INTRODUCTION TO CHIRAL EFFECTIVE FIELD THEORY ©

bira
U. van Kolck

Institut de Physique Nucléaire d'Orsay
and *University of Arizona*

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Outline

- Effective Field Theories
  - Introduction
  - What is effective
  - Example: NRQED
  - Summary
- QCD at Low Energies

Reference:

U. van Kolck,

Effective field theories of loosely bound nuclei,

in *The Euroschool on Exotic Beams, Vol. IV*

C. Scheidenberger and M. Pfützer (eds.), Springer, Berlin Heidelberg (2014)

Lect. Notes Phys. **879** (2014) 123

Wanted

Dead ◆or◆ Alive

FORMULATION OF NUCLEAR PHYSICS CONSISTENT WITH STANDARD MODEL (SM) OF PARTICLE PHYSICS

Reward

understanding emergence of complexity
at the most fundamental level:
nucleus made out of quarks and gluons interacting
strongly (QCD), yet exhibiting many regularities

◆

use of nuclei as laboratories
for physics beyond the SM

Beware

coupling constants not small: not an easy problem!

Not an easy problem…

“There are few problems in nuclear theoretical physics which have attracted more attention than that of trying to determine the fundamental interaction between two nucleons.
It is also true that scarcely ever has the world of physics owed so little to so many …
… It is hard to believe that many of the authors are talking about the same problem or, in fact, that they know what the problem is.”

M. L. Goldberger

Midwestern Conference on Theoretical Physics, Purdue University, 1960

Nuclear Physics

The canons of tradition

I Nuclei are essentially made out of non-relativistic nucleons in two isospin states (protons and neutrons)

II The interaction potential is mostly two-body, but there is evidence for smaller three-body forces

III Isospin is a good symmetry, except for the Coulomb interaction, breaking in two-nucleon scattering lengths, and smaller effects

IV External probes (*e.g.* photons) interact mainly with each nucleon, but there is evidence for smaller two-nucleon currents

but…

WHY?

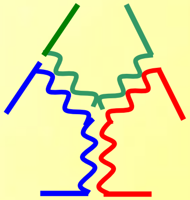
Quantum Chromodynamics

On the road to infrared slavery

1. Up, down quarks are relatively light, $m_{u,d} \sim 5\ \mathrm{MeV}$, and thus relativistic

2. The interaction is a multi-gluon, and thus a multi-quark, process

   e.g.



3. Isospin symmetry is not obvious: $\varepsilon = \dfrac{m_d - m_u}{m_d + m_u} \sim \dfrac{1}{3}$

4. External probes can interact with collection of quarks

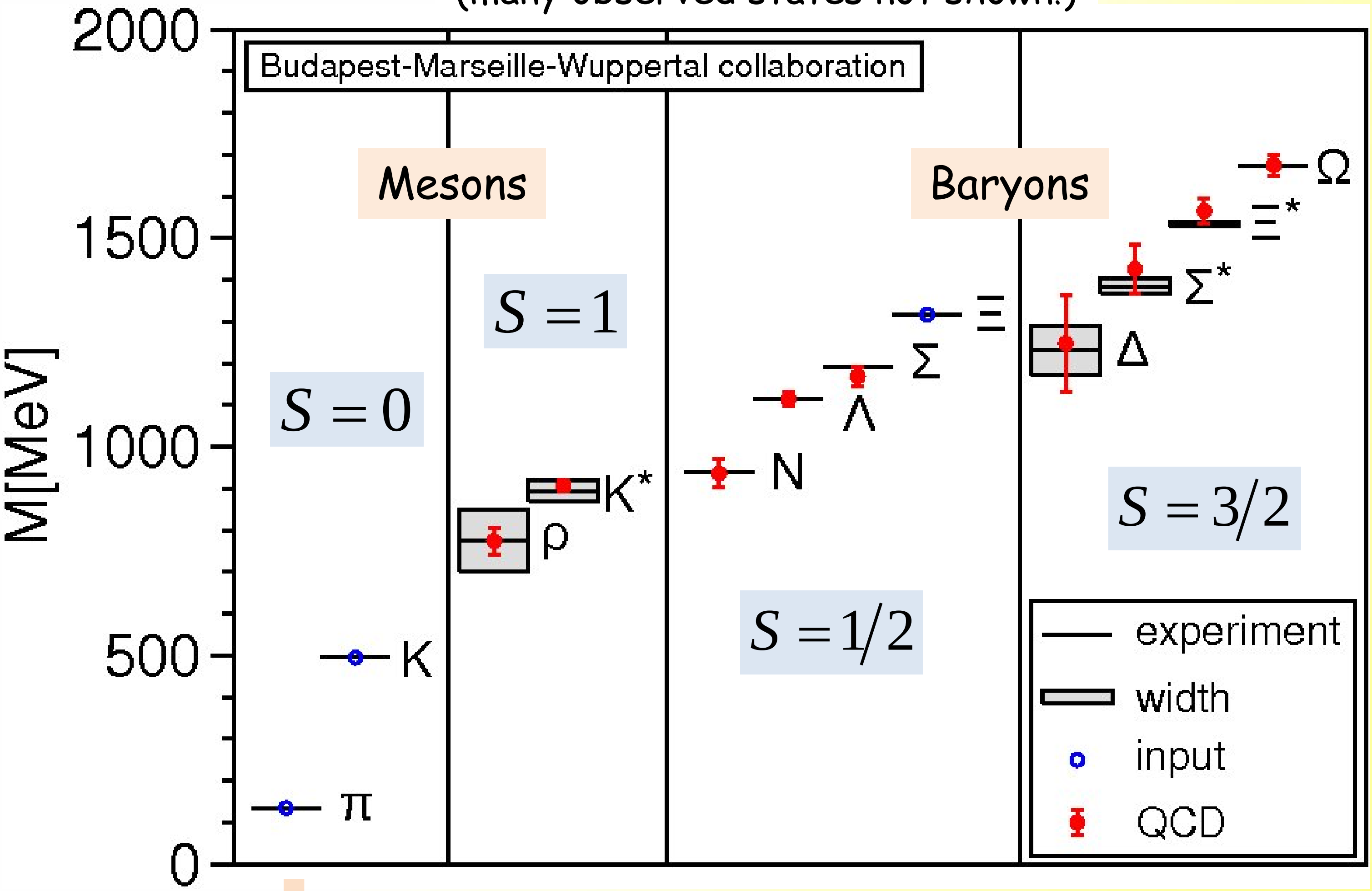
difficulty

quarks and gluons **not** the most convenient degrees of freedom at low energies

How does nuclear structure emerge from QCD?

Strongly interacting particles (hadrons)

(many observed states not shown!)



