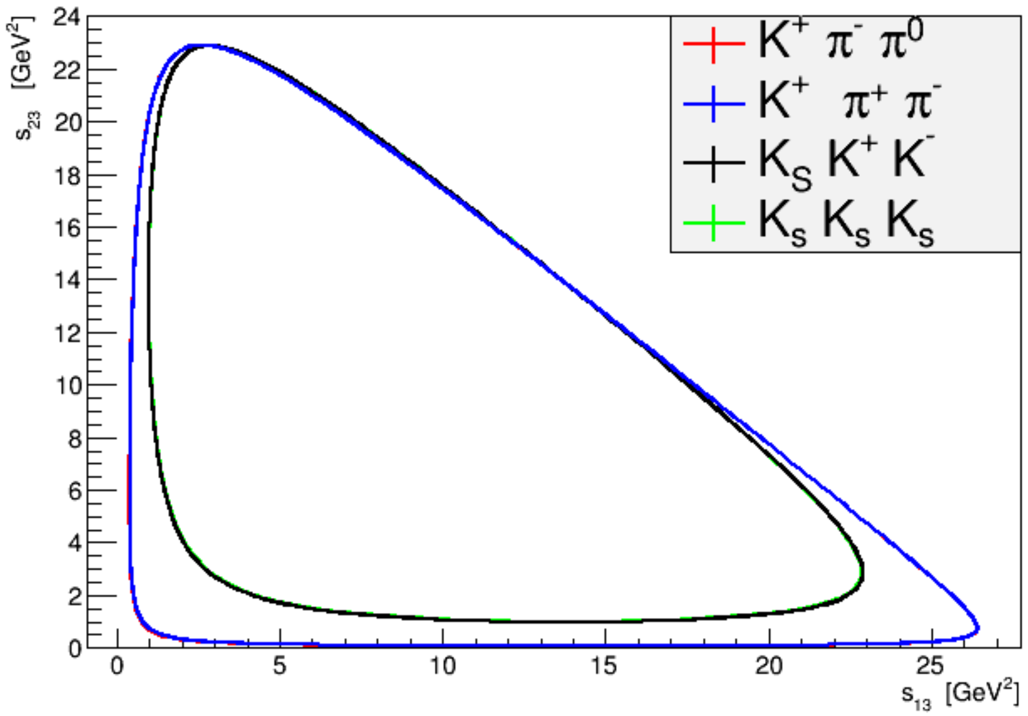
Choice of points on the DP

Fully symmetrised amplitudes

$$A_{\mathrm{fs}}(s_{12}, s_{13}) = \frac{1}{\sqrt{6}}(A(s_{12}, s_{13}) + A(s_{12}, s_{23}) + A(s_{13}, s_{23}) + A(s_{13}, s_{12}) + A(s_{23}, s_{12}) + A(s_{23}, s_{13}))$$



The fully symmetric DP is divided into 6 regions containing the same information.



Kinematic boundaries of the different modes

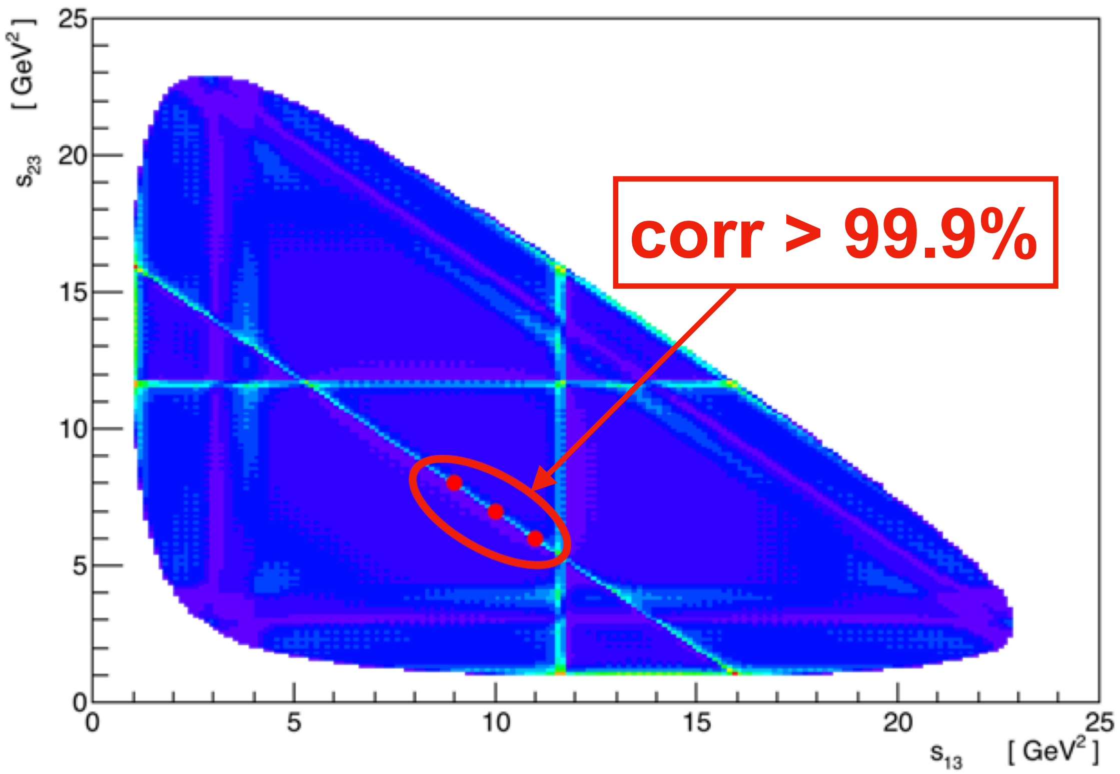
The information we can use is limited by the size of $B^0 \to K_S K_S K_S$ DP (smallest one).

