# Factorization and the Spectral Function Formalism for Lepton-Nucleus Cross Section

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# PREAMBLE

- ⋆ Atomic nuclei are complex many-body systems, whose response to an external probe involves a variety of different reaction mechanism
- ⋆ At small to moderate momentum transfer, typically $q \lesssim 500$ MeV, non relativistic nuclear many body-theory provides a consistent framework to carry out accurate *ab initio*—i.e. parameter free—calculations
- ⋆ At large momentum transfer, the non relativistic treatment breaks down, and more approximate approaches are required to describe the elementary interaction vertex, as well as the hadronic final state
- ⋆ The impulse approximation, naturally leading to the factorization scheme and the spectral function formalism, allows for a consistent description of a variety of reaction mechanisms
- ⋆ Being inherently modular, the formalism based on factorization is ideally suited for implementation in simulation codes

# The Lepton-Nucleus x-Nection

⋆ Consider, for example, the cross section of the process

$$\ell + A \to \ell' + X$$

at fixed beam energy. Note that this constraint does not apply to the case of neutrino scattering

$$d\sigma_A \propto L_{\mu\nu} W_A^{\mu\nu}$$

- $L_{\mu\nu}$ is fully specified by the lepton kinematical variables
- The determination of the nuclear response

$$W_A^{\mu\nu} = \sum_X \langle 0 | J_A^{\mu\,\dagger} | X \rangle \langle X | J_A^\nu | 0 \rangle \delta^{(4)}(P_0 + k - P_X - k')$$

involves

  - the ground state of the target nucleus, $|0\rangle$
  - all relevant hadronic final states, $|X\rangle$
  - the nuclear current operator

$$J_A^\mu = \sum_i j_i^\mu + \sum_{j>i} j_{ij}^\mu$$

# The Non Relativistic Regime

- ⋆ In the low-energy regime quasi elastic scattering leading to final states involving nucleons only, i.e.

$$|X\rangle = |(A-1)^{\star}\, p\rangle \, , |(A-2)^{\star}\, pp\rangle \ldots$$

  is the dominant reaction mechanism

- ⋆ at low to moderate momentum transfer, typically in the range $|\mathbf{q}| \lesssim 500$ MeV, the non relativistic approximation cal be employed to carry out highly accurate *ab initio* calculations based on realistic nuclear Hamiltonians, strongly constrained by phenomenology

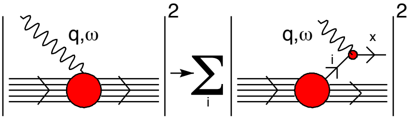
$$H = \sum_i \frac{\mathbf{p_i}^2}{2m} + \sum_{j>i} v_{ij} + \sum_{k>j>i} V_{ijk} \, ,$$

  and consistent nuclear current operators $J_A^{\mu}$

- ⋆ The non relativistic approach has been widely employed to describe the electromagnetic and weak responses of light and medium-heavy nuclei

# The Impulse Approximation (IA) Regime

- ⋆ at large momentum transfer, the final state and the current operator can no longer be described within the non relativistic approximation
- ⋆ for $\lambda \ll d_{\rm NN} \sim 1.6$ fm, the average nucleon-nucleon distance in the target nucleus, nuclear scattering reduces to the incoherent sum of scattering processes involving individual nucleons




- ⋆ Basic assumptions
  - ▷ $J_A^\mu(q) \approx \sum_i j_i^\mu(q)$ (single-nucleon coupling)
  - ▷ $|X\rangle \to |\mathbf{p}\rangle \otimes |n_{(A-1)}, \mathbf{p_n}\rangle$ (factorization of the final state)
- ⋆ As a zero-th order approximation, Final State Interactions (FSI) and processes involving two-nucleon Meson-Exchange Currents (MEC) are neglected (more on this later)

# The IA Cross Section

- ⋆ Factorisation allows to rewrite the nuclear transition matrix element in the form

$$\langle X|J_A^\mu|0\rangle \to \sum_i \int d^3k \; M_n(\mathbf{k})\langle \mathbf{k}+\mathbf{q}|j_i^\mu|\mathbf{k}\rangle$$

  - The nuclear amplitude $M_n$ describes initial sate properties, independent of momentum transfer
  - The matrix element of the current between free-nucleon states can be computed exactly using the fully relativistic expression

- ⋆ Nuclear x-section

$$d\sigma_A = \int d^3k dE \; d\sigma_N \; P(\mathbf{k}, E)$$

- ⋆ The spectral function $P(\mathbf{k}, E)$ describes the probability of removing a nucleon of momentum $\mathbf{p}$ from the nuclear ground state, leaving the residual system with excitation energy $E$
- ⋆ The lepton-nucleon cross section $d\sigma_N$ can be obtained—at least in principle—from proton and deuteron data