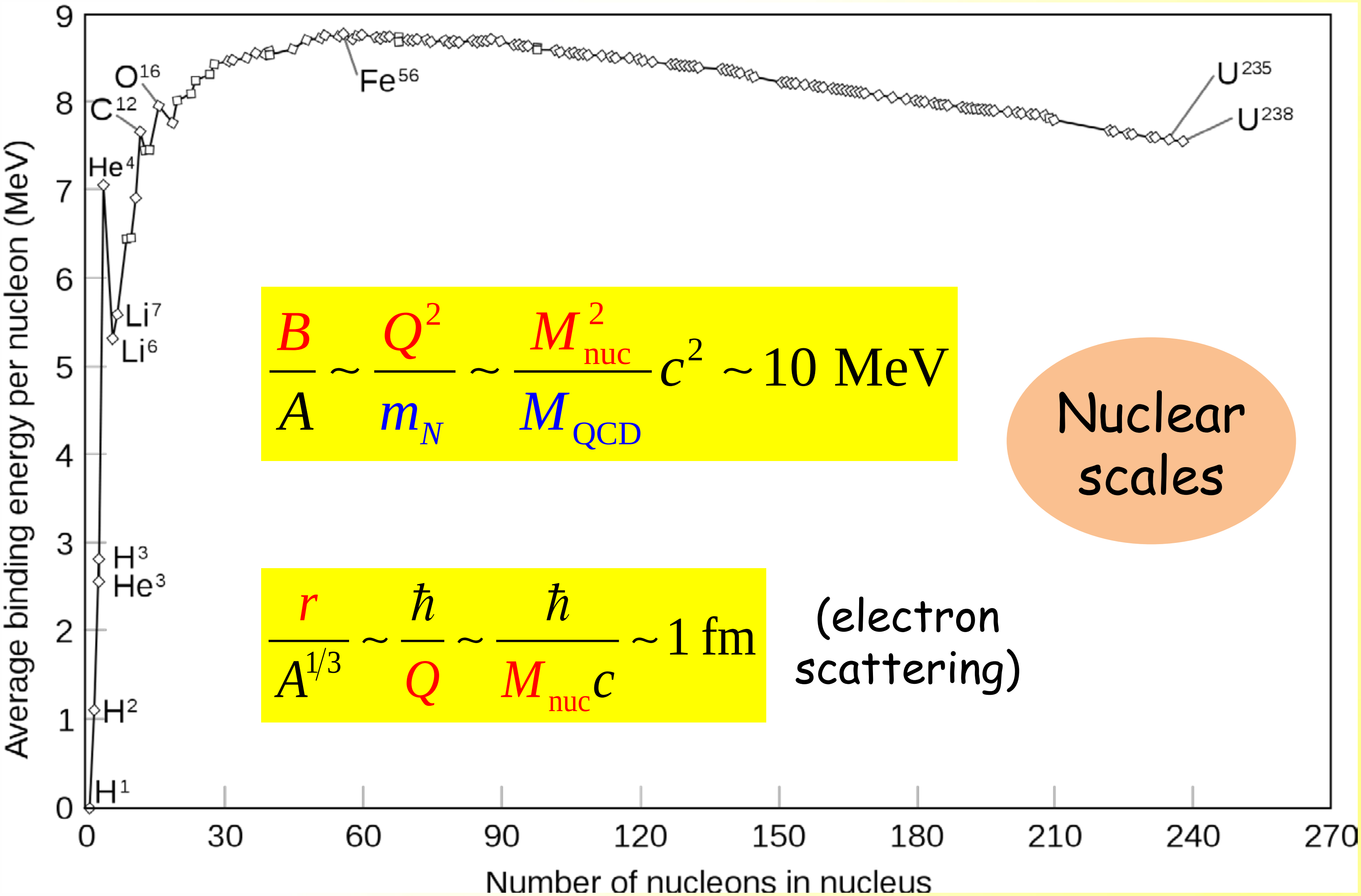
QCD Scale

Exception: pion

$m_\pi \simeq 140\ \mathrm{MeV}/c^2 \ll M_{\mathrm{QCD}}$

we'll return to it!

$$M_{\mathrm{QCD}} \sim 1000\ \mathrm{MeV}/c^2$$
$$= 1\ \mathrm{GeV}/c^2$$



$$\frac{B}{A} \sim \frac{Q^2}{m_N} \sim \frac{M_{\text{nuc}}^{\;2}}{M_{\text{QCD}}} c^2 \sim 10\ \text{MeV}$$

Nuclear
scales

$$\frac{r}{A^{1/3}} \sim \frac{\hbar}{Q} \sim \frac{\hbar}{M_{\text{nuc}} c} \sim 1\ \text{fm}$$

(electron
scattering)

$$Q \sim M_{\text{nuc}} c \sim 100\ \text{MeV}/c$$

Multi-scale problems

H atom

$$H=\left(\frac{p^2}{2m_e}-\frac{\alpha\hbar c}{r}\right)\left[1+\mathcal{O}\left(\alpha;\frac{p^2}{m_e^2c^2};\frac{\hbar^2}{m_e^2c^2r^2}\right)\right]\qquad \alpha\equiv\frac{e^2}{4\pi\hbar c}\cong\frac{1}{137}\ll 1$$

$$\left.\begin{array}{l} r\sim R\\ p\sim\dfrac{\hbar}{R}\end{array}\right\}\ E(R)\sim\left(\frac{\hbar^2}{2m_eR^2}-\frac{\alpha\hbar c}{R}\right)$$

$$\frac{dE(R)}{dR}=0\ \Rightarrow\ R=\frac{\hbar}{\alpha m_e c}$$

Three scales

$$\left\{\begin{array}{l} m_ec^2=0.5\text{ MeV}\\ pc\sim\alpha m_ec^2=3.6\text{ keV}\\ -E\sim\dfrac{p^2}{2m_e}\sim\dfrac{1}{2}\alpha^2m_ec^2=13.6\text{ eV}\end{array}\right.$$