Choice of points on the DP

The use of several points allows:

- Improving the validity of flavour SU(3) hyp.
- Using the maximum amount of information.

Extract γ using the maximum possible number of points on the DP.



In practice, due to very high correlations between certain points we are limited to the use of 3 simultaneous points.

Cov matrix: 33x33 (39x39)

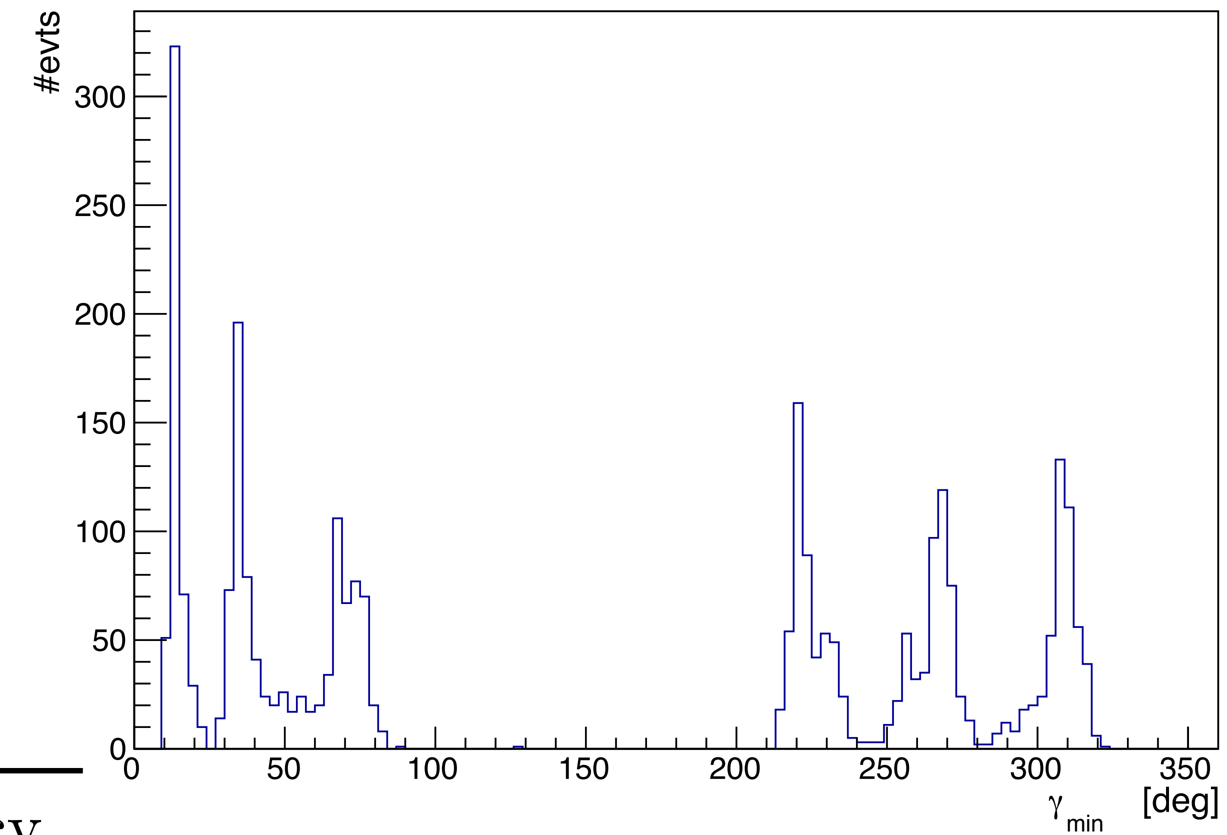
Method for extracting the results

- Several hundred combinations of 3 points randomly scattered over the DP.
- For each set of points: scan on the value of γ (500 fits with random initial parameters).
- Extract minima and statistical uncertainties for each scan.
- Combine results of all scans.
- Estimate systematic uncertainties.

Baseline results: extraction of γ using 4 modes

- $\alpha_{SU(3)}$ fixed to 1 in the fit.
- 501 sets of random 3-points combinations (correlations < 70%).
- 500 fits randomising the initial values of the parameters per set.
- Fit convergence = 100%.