# Nuclear Spectral Function

- The analytic structure of the two-point Green's function—dictated by the Källèn-Lehman representations—is reflected by the spectral function

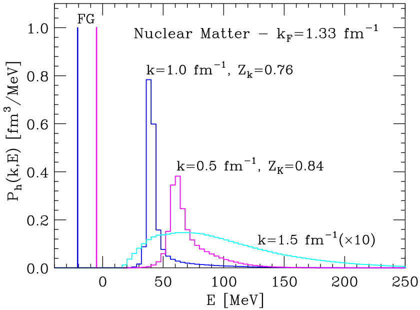
$$P(\mathbf{k}, E) = P_{MF}(\mathbf{k}, E) + P_{\text{corr}}(\mathbf{k}, E)$$
$$= \sum_{h \in \{F\}} Z_h |M_h(\mathbf{k})|^2 F_h(E - e_h) + P_{\text{corr}}(\mathbf{k}, E) \qquad (1)$$

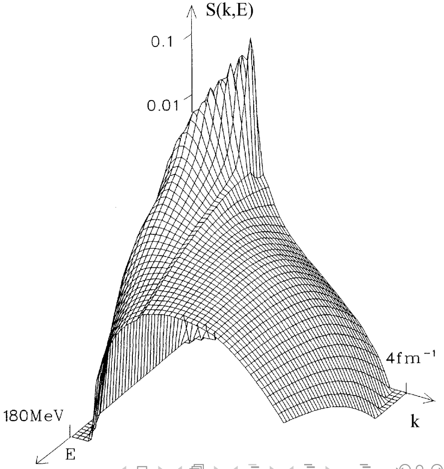
  - $Z_h M_h(\mathbf{k}) = \langle h | a_{\mathbf{k}} | 0 \rangle$
  - Energy dependence of $P_{MF}(\mathbf{k}, E)$ described by the function $F_h(E - e_h)$, sharply peaked around $E = e_h$
  - $P_{\text{corr}}(\mathbf{k}, E)$ is a *smooth* contribution arising from correlations

- In Mean Field Approximation
  - $M_h(\mathbf{k}) = \langle h | a_{\mathbf{k}} | 0 \rangle \to \phi_h(\mathbf{k})$, the momentum-space wave function of single-particle state $h$
  - Spectroscopic factors $Z_h \to 1$
  - Energy distribution $F_h(E - e_h) \to \delta(E - e_h)$
  - $P_{\text{corr}}(\mathbf{k}, E) \to 0$

# SYMMETRIC NUCLEAR MATTER AT EQUILIBRIUM DENSITY

- ⋆ Calculation carried out at 2nd order in CBF perturbation theory



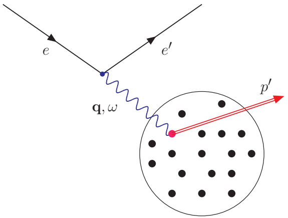



# Obtaining $P(\mathbf{k}, E)$ from Electron Scattering Data

- Consider the $(e, e'p)$ Reaction

$$e + A \to e' + p + (A - 1)$$

in which both the outgoing electron and the proton, carrying momentum $p'$, are detected in coincidence, and the recoiling nucleus can be left in a any (bound or continuum) state $|n\rangle$ with energy $E_n$




- In the absence of final state interactions (FSI)—which can be taken into account as corrections—the *measured* missing momentum and missing energy can be identified with the momentum of the knocked out nucleon and the excitation energy of the recoiling nucleus, $E_n - E_0$

$$\mathbf{p}_m = \mathbf{p}' - \mathbf{q} \quad , \quad E_m = \omega - T_{\mathbf{p}'} - T_{A-1} \approx \omega - T_{\mathbf{p}'}$$

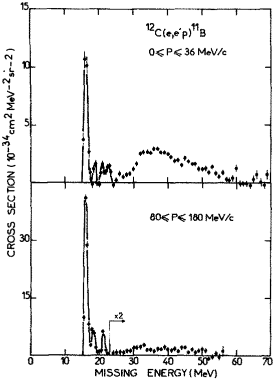
and the differential cross section is given by

$$\frac{d\sigma_A}{dE_{e'} d\Omega_{e'} dE_{p'} d\Omega_{p'}} \propto \sigma_{ep} P(p_m, E_m)$$

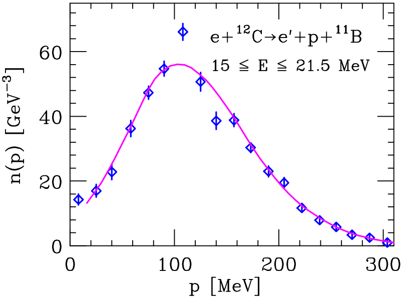
# $^{12}C(e, e'p)$ at Moderate Missing Energy