α

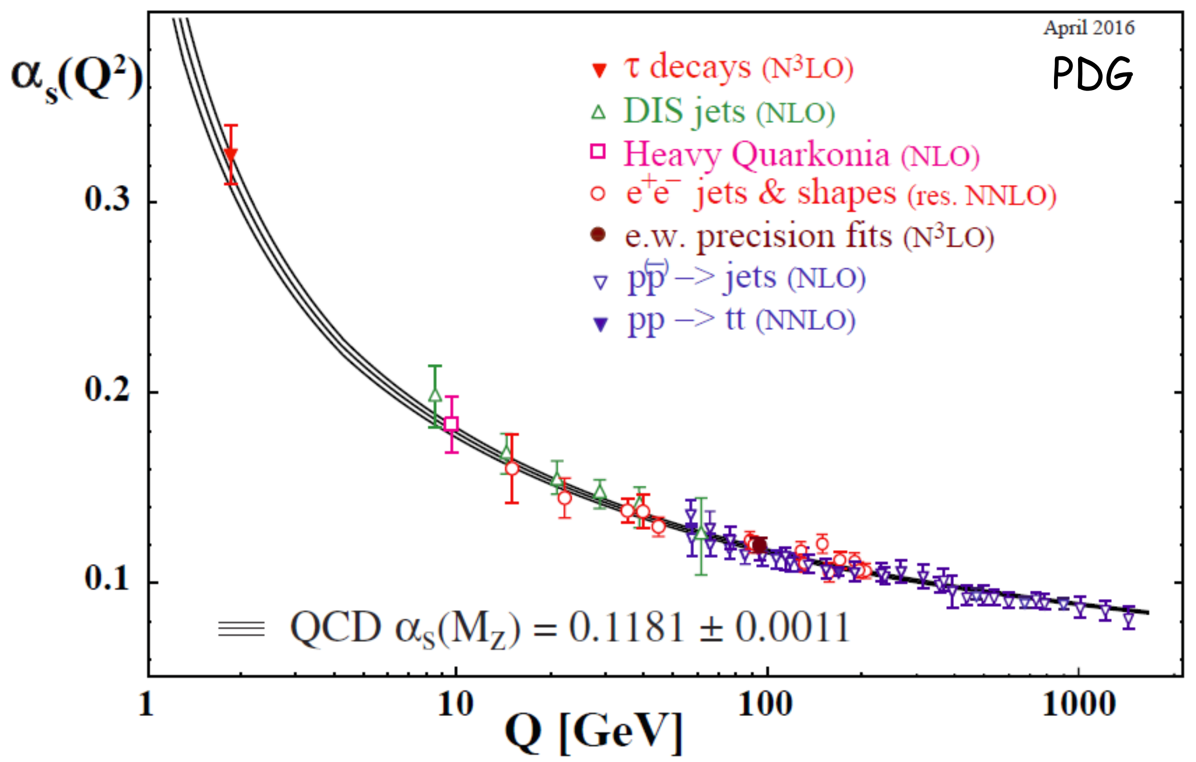
α

(from now on, units such that $\hbar=1, c=1$)

However…

no obvious small coupling
in nuclear forces.

~ 1



QCD
“fine-structure”
constant

Needed:

method that does <u>not</u>
rely on small couplings

$\sim M_{\rm QCD}$

➡ EFFECTIVE FIELD THEORY

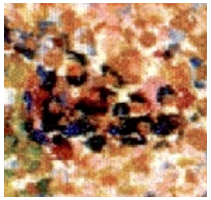
"I do not believe that scientific progress is always
best advanced by keeping an altogether open mind.
It is often necessary to forget one's doubts and to follow the
consequences of one's assumptions wherever they may lead ---
the great thing is not to be free of theoretical prejudices,
but to have the right theoretical prejudices.
And always, the test of any theoretical preconception
is in where it leads."

S. Weinberg

The First Three Minutes,
1972

Ingredients

- Relevant degrees of freedom



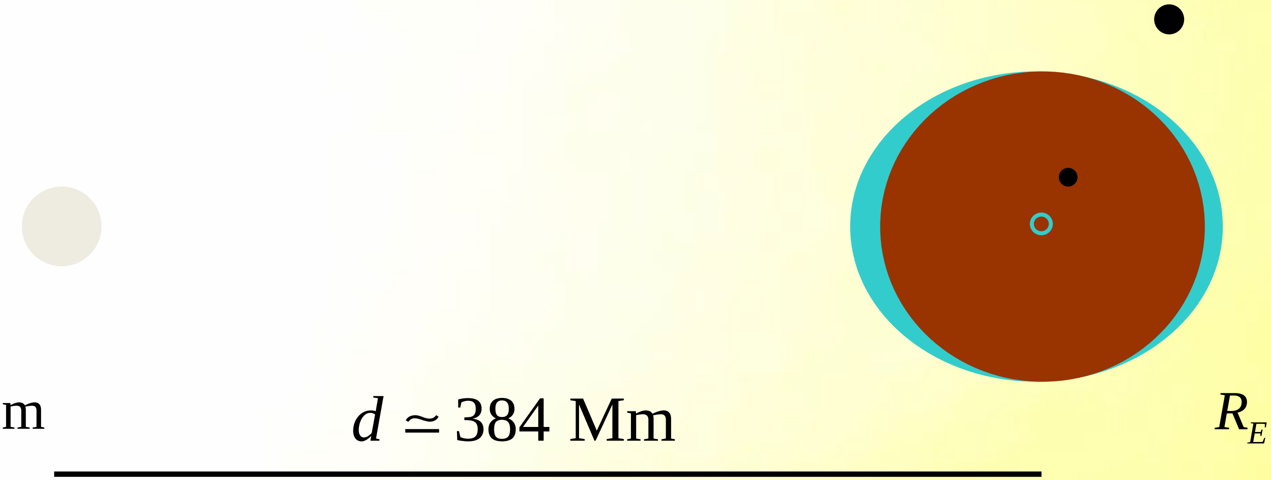
Ingredients

- Relevant degrees of freedom

  choose the coordinates that fit the problem

- All possible interactions

Example: Earth-moon-satellite system



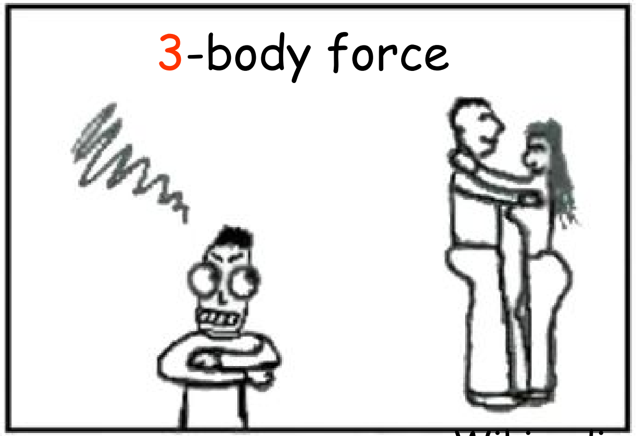
$R_m \simeq 1.7\ \mathrm{Mm}$

$d \simeq 384\ \mathrm{Mm}$

$R_E \simeq 6.4\ \mathrm{Mm}$

2-body forces →2+3-body forces

change in resolution



Wikipedia

Ingredients

- Relevant degrees of freedom

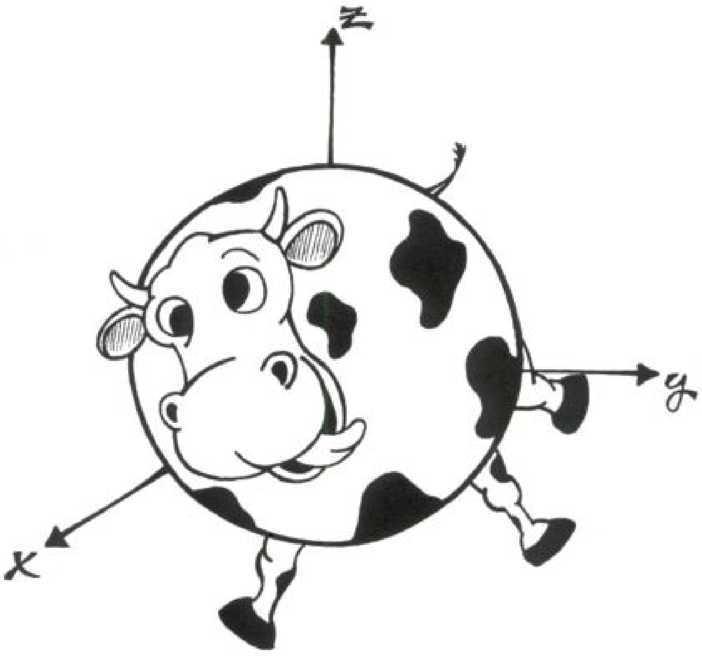
  choose the coordinates that fit the problem