Histogram of the central values of the minima extracted from the 501 sets of points.



Preferred values for γ: central values (μ) and statistical uncertainties (σ_L, σ_R).

	μ	σ_L	σ_R	frequency
minimum 1	12.9°	4.3 °	8.4°	484
minimum 2	36.6 °	6.1°	6.6°	474
minimum 3	68.9 °	8.6 °	8.6°	461
minimum 4	223.2°	7.5°	10.9°	499
minimum 5	266.4°	10.8°	9.2°	487
minimum 6	307.5°	8.1°	6.9°	488

$$\gamma_{SM} = 72.1^{\circ}{}^{+5.4^{\circ}}_{-5.8^{\circ}}$$

Results

- 6 possible values for γ.
- 3rd minimum compatible with SM.
- Statistical error of the order of 10°.

Systematic uncertainties

Influence of "poorly resolved" minima

- To combine the results obtained from the different sets of 3 points we average on the central values of the minima.
- Some minima are not deep enough to extract statistical uncertainties. They are labelled as "**poorly resolved minima**" and are **not included** in the average for the baseline result.
- The central value including all the minima, μ^{all}, is used to assign a systematic uncertainty

$$\text{Syst1} = |\mu - \mu^{all}|$$

Influence flavour SU(3) breaking

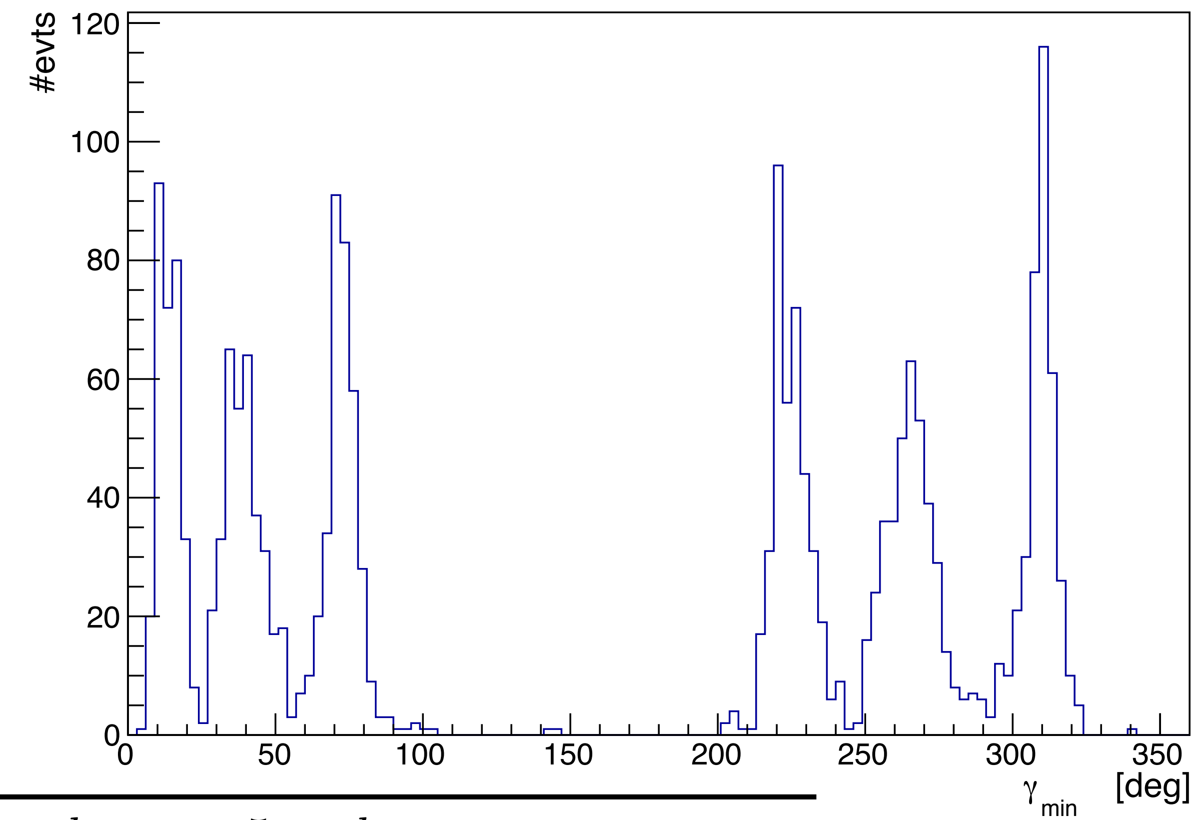
- So far we do not take into account flavour SU(3) breaking.
- γ is re-extracted with 5 modes, letting $\alpha_{SU(3)}$ float in the fit. [next slide]
- Central values found with 5 modes are used to assign a systematic uncertainty

$$\text{Syst2} = |\mu^{4modes} - \mu^{5modes}|$$

Extraction of γ using 5 modes

- $\alpha_{SU(3)}$ free in the fit.
- 401 sets of random 3-points combinations (correlations < 80%).
- 500 fits randomising the initial values of the parameters per set.
- Fit convergence ≳ 80%.