⋆ At moderate missing energy the recoiling nucleus is left in a bound state, e.g. $|^{11}B(3/2^-),\ p\rangle$, $|^{11}B(1/2^-),\ p\rangle$

- Missing energy spectrum of $^{12}C$ measured at Saclay in the 1970s



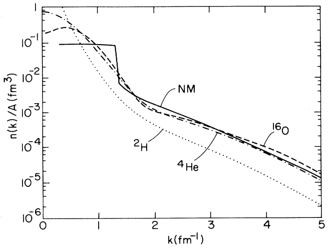

- $P$- state momentum distribution.




⋆ The spectroscopic factors turn out to be significantly less than unity

# THE LOCAL DENSITY APPROXIMATION (LDA)

- ⋆ Bottom line: accurate theoretical calculations show that the tail of the momentum distribution, arising from the continuum contribution to the spectral function, turns out to be largely $A$-independent for $A > 2$




- ⋆ Spectral functions of nuclei can be obtained within the Local Density Approximation (LDA)

$$P_{\rm LDA}(\mathbf{k},E) = P_{\rm MF}(\mathbf{k},E) + \int d^3r\, \rho_A(r)\, P^{NM}_{corr}(\mathbf{k},E;\rho=\rho_A(r))$$

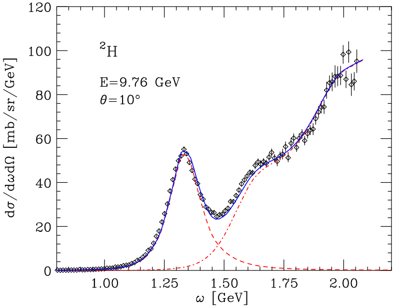
using the Mean Field (MF), or shell model, contributions obtained from $(e,e'p)$ data

- ⋆ The continuum contribution $P^{NM}_{corr}(\mathbf{k},E)$ is computed for uniform nuclear matter at different densities using accurate theoretical approaches

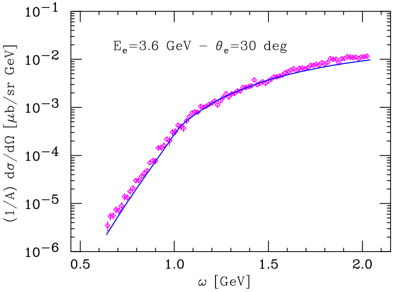
# Comparison to $e + A \to e' + X$ Data

- ⋆ From deuteron to infinite matter. Treatment of elastic and inelastic channels fully consistent
- ⋆ Results corrected for FSI effects in elastic channel

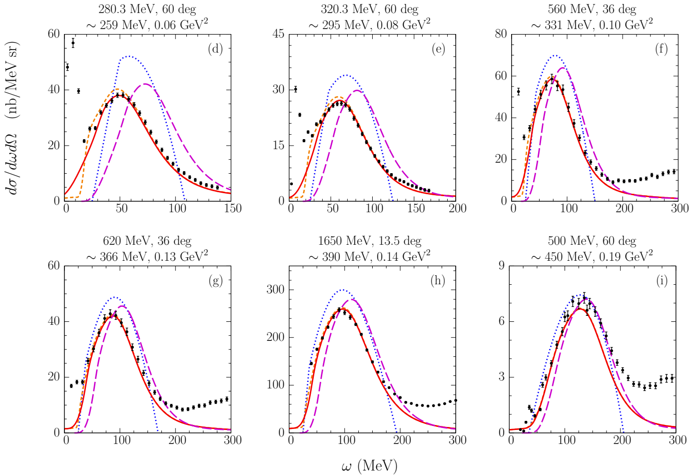
⋆ $A = 2$ (SLAC data)




⋆ $A \to \infty$ (extrapolation of SLAC data)



★ $e + {}^{12}\mathrm{C} \to e' + X$ quasi elastic cross section computed within the IA including corrections arising from FSI.