- All possible interactions

  what is not forbidden is compulsory

- Symmetries

A farmer is having trouble with a cow whose milk has gone sour. He asks three scientists—a biologist, a chemist, and a physicist—to help him. The biologist figures the cow must be sick or have some kind of infection, but none of the antibiotics he gives the cow work. Then, the chemist supposes that there must be a chemical imbalance affecting the production of milk, but none of the solutions he proposes do any good either. Finally, the physicist comes in and says, "First, we assume a spherical cow…"



$$\sum_{ij}\alpha_{ij}u_iv_j \to \vec{u}\cdot\vec{v} \qquad +\sum_{ij}\delta\alpha_{ij}u_iv_j$$

no, say, u_1v_2

$$\left|\delta\alpha_{ij}\right| \ll 1$$

amenable to perturbation theory

Ingredients

- Relevant degrees of freedom

  choose the coordinates that fit the problem

- All possible interactions

  what is not forbidden is compulsory

- Symmetries

  not everything is allowed

- Naturalness

't Hooft '79

After scales have been identified,

the remaining, dimensionless parameters are

$$\mathcal{O}(1)$$

unless suppressed by a symmetry

cow

non-sphericity...

Occam's razor:

simplest assumption, to be revised if necessary

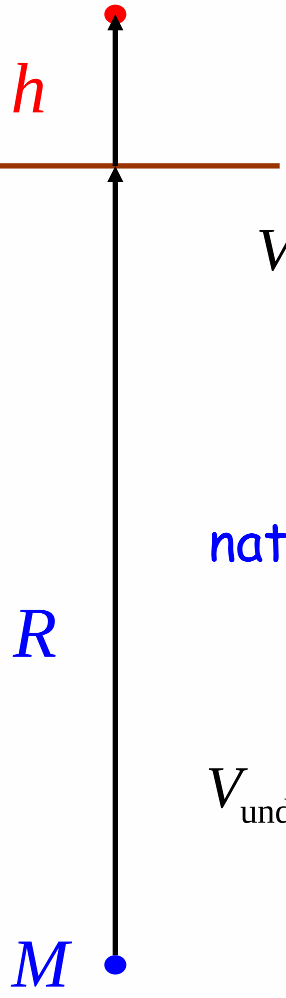
fine-tuning

➡ Expansion in powers of $\frac{E}{E_{\text{und}}}$

energy of probe

energy scale of underlying theory

A classical example: the flat Earth
light object near surface of a large body



$$E \sim mgh \ll E_{\rm und} \equiv mgR$$

d.o.f.: mass m

sym: $V_{\rm eff}(h,x,y) = V_{\rm eff}(h)$

$$V_{\rm eff}(h) = m\sum_{i=0}^{\infty} g_i h^i = \text{const} + mg\left\{h + \frac{g_2}{g}h^2 + \ldots\right\}$$

parameters

(neglecting quantum corrections…)

naturalness:
$$\frac{mg_{i+1}h^{i+1}}{mg_i h^i} = \frac{E}{E_{\rm und}}\times\mathcal{O}(1) = \frac{h}{R}\times\mathcal{O}(1) \Longleftrightarrow g_{i+1} = \mathcal{O}\left(\frac{g}{R^i}\right)$$

$$\Updownarrow$$

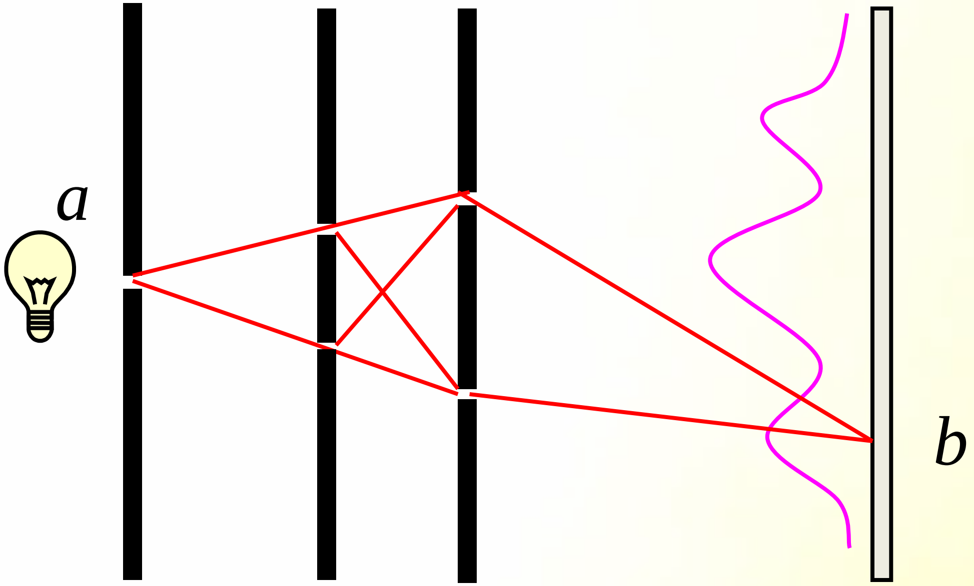
$$V_{\rm und}(h) = -GMm\frac{1}{R+h} = m\underbrace{\left(\frac{GM}{R^2}\right)}_{\equiv g}\sum_{i=0}^{\infty}\left(\frac{-1}{R}\right)^{i-1}h^i \Rightarrow g_{i+1} = (-1)^i\frac{g}{R^i}$$