Histogram of the minima extracted from the 401 sets of points.



Preferred values for γ: central values (μ) and statistical uncertainties (σ_L, σ_R).

	μ	σ_L	σ_R	$\lvert\mu - \mu^{all}\rvert$	$\lvert\mu^{4modes} - \mu^{5modes}\rvert$	frequency
minimum 1	11.9	5.8	9.1	1.3	1.0	306
minimum 2	39.2	6.3	6.7	1.2	2.6	329
minimum 3	71.3	9.5	9.3	0.4	2.4	372
minimum 4	223.9	7.4	9.5	0.1	0.7	383
minimum 5	265.0	11.0	10.0	1.2	1.3	378
minimum 6	308.4	8.8	7.0	0.6	0.9	391

Central values and statistical uncertainties are compatible with those obtained extracting γ with 4 modes.

Summary of systematic uncertainties

	Poorly resolved minima	Flavour SU(3) breaking
Minimum 1	0.8°	1.0°
Minimum 2	0.3°	2.6°
Minimum 3	0.2°	2.4°
Minimum 4	0.7°	0.7°
Minimum 5	1.4°	1.3°
Minimum 6	0.7°	0.9°

→ **Statistical uncertainties dominate ($\approx 10^\circ$)**

Test no 1 of flavour SU(3) breaking

- From the theoretical expressions for the amplitudes:

$$A(B^0 \to K^+ K^0 K^-)_{\rm fs} = \alpha_{SU(3)} A(B^+ \to K^+ \pi^+ \pi^-)_{\rm fs}$$

- If flavour SU(3) symmetry is conserved, $\alpha_{SU(3)} = 1$, and thus these amplitudes are equal.

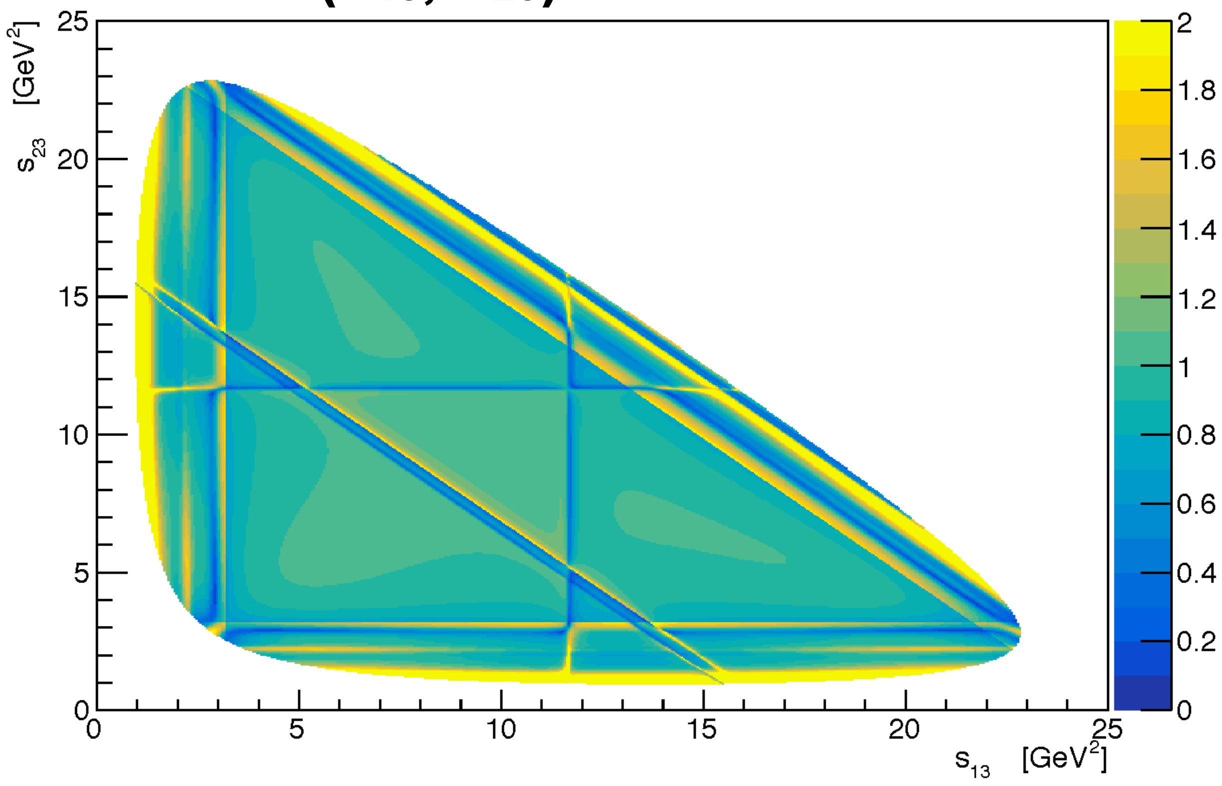
- We define the ratio $R(s_{13}, s_{23})$

$$R(s_{13}, s_{23}) = \frac{A^{K^+\pi^+\pi^-}(s_{13}, s_{23}) + \bar{A}^{K^+\pi^+\pi^-}(s_{13}, s_{23})}{A^{K_S K^+ K^-}(s_{13}, s_{23}) + \bar{A}^{K_S K^+ K^-}(s_{13}, s_{23})}$$