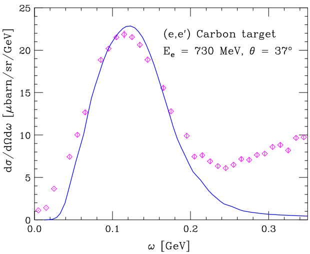


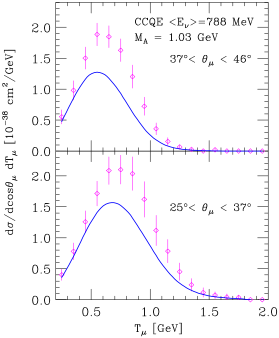

★ Recall: no adjustable parameters involved

# $eA$ VS $\nu A$ CROSS SECTION: THE ISSUE OF FLUX AVERAGE

▷ MiniBooNe CCQE cross section

▷ Electron scattering

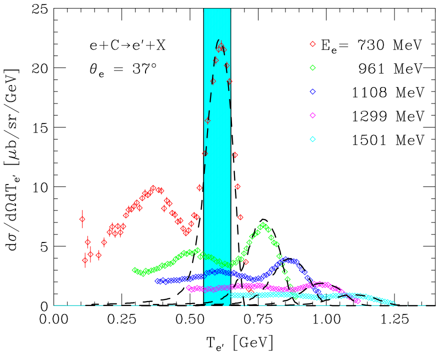



- Theoretical calculations carried out using the same spectral function and vector form factors employed to describe the electron scattering cross section and setting $M_A = 1.03$
- Owing to flux average, reaction mechanisms other than single-nucleon knock out contribute to the cross section

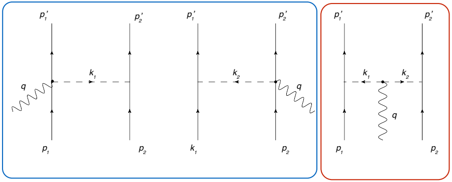
# "FLUX-AVERAGED" ELECTRON SCATTERING X-SECTION

- The electron scattering x-section off Carbon at $\theta_e$= 37° has been measured for a number of beam energies



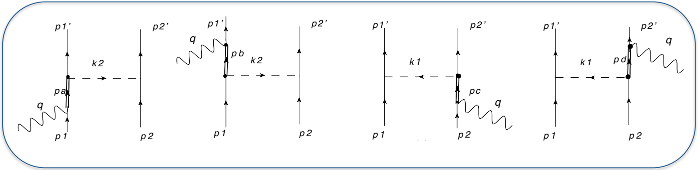

- reaction mechanisms ohter than single-nucleon knock-out contribute to the "flux averaged" cross section (for MiniBooNE $\Phi(0.7)/\Phi(1.0) \approx 0.8$)

# CORRECTIONS TO THE IA: MESON-EXCHANGE CURRENTS




Seagull
or
contact
term

Pion
in
flight
term



# THE EXTENDED FACTORISATION *ansatz*

- Highly accurate and consistent calculations of processes involving MEC can be carried out in the non relativistic regime
- Fully relativistic MEC used within independent particle models, such as the Fermi gas model
- Using relativistic MEC and a realistic description of the nuclear ground state requires the extension of the IA scheme to two-nucleon emission amplitudes
  - Rewrite the hadronic final state $|n\rangle$ in the factorized form

$$|n\rangle \to |\mathbf{p},\mathbf{p}'\rangle \otimes |n_{(A-2)}\rangle = |n_{(A-2)},\mathbf{p},\mathbf{p}'\rangle$$

$$\langle X|j^{\mu}_{ij}|0\rangle \to \int d^3k d^3k' M_n(\mathbf{k},\mathbf{k}') \langle \mathbf{p}\mathbf{p}'|j^{\mu}_{ij}|\mathbf{k}\mathbf{k}'\rangle \, \delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}-\mathbf{p}-\mathbf{p}')$$

The amplitude

$$M_n(\mathbf{k},\mathbf{k}') = \langle n_{(A-2)},\mathbf{k},\mathbf{k}'|0\rangle$$

is independent of $q$ , and can be obtained from non relativistic many-body theory

# Two-Nucleon Spectral Function

- ⋆ Calculations have been carried out for uniform isospin-symmetric nuclear matter

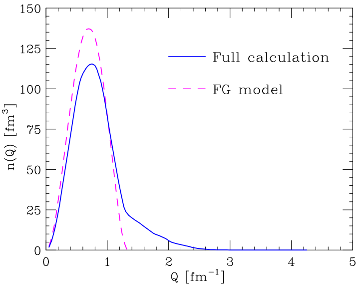
$$P(\mathbf{k}_1, \mathbf{k}_2, E) = \sum_n |M_n(k_1, k_2)|^2 \delta(E + E_0 - E_n)$$