$h \ll R$

itself the first term in a low-energy EFT of general relativity…

Going a bit deeper…

A short path to quantum mechanics



$$P = \left| A_1 + A_2 + A_3 + A_4 \right|^2$$

sum over
all paths

Path Integral

Feynman '48

RULE

$$A_i \propto \exp\left(i \int_a^b dt\, \mathcal{L}(q(t)) \right)$$

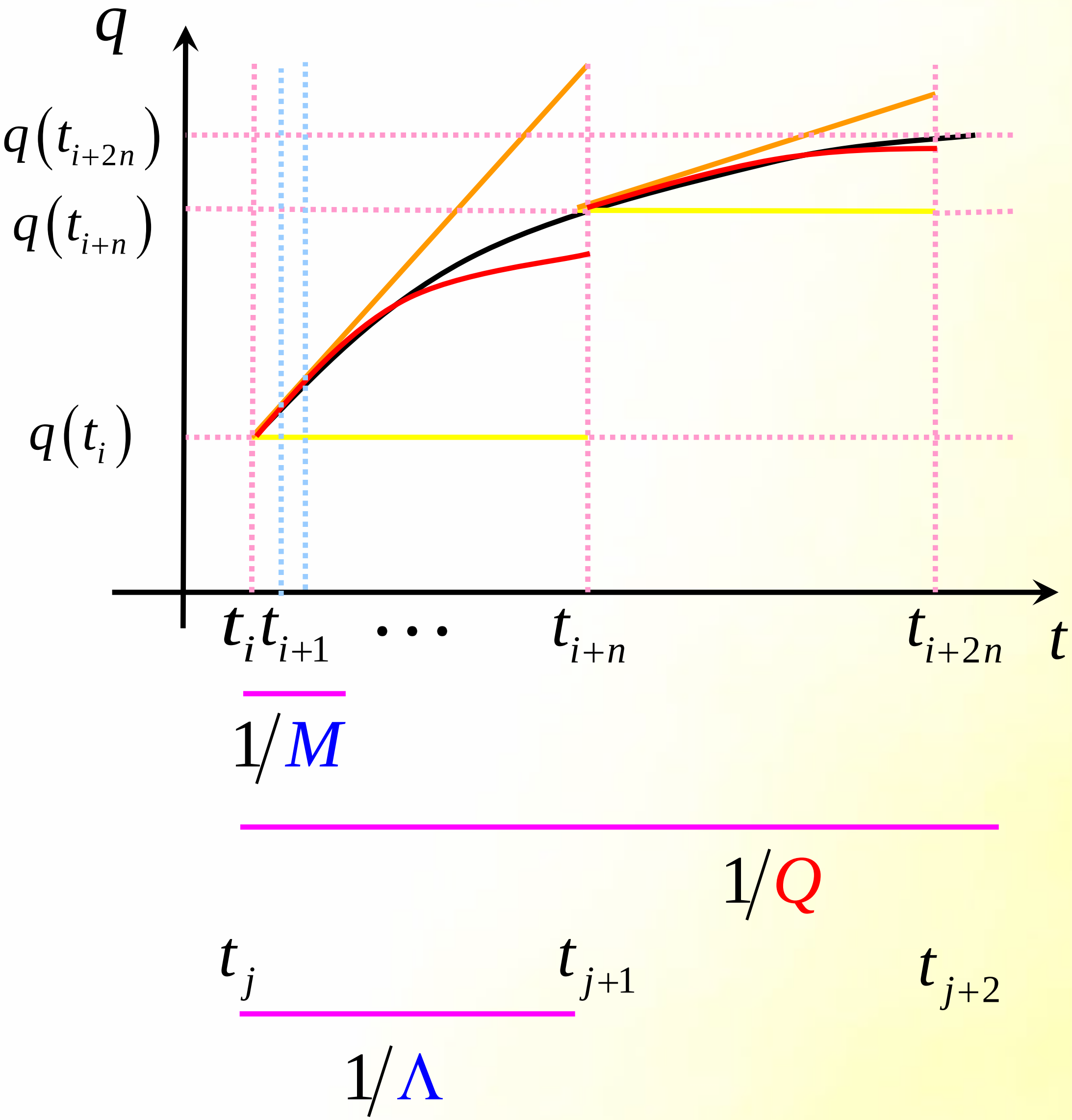
each path contributes a phase
given by the classical action

$$A = \int \boxed{Dq} \exp\left(i \int dt\, \mathcal{L}(q(t)) \right)$$

$$\prod_i \int dq(t_i)$$

classical path $\delta\left(\int dt\, \mathcal{L}(q(t)) \right) = 0$

EFFECTIVE THEORY



scale of fine-structure
of dynamics

scale of variation
of long-range dynamics

coarse-graining scale
(cutoff)

$$\mathcal{L}(q(t_i)) \to \mathcal{L}\left(q(t_i) + \left.\frac{dq}{dt}\right|_{t_i}(t-t_i) + \frac{1}{2}\left.\frac{d^2q}{dt^2}\right|_{t_i}(t-t_i)^2 + \ldots\right)$$

More generally,

$$A=\int Dq\,\exp\left(i\int dt\,\mathcal{L}_{\text{und}}(q)\right)$$

$$\times\int D\tilde{q}\;\delta\left(\tilde{q}-f_{\Lambda}(q)\right)\quad\Leftarrow\quad\prod_{i}\int d\tilde{q}(t_i)\,\delta\left(\tilde{q}(t_i)-f\left(q(t_i)\right)\right)$$

$$=\int D\tilde{q}\;\exp\left(i\int dt\,\mathcal{L}_{\text{EFT}}(\tilde{q})\right)$$

$$\mathcal{L}_{\text{EFT}}(\tilde{q})=\sum_{d,n=0}^{\infty}c_{d+n}(M,\Lambda)\,O_{d+n}\left(\tilde{q},\left(\frac{d^d\tilde{q}}{dt^d}\right)^n\right)$$

Naturalness

$$c_{d+n}\sim\frac{c_0}{M^{d+n}}$$

e.g.

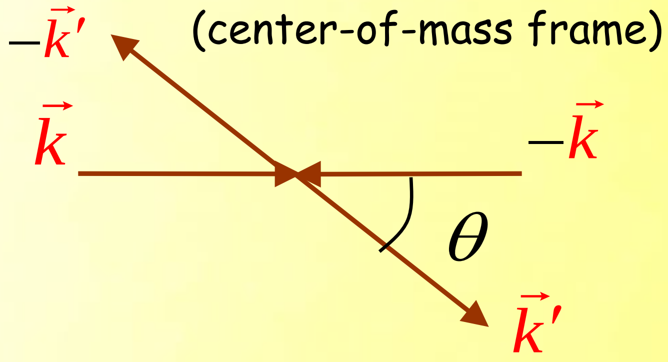
$$V_{\text{EFT}}(\tilde{q})=c_0\tilde{q}^4+c_2\tilde{q}^2\left(\frac{d\tilde{q}}{dt}\right)^2+\ldots$$

Observables ~ expansion in $\frac{Q}{M}$

All information is in the S matrix…

Heisenberg '43
…

elastic scattering
(for simplicity)



$\left|\vec{k}'\right| = \left|\vec{k}\right| \equiv k$ (conservation of energy)

θ : given by certain probability amplitude – the "scattering amplitude"

$$T(k,\theta) = \sum_{l=0}^{\infty} T_l(k) P_l(\cos\theta)$$

angular momentum

partial-wave amplitude

Legendre polynomial

parametrized by phase shift $\delta_l(k)$

$T_l^{-1}(\kappa_r + i\kappa_i) = 0$

$\kappa_r = 0$ bound states ➡ $E = -B < 0$

$\kappa_i \leq 0$ resonances ➡ $E = E_R - i\Gamma_R/2$

characteristic external momentum

$$\left\{\begin{array}{l} T = T^{(\infty)}(\overset{\downarrow}{Q}) \sim \underbrace{N(M)}_{\text{normalization}} \displaystyle\sum_{\nu=\nu_{\min}}^{\infty} \sum_{i} \tilde{c}_{\nu,i}(\Lambda)\left[\frac{Q}{M}\right]^{\nu} \underbrace{F_{\nu,i}\left(\frac{Q}{m};\frac{Q}{\Lambda}\right)} \\[2ex] \dfrac{\partial T}{\partial \Lambda} = 0 \end{array}\right.$$

“non-analytic”,
from the solution of
a dynamical equation
(e.g. Schrödinger eq.)

$$\nu = \nu(d, n, \ldots)$$

“power counting”

For $k \sim m$, truncate consistently with RG invariance

so as to allow systematic improvement (perturbation theory):

$$T = T^{(\bar{\nu})}\left[1 + \mathcal{O}\left(\frac{Q}{M}, \frac{Q}{\Lambda}\right)\right] \qquad \frac{\Lambda}{T^{(\bar{\nu})}} \frac{\partial T^{(\bar{\nu})}}{\partial \Lambda} = \mathcal{O}\left(\frac{Q}{\Lambda}\right)$$