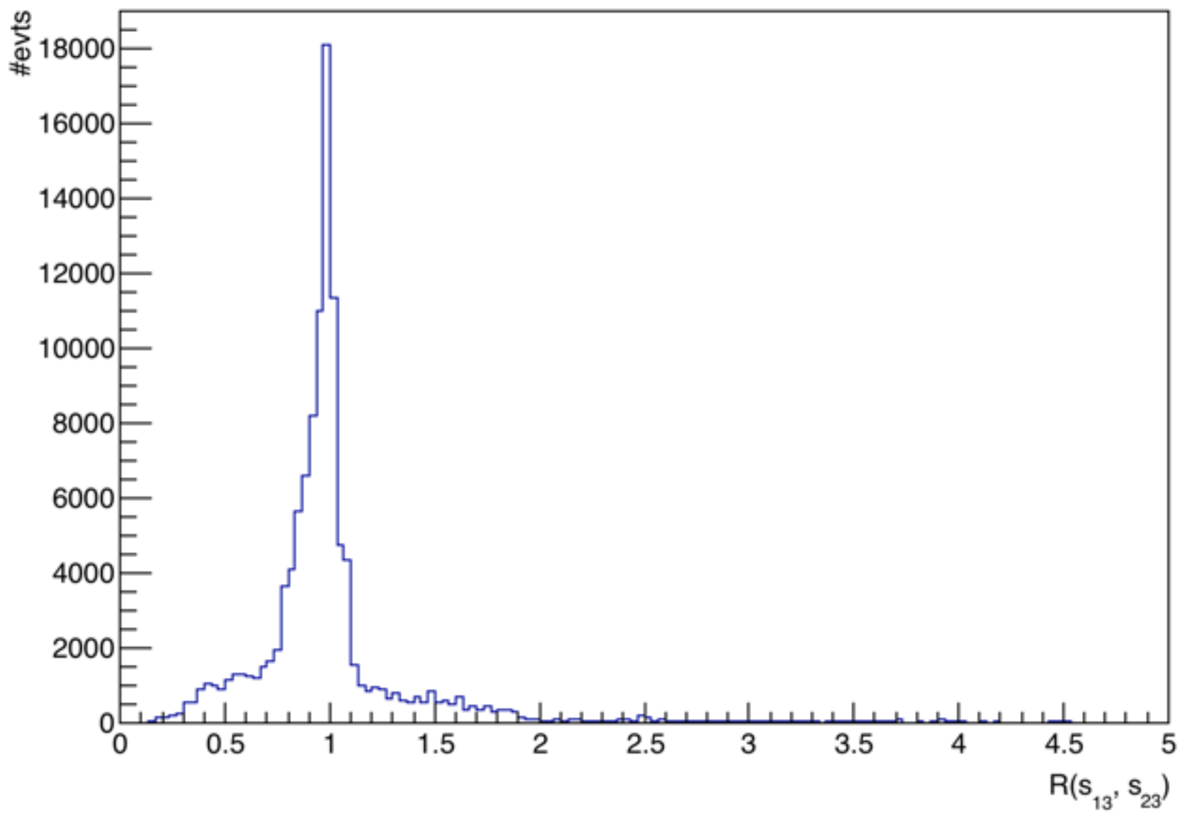
Hypothesis:
- Flavour SU(3) symmetry is conserved when averaging over many points over the DP.