$$n(\mathbf{k}_1, \mathbf{k}_2) = \int dE \, P(\mathbf{k}_1, \mathbf{k}_2, E)$$

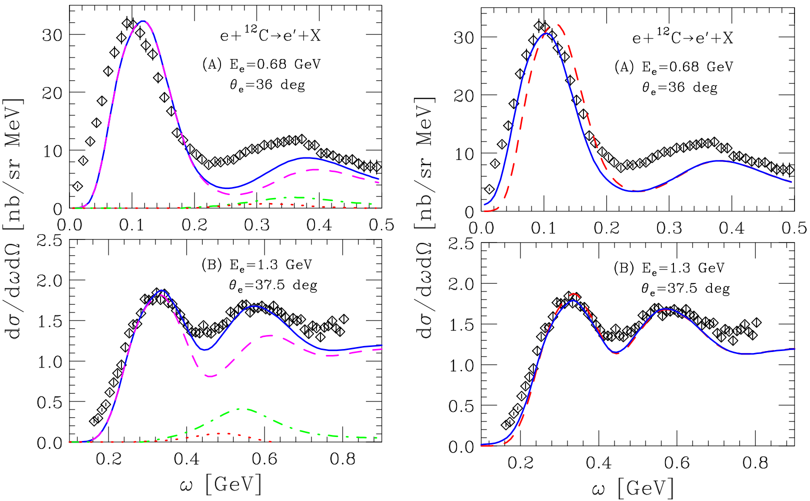
- ⋆ Relative momentum distribution

$$n(\mathbf{Q}) = 4\pi |\mathbf{Q}|^2 \int d^3q \, n\left(\frac{\mathbf{Q}}{2} + \mathbf{q}, \frac{\mathbf{Q}}{2} - \mathbf{q}\right)$$

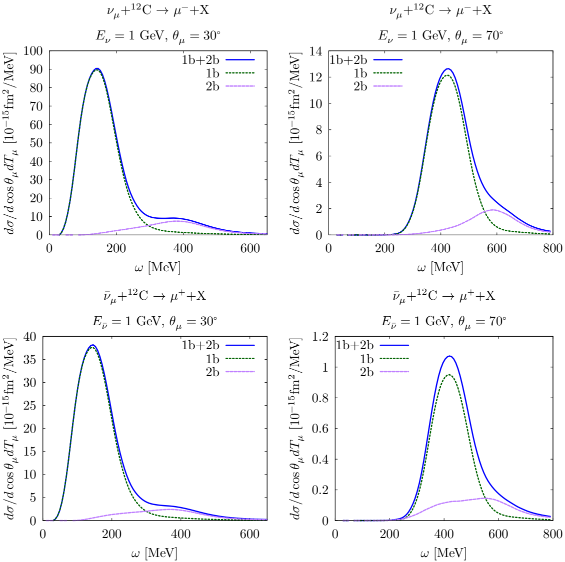
$$\mathbf{q} = \mathbf{k}_1 + \mathbf{k}_2 \ , \ \ \mathbf{Q} = \frac{\mathbf{k}_1 - \mathbf{k}_2}{2}$$