“second quantization”:

+ Lorentz invariance

$$q(t) \rightarrow \psi(\vec{r},t), \psi^*(\vec{r},t)$$

representation of $\mathrm{SO}(3,1)$

$$dt \rightarrow dt\, d^3r$$

$$\equiv d^4x$$

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{r}}$$

$$\rightarrow \frac{\partial}{\partial x^\mu}$$

EFFECTIVE FIELD THEORIES

Euler + Heisenberg ’36
Weinberg ’67 … ’79
Wilson, early 70s
…

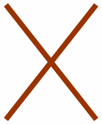
$$A=\int D\psi D\psi^{*}\exp\left(i\int d^{4}x\ \{\mathcal{L}_{\text{free}}(\psi)+\mathcal{L}_{\text{int}}(\psi)\}\right)$$

$$=\int D\psi D\psi^{*}\left\{1+i\int d^{4}x\mathcal{L}_{\text{int}}(\psi)+\left[i\int d^{4}x\mathcal{L}_{\text{int}}(\psi)\right]^{2}+\dots\right\}\exp\left(i\int d^{4}x\mathcal{L}_{\text{free}}(\psi)\right)$$

momentum space ⇩ ⇩ ⇩

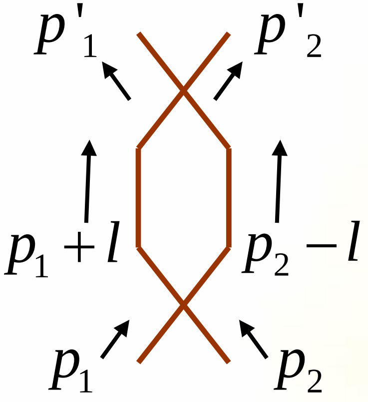
(skip many steps…)

$$\mathcal{L}_{\text{int}}=\frac{\lambda}{4}\left(\psi^{*}\psi\right)^{2}$$



$$=i\lambda$$

$$=\frac{i}{p^{2}-m^{2}+i\varepsilon}$$



$$=\int\frac{d^{4}l}{(2\pi)^{4}}\ i\lambda\ \frac{i}{(p_{1}+l)^{2}-m^{2}+i\varepsilon}\ \frac{i}{(p_{2}-l)^{2}-m^{2}+i\varepsilon}\ i\lambda$$

$$=\dots$$

but divergent from high-momentum region…

➡ needs a cutoff to separate high and low momenta

EFFECTIVE FIELD THEORIES

Euler + Heisenberg '36
Weinberg '67 … '79
Wilson, early 70s
…

Two possibilities:

- know <u>and</u> can solve underlying theory --

  get c_i 's in terms of parameters in $\mathcal{L}_{\text{und}}$
- know <u>but</u> cannot solve, or do <u>not</u> know, underlying theory --

  invoke Weinberg's "folk theorem":

Weinberg '79

> "The quantum field theory generated by the most general Lagrangian with some assumed symmetries will produce the most general *S* matrix incorporating quantum mechanics, Lorentz invariance, unitarity, cluster decomposition and those symmetries, with no further physical content."

Note: proven only for scalar field with Z_2 symmetry in E_4, Ball + Thorne '94
but no known counterexamples

Bira's EFT Recipe

1. identify degrees of freedom and symmetries
2. construct most general Lagrangian
3. run the methods of field theory

what is not forbidden
is mandatory!

- compute Feynman diagrams with all momenta $Q < \Lambda$ ("regularization")
- relate $c_i(\Lambda), \Lambda$ to observables, which should be independent of Λ ("renormalization")

not a model form factor

➡ controlled expansion in $\frac{Q}{M} \times \mathcal{O}(1)$

"naturalness": what else?
unless suppressed by symmetry…

contrast to models, which have fewer, but *ad hoc*, interactions;
useful in the identification of relevant degrees of freedom
and symmetries, but plagued with uncontrolled errors

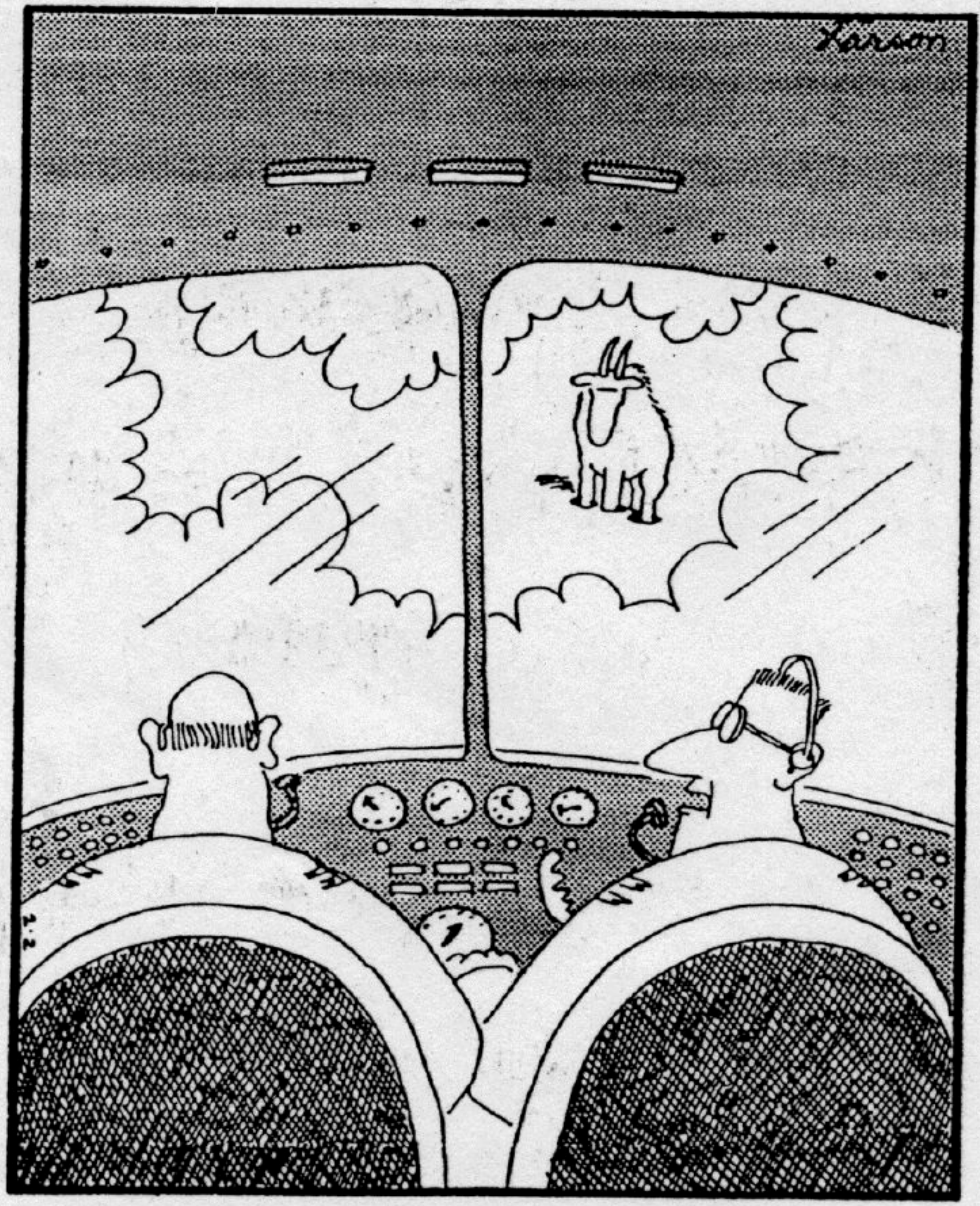
“A significant change in physicists’ attitude towards what should be taken as a guiding principle in theory construction is taking place in recent years in the context of the development of EFT.