Test no 1 of flavour SU(3) breaking

$R(s_{13}, s_{23})$ over the DP



Histogram of the values of $R(s_{13}, s_{23})$

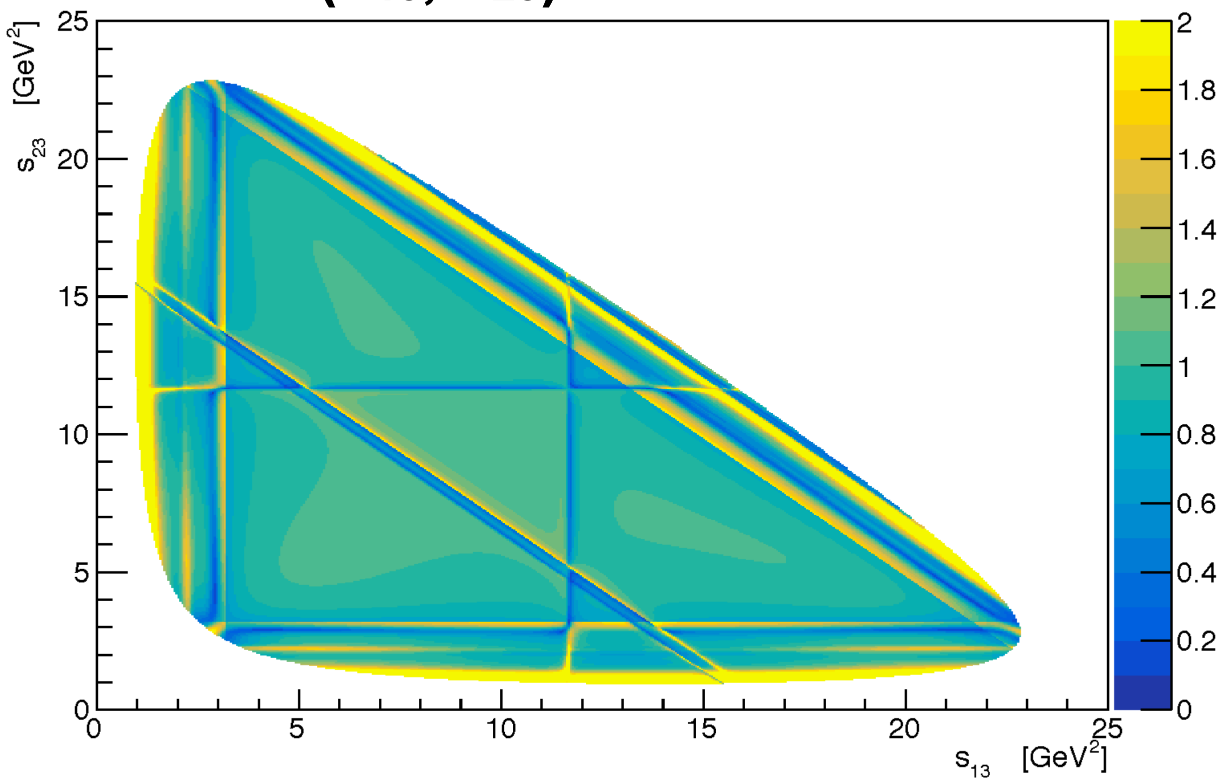


Remarks:

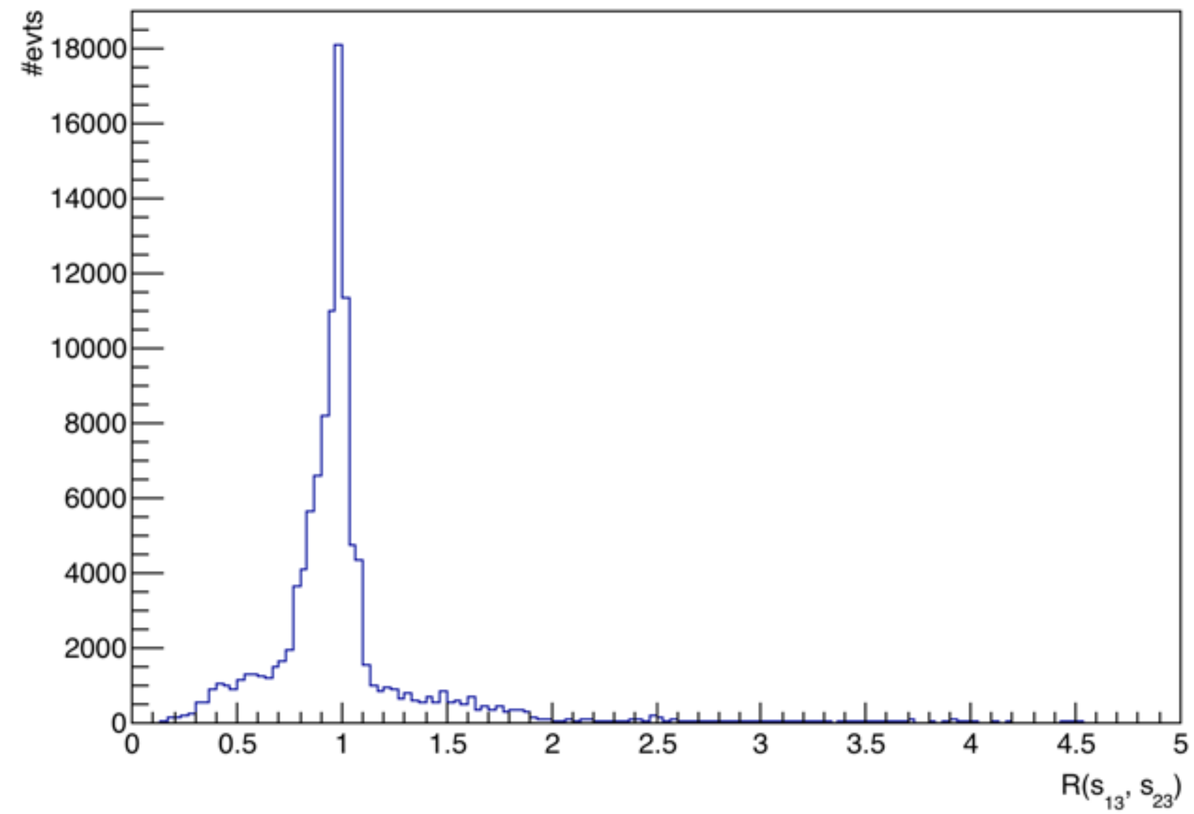
- $R(s_{13}, s_{23})$ varies over the DP, especially near resonances
- $< R(s_{13}, s_{23}) > = 1.03 \approx 1$