# MEC CONTRIBUTION TO $e + {}^{12}\mathrm{C} \to e' + X$

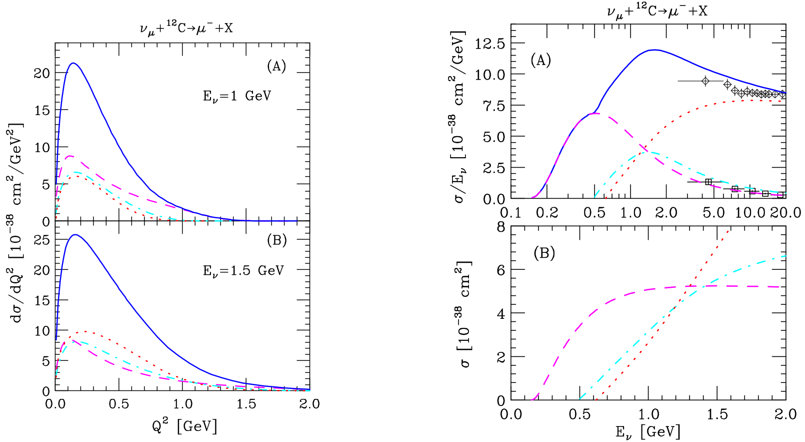


# MEC Contribution to $\nu_\mu + {}^{12}\mathrm{C} \to \mu^- + X$



# INELASTIC CONTRIBUTION TO $\nu_\mu + {}^{12}\mathrm{C} \to \mu^- + X$

- ⋆ Factorization + LDA spectral function

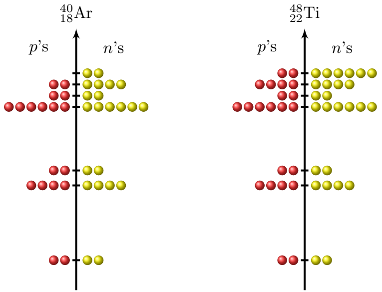


# Summary & Prospects

- ⋆ The formalism based on factorization and nuclear spectral functions—extensively applied to study electron-nucleus scattering—is approaching the level of maturity needed to perform meaningful calculations of neutrino-nucleus interactions
- ⋆ Factorization allows to combine an accurate treatment of the initial state within Nuclear Many-Body Theory with a fully relativistic treatment of the interaction vertex
- ⋆ Being based on intrinsic properties of the target, the formalism can be applied to obtain a consistent description of a variety of reaction channels, and appears to be easily implementable in generators
- ⋆ Developments needed for applications to flux-integrated neutrino cross sections include a study of processes leading to the collective excitations of the nuclear target and the treatment of final state interactions in inelastic channels
- ⋆ In the winter of 2017, JLab experiment E12-14-012 has collected $\mathrm{Ar}(e,e'p)$ and $\mathrm{Ti}(e,e'p)$ data, to be used to obtain the argon spectral functions within LDA

# The E12-14-012 Experiment: Why Argon and Titanium?

- ⋆ The reconstruction of neutrino and antineutrino energy in liquid argon detectors will require the understanding of the spectral functions describing both protons and neutrons
- ⋆ The Ar$(e, e'p)$ cross section only provides information on proton interactions. The information on neutrons can be obtained from the Ti$(e, e'p)$, exploiting the pattern of shell model levels



# Kinematic Neutrino Energy Reconstruction

- In the charged current quasi elastic (CCQE) channel, assuming single nucleon single knock, the relevant elementary process is

$$\nu_\ell + n \to \ell^- + p$$

- The *reconstructed* neutrino energy is

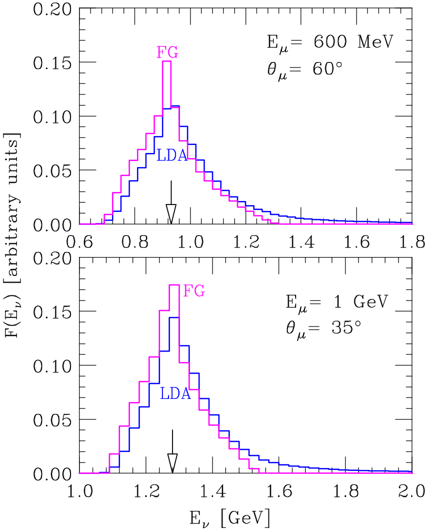
$$E_\nu = \frac{m_p^2 - m_\mu^2 - {E_n}^2 + 2E_\mu E_n - 2\mathbf{k}_\mu \cdot \mathbf{p}_n + |{\mathbf{p}_n}^2|}{2(E_n - E_\mu + |\mathbf{k}_\mu| \cos\theta_\mu - |\mathbf{p}_n| \cos\theta_n)} ,$$

  where $|\mathbf{k}_\mu|$ and $\theta_\mu$ are measured, while $\mathbf{p}_n$ and $E_n$ are the *unknown* momentum and energy of the interacting neutron

- Existing simulation codes routinely use $|\mathbf{p}_n| = 0$ , $E_n = m_n - \epsilon$ , with $\epsilon \sim 20$ MeV for carbon and oxygen, or the Fermi gas (FG) model

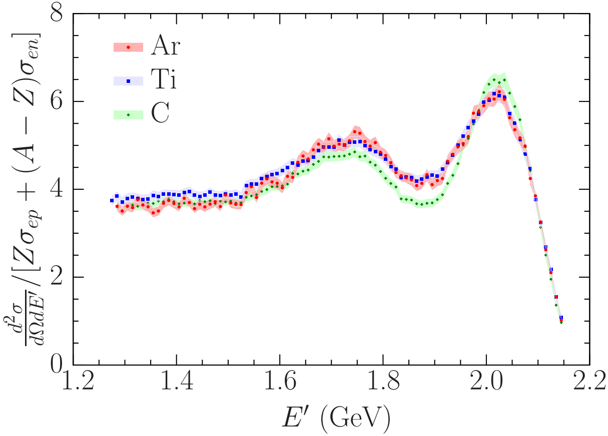
# Reconstructed Neutrino Energy in the CCQE Channel

- Neutrino energy reconstructed using $2 \times 10^4$ pairs of $\{\mathbf{p}, E\}$ values sampled from the LDA and FG oxygen spectral functions
- The average value $\langle E_\nu \rangle$ obtained from the realistic spectral function turns out to be shifted towards larger energy by an amount $\Delta E_\nu \sim 70$ MeV



# FIRST RESULTS FROM JLAB E12-14-012

- $(e, e')$ cross section at $E$ =2.222 GeV and $\theta_e$ =15.541 deg
- Titanium data published in PRC **98**, 014617 (2018), Argon paper in preparation