For many years (…) renormalizability has been taken as a necessary requirement.

Now, considering the fact that experiments can probe only a limited range of energies, it seems natural to take EFT as a general framework for analyzing experimental results.”

T. Y. Cao

in Renormalization, From Lorentz to Landau (and Beyond), L.M. Brown (ed.), 1993



"Say. . . . What's a mountain goat doing way up here in a cloud bank?"

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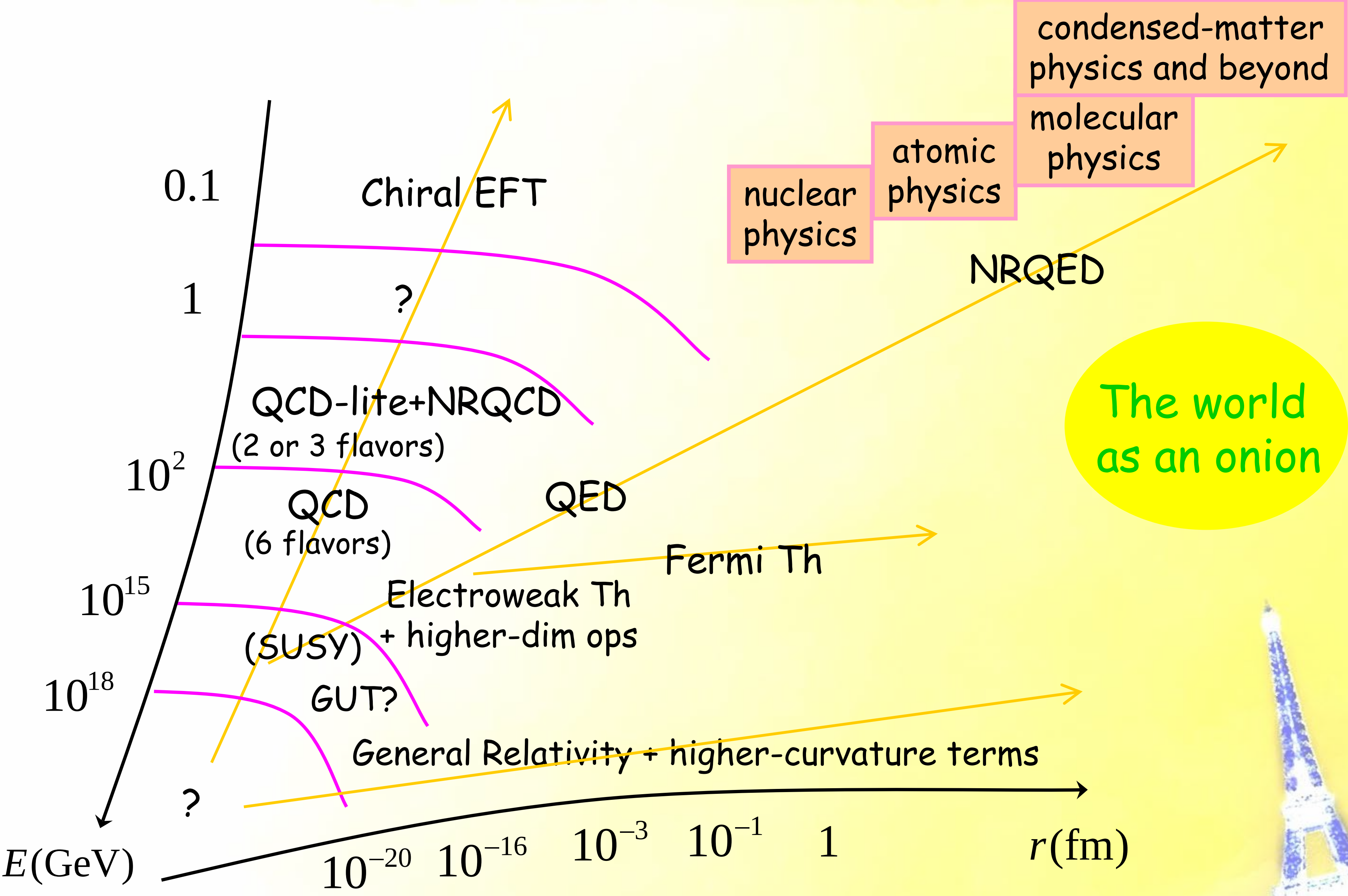
Time for a paradigm change, perhaps?



condensed-matter physics and beyond

molecular physics

atomic physics

nuclear physics

NRQED

The world as an onion

Chiral EFT

?

QCD-lite+NRQCD
(2 or 3 flavors)

QCD
(6 flavors)

QED

Fermi Th

Electroweak Th
+ higher-dim ops

(SUSY)

GUT?

General Relativity + higher-curvature terms

?

0.1

1

10^2

10^{15}

10^{18}

$E(\mathrm{GeV})$

10^{-20} 10^{-16} 10^{-3} 10^{-1} 1 $r(\mathrm{fm})$

A quantum example: non-relativistic QED (NRQED)

Caswell + Lepage '86

…

single fermion ψ of mass M, massless spin-1 boson A_μ

Lorentz, parity, time-reversal,

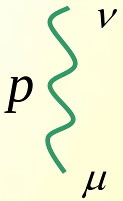
and U(1) gauge invariance

$$\begin{bmatrix} D_\mu = \partial_\mu - ieA_\mu \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{bmatrix}$$

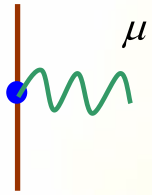
$$\mathcal{L}_{\text{und}} = \bar{\psi}\left(i\not{D} - M\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \ldots$$

the first terms in
a low-energy EFT of the SM…

$$= \frac{i}{\not{p} - M + i\varepsilon}$$



$$= \frac{-i\,\eta_{\mu\nu}}{p^2 + i\varepsilon}$$



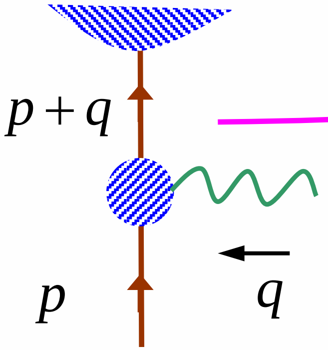
$$= ie\gamma_\mu$$

$\Rightarrow$ interactions $\propto e = \sqrt{4\pi\alpha} \sim \frac{1}{3}$

perturbation theory

How do E&M bound states arise?