Test no 1 of flavour SU(3) breaking

$R(s_{13}, s_{23})$ over the DP



Histogram of the values of $R(s_{13}, s_{23})$



Remarks:

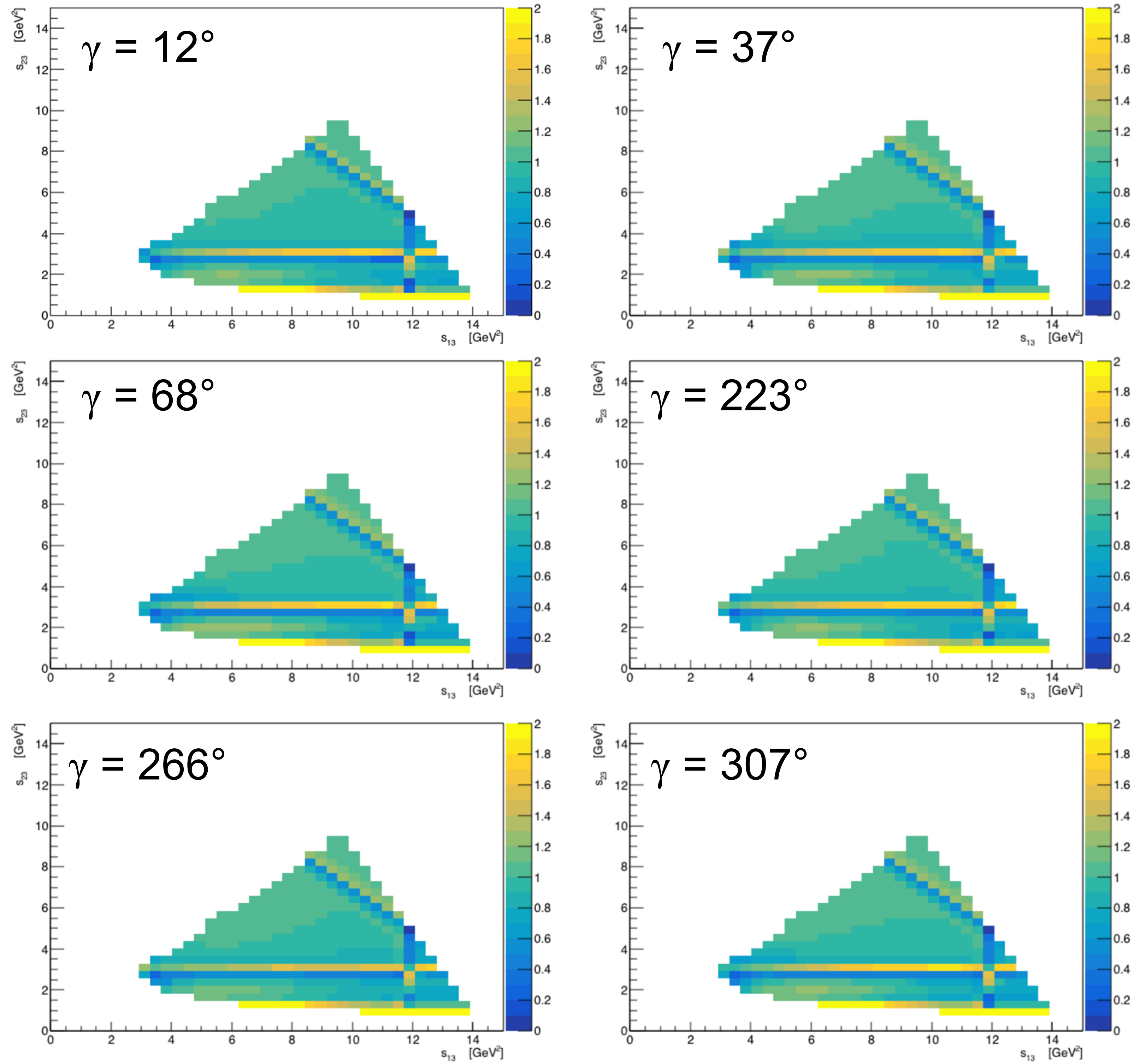
- $R(s_{13}, s_{23})$ varies over the DP, especially near resonances → **as expected.**
- $< R(s_{13}, s_{23}) > = 1.03 \approx 1$ → **as expected.**

The hypothesis of flavour SU(3) symmetry conserved "on average" holds.