# QUESTIONS & (TENTATIVE) ANSWERS

- **Q**: What is the role of relativity in neutrino-nucleus collisions?
- **A**: Relativity plays an important role. The formalism based on factorization and nuclear spectral functions is ideally suited to take into account the effects of relativistic kinematics, as well as the appearance of hadrons other than nucleons in the final states

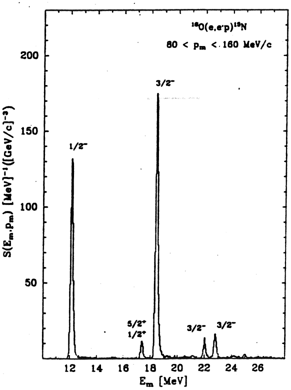
- **Q**: What new electron scattering measurements can be useful for this program?
- **A**: Measurements of the response function of oxygen and argon (& titanium) at low to moderate energy may provide information useful to test theoretical model in the regime relevant to supernova neutrinos, as well as to pin down the limit of applicability of the impulse approximation paradigm

- **Q**: How can better theory be implemented in neutrino generators simulations?
- **A**: In the impulse approximation regime the implementation is conceptually straightforward. The main issue is the development of a framework allowing to include corrections (MEC, collective excitations. . . ) in a consistent fashion

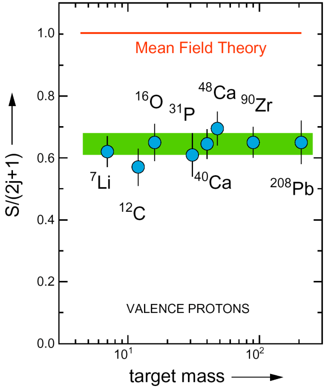
Backup slides

# $(e, e'p)$ Cross Section at Moderate $p_m$ and $E_m$

- ⋆ The spectroscopic lines corresponding to the energies of the shell model states are clearly seen in missing energy spectra

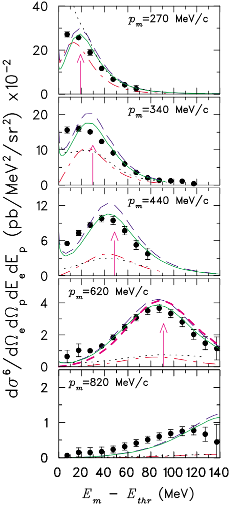



- ⋆ The integrated strengths yielding their normalisations are significantly below unity

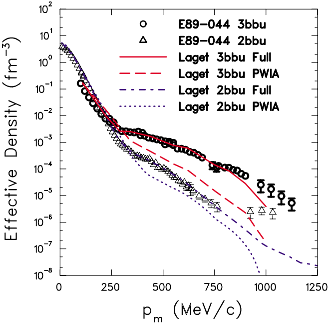


# $(e, e'p)$ Studies of the Correlation Strength

- $^3\text{He}(e, e'p)$ at large $|\mathbf{p}_m|$ and $E_m$ in JLab hall A: strong energy-momentum correlation observed.

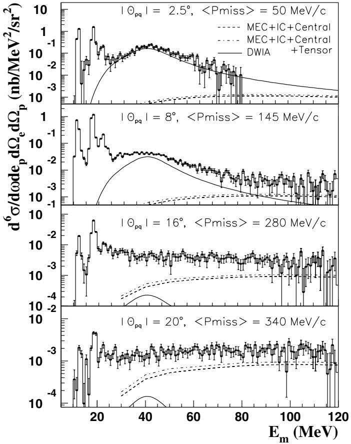



$$n(k) = \int dE\, P(\mathbf{k}, E)$$



# Large $|\mathbf{p}_m|$ and $E_m$ Strength in Oxygen

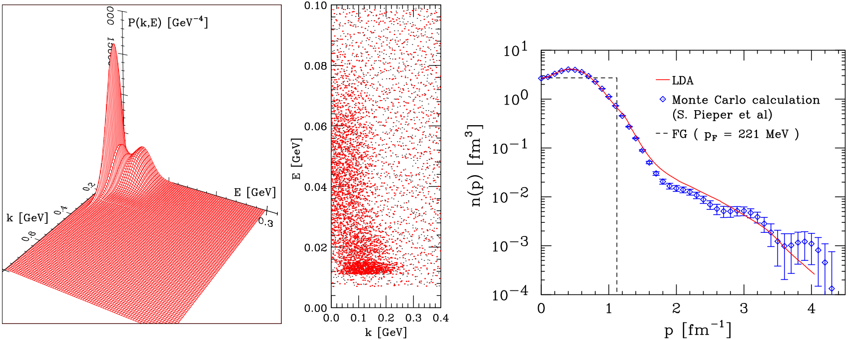
- $|\mathbf{p}_m|$-evolution of missing energy spectrum in Oxygen. Hall A data