$Q \ll M$



$$= \frac{i}{\not{p} + \not{q} - M + i\varepsilon} = \frac{i\left(p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} + \not{q} + M\right)}{\left(p^0+q^0\right)^2 - \left(\vec{p}+\vec{q}\right)^2 - M^2 + i\varepsilon}$$

$$= \frac{i\left(p^0\gamma^0 + M - \vec{p}\cdot\vec{\gamma} + \not{q}\right)}{2p^0q^0 + q^{02} - 2\vec{p}\cdot\vec{q} - \vec{q}^{\,2} + i\varepsilon}$$

$$= \frac{i}{q^0 + i\varepsilon}\,\frac{\left(1+\gamma^0\right)}{2} + \dots$$

$$|\vec{p}| \sim |\vec{q}| = \mathcal{O}(Q)$$

$$q^0 = |\vec{q}| = \mathcal{O}(Q)$$

$$p^0 = \sqrt{\vec{p}^{\,2} + M^2} = M + \mathcal{O}\left(\frac{Q^2}{M}\right)$$

$$P_\pm \equiv \frac{1 \pm \gamma^0}{2} \qquad P_\pm P_\pm = P_\pm, \;\; P_\pm P_\mp = 0$$

projector onto ± energy states

“heavy-fermion formalism”

Georgi ’90

$$\Psi_\pm \equiv e^{iMt} P_\pm \psi \Leftrightarrow \;\; \psi = \left(P_+ + P_-\right)\psi = e^{-iMt}\left(\Psi_+ + \Psi_-\right)$$

annihilates particles
creates antiparticles

$$\mathcal{L}_{\text{und}} = \bar{\Psi}_+ iD_0\Psi_+ - \bar{\Psi}_- i\vec{\gamma}\cdot\vec{D}\Psi_+ + \bar{\Psi}_+ i\vec{\gamma}\cdot\vec{D}\Psi_- - \bar{\Psi}_-\left(iD_0 + 2M\right)\Psi_- - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$+ \text{ other, heavy d.o.f.s}$$

$$Z = \int DA \int D\Psi_+ \int D\Psi_- \exp\left(i\int d^4x\,\mathcal{L}_{\text{und}}(\Psi_{+,}\Psi_-, A)\right) \times \int D\Psi\,\delta\left(\Psi - \Psi_+\right)$$

$$= \int DA \int D\Psi \;\exp\left(i\int d^4x\,\mathcal{L}_{\text{EFT}}(\Psi, A)\right)$$

complete square,
do Gaussian integral

Mannel, Roberts
+Ryzak '91

$$\mathcal{L}_{\text{EFT}} = \bar{\Psi}\, iD_0\Psi + \frac{1}{2M}\bar{\Psi}\vec{D}^2\Psi + \frac{e}{2M}\bar{\Psi}\sigma_i\Psi\varepsilon_{ijk}F^{jk} + \ldots - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \ldots$$

non-relativistic expansion

Pauli term

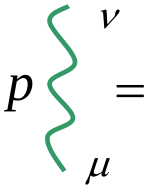
$$+ \frac{e\enclose{circle}{\kappa}}{2M}\bar{\Psi}\sigma_i\Psi\,\varepsilon_{ijk}F^{jk} + \ldots$$

anomalous
magnetic moment
$=\mathcal{O}(1)$

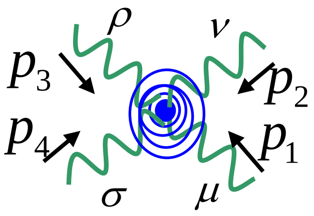
most general Lag with Ψ, *A*
invariant under U(1) gauge, parity, time-reversal,
and Lorentz transformations

$$\mathcal{L}_{\text{EFT}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \underline{\frac{a}{M^4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{b}{M^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2} + \ldots$$

Euler + Heisenberg '36



$$= \frac{-i\,\eta_{\mu\nu}}{p^2 + i\varepsilon}$$



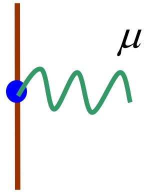
$$= \frac{i}{M^4}\left\{a\left[\eta_{\mu\rho}\eta_{\nu\sigma}\,p_1\cdot p_3\,p_2\cdot p_4 + \ldots\right] + b\left[\ldots\right]\right\}$$

$$+\bar{\Psi}\,iv\cdot D\Psi + \frac{1}{2M}\bar{\Psi}\left((v\cdot D)^2 - D^2\right)\Psi + \frac{e}{M}(1+\kappa)\bar{\Psi}v_\alpha S_\beta\Psi\,\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu} + \ldots$$

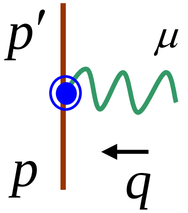
$$v \equiv \left(1,\vec{0}\right)$$

$$S \equiv \left(0,\frac{\vec{\sigma}}{2}\right)$$

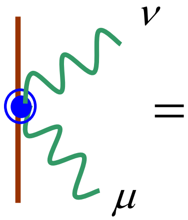
$$= \frac{i}{v\cdot p + \frac{1}{2M}\left(p^2 - (v\cdot p)^2\right) + \ldots + i\varepsilon}$$



$$= ie\,v_\mu$$

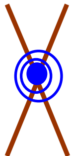


$$= \frac{e}{2M}\left\{i\left(p + p'\right)_\mu + 2(1+\kappa)\,\varepsilon_{\mu\nu\alpha\beta}v^\nu S^\alpha q^\beta\right\}$$



$$= i\frac{e^2}{M}\left(\eta_{\mu\nu} - v_\mu v_\nu\right)$$

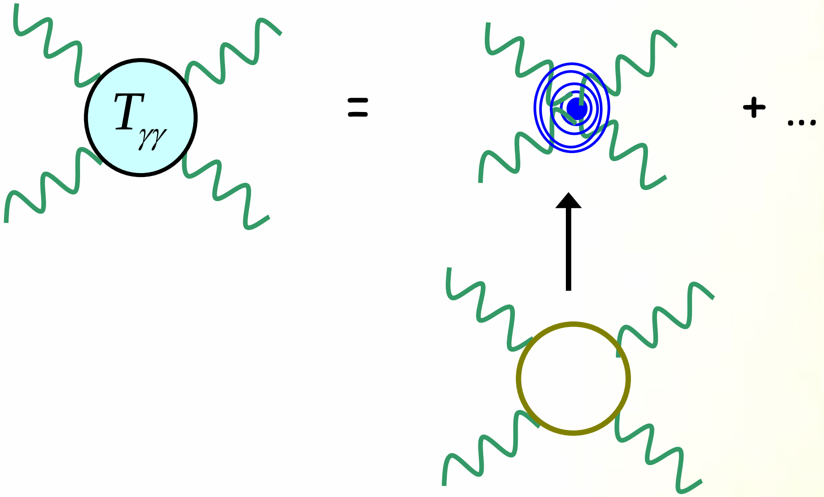
$$+\frac{\gamma_0^{(0)}}{M^2}\bar{\Psi}\Psi\,\bar{\Psi}\Psi + \frac{\gamma_0^{(1)}}{M^2}\bar{\Psi}S\Psi\cdot\bar{\Psi}S\Psi + \ldots$$



$$= \frac{i}{M^2}\left(\gamma_