Test no 2 of flavour SU(3) breaking

- Extract $\alpha_{SU(3)}$ value by a fit at several single points (≈ 400) over the DP fixing γ to the values of the 6 minima we found previously.

γ_i	$\langle \alpha_{SU(3)} \rangle_i$
12°	1.06
37°	1.06
68°	1.05
223°	1.06
266°	1.05
307°	1.05



The hypothesis of flavour SU(3) symmetry conserved "on average" holds.

Summary and results

- We studied a method for extracting γ from charmless 3-body decays relying on flavour SU(3) symmetry.
- Using BABAR results:
  - 6 values for γ (1 consistent with SM).
  - Well separated, no overlap.
  - Statistical error about 10° (BABAR results only!).
  - Statistical error dominates over Systematics.

$$\gamma_1 = 12.9^{\circ}{}^{+8.4^{\circ}}_{-4.3^{\circ}} \ \text{(stat)} \pm 1.3^{\circ} \ \text{(syst)},$$
$$\gamma_2 = 36.6^{\circ}{}^{+6.6^{\circ}}_{-6.1^{\circ}} \ \text{(stat)} \pm 2.6^{\circ} \ \text{(syst)},$$
$$\gamma_3 = 68.9^{\circ}{}^{+8.6^{\circ}}_{-8.6^{\circ}} \ \text{(stat)} \pm 2.4^{\circ} \ \text{(syst)},$$
$$\gamma_4 = 223.2^{\circ}{}^{+10.9^{\circ}}_{-7.5^{\circ}} \ \text{(stat)} \pm 1.0^{\circ} \ \text{(syst)},$$
$$\gamma_5 = 266.4^{\circ}{}^{+9.2^{\circ}}_{-10.8^{\circ}} \ \text{(stat)} \pm 1.9^{\circ} \ \text{(syst)},$$
$$\gamma_6 = 307.5^{\circ}{}^{+6.9^{\circ}}_{-8.1^{\circ}} \ \text{(stat)} \pm 1.1^{\circ} \ \text{(syst)}.$$

- The paper is on the arXiv: arXiv:1812.06194

Perspectives

The results of this study are very encouraging and we are following up in this direction.

- Take into account other symmetry states (under way):
  - totally anti-symmetric states
  - mixed states

  } may help to decrease the statistical uncertainties and reduce the number of solutions.

- Interesting longer term possibility: dedicated analysis in a single experiment (LHCb, BELLE 2...) or even joint analysis?

BACKUP

Observables

Observables as functions of the theoretical parameters

$A = ae^{i\phi_a}, B = be^{i\phi_b}, C = ce^{i\phi_c}$ and $D = de^{i\phi_d}$

$\phi_a = 0$

$$X^{th}_{K^+\pi^+\pi^-}(s_1,s_2) = a^2 + (\kappa b)^2 + c^2 + 2ac\cos\phi_c\cos\gamma - 2\kappa ab\cos\phi_b - 2\kappa bc\cos(\phi_b-\phi_c)\cos\gamma$$
$$Y^{th}_{K^+\pi^+\pi^-}(s_1,s_2) = -2\left(ac\sin\phi_c + \kappa bc\sin(\phi_b-\phi_c)\right)\sin\gamma$$
$$X^{th}_{K_SK^+K^-}(s_1,s_2) = \alpha^2_{SU(3)}X^{th}_{K^+\pi^+\pi^-}$$
$$Y^{th}_{K_SK^+K^-}(s_1,s_2) = \alpha^2_{SU(3)}Y^{th}_{K^+\pi^+\pi^-}$$
$$Z^{th}_{K_SK^+K^-}(s_1,s_2) = \alpha^2_{SU(3)}\left(-c^2\cos\gamma - ac\cos\phi_c + \kappa bc\cos(\phi_b-\phi_c)\right)\sin\gamma$$
$$X^{th}_{K_S\pi^+\pi^-}(s_1,s_2) = a^2 + (\kappa d)^2 + d^2 + 2ad\cos\phi_d\cos\gamma - 2\kappa ad\cos\phi_d - 2\kappa d^2\cos\gamma$$
$$Y^{th}_{K_S\pi^+\pi^-}(s_1,s_2) = -2ad\sin\phi_d\sin\gamma$$
$$Z^{th}_{K_S\pi^+\pi^-}(s_1,s_2) = \left(-d^2\cos\gamma - ad\cos\phi_d + \kappa d^2\right)\sin\gamma$$
$$X^{th}_{K^+\pi^+\pi^0}(s_1,s_2) = \frac{1}{2}\left(b^2 + \kappa^2c^2 - 2\kappa bc\cos\gamma\cos(\phi_b-\phi_c)\right)$$
$$Y^{th}_{K^+\pi^+\pi^0}(s_1,s_2) = \kappa bc\sin\gamma\sin(\phi_b-\phi_c)$$
$$X^{th}_{K_SK_SK_S}(s_1,s_2) = 2\alpha^2_{SU(3)}a^2$$