- The determination of the spectral function at large missing energy and missing momentum is hindered by significant FSI and MEC effects

# SPECTRAL FUNCTION AND MOMENTUM DISTRIBUTION OF $^{16}$O

- ⋆ $n(k) = \int dE\, P(k,E)$




- ⋆ shell model states account for $\sim 80\%$ of the strenght
- ⋆ the remaining $\sim 20\%$ , arising from NN correlations, is located at high momentum and large removal energy

# Determination of the Spectroscopic Factor

- The spectroscopic factor of the $p$-state with $j = 3/2$ is obtained from

$$Z_p = \frac{(2j+1)}{Z} \int_{\Delta k} \frac{d^3k}{(2\pi)^3} \int_{\Delta E} dE \, P_{\rm expt}(|\mathbf{k}|, E) = 0.625$$

with

$$\Delta k \equiv [0\text{–}310] \ \mathrm{MeV} \quad , \quad \Delta E \equiv [15\text{–}22.5] \ \mathrm{MeV}$$

- Models based on the mean field approximation predict $Z_p = 1$
- The deviation of $Z_p$ from unity implies that dynamical effects not taken into account within the independent particle picture reduce the average number of protons occupying the $j = 3/2$ $p$-state from $4$ to $2.5$
- The result obtained from the LDA analysis is within 2% of the experimental value

# Early Studies of the Correlation Strength

- The $(e, e'p)$ cross section at large $E_m$ and $p_m$ , tipically $E_m \gtrsim 50$ MeV and $p_m \gtrsim 250$ MeV, gives access to the *correlation strength*. Strong energy–momentum correlation clearly observed.

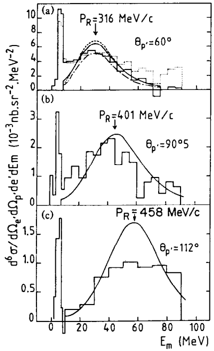



Fig. 5. Missing energy spectra form $^3$He(e, e$'$p), showing evidence for an interaction on a two-nucleon correlated pair

CEBAF PROPOSAL COVER SHEET

This Proposal must be mailed to:

CEBAF
Scientific Director's Office
12000 Jefferson Avenue
Newport News, VA 23606

and received on or before OCTOBER 31, 1989

A. TITLE: Selected studies of the $^3$He and $^4$He nuclei through electrodisintegration at high momentum transfer

B. CONTACT PERSON: J. MOUGEY

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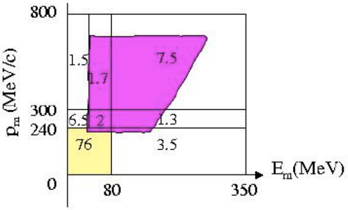
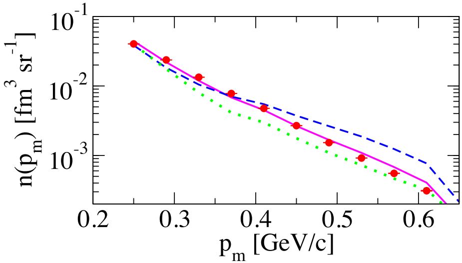
We propose to use the CEBAF Hall A High Resolution Spectrometer pair to study selective aspects of the electromagnetic response of $^3$He and $^4$He through (e,e'p) coincidence measurements at $Q^2$ values from 0.4 to 4.1(GeV/$c$)$^2$. In Part I, we propose to study the single nucleon structure of the He isotopes with special emphasis on high momenta (up to ~ 0.6 GeV/$c$) by the separation of the $R_L$, $R_T$ and $R_{LT}$ response functions. The $Q^2$ dependence of the reaction will be examined in Part II by performing longitudinal/transverse (L/T) separations for protons emitted along $\vec{q}$, up to $Q^2 = 4.11$(GeV/$c$)$^2$ at quasifree kinematics ($p_m = 0$) and for $Q^2 = 0.5$ and 1.(GeV/$c$)$^2$ at $p_m = \pm 0.3$GeV/$c$. In Part III, we focus on the continuum region to study correlated nucleon pairs. Measurements at $Q^2 = 1.$ (GeV/$c$)$^2$ and recoil momenta up to 1 GeV/$c$ are proposed, including separations of the in-plane structure functions for $p_m < 680$ MeV/$c$.

## Comparison to the Measured Correlation Strength

- The correlation strength in carbon has been investigated in JLab Hall C by the E97-006 Collaboration




- Measured correlation strength (Rohe et al, 2005)

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|---|---|
| Experiment | 0.61 ±0.06 |
| Greens function theory [3] | 0.46 |
| CBF theory [2] | 0.64 |
| SCGF theory [4] | 0.61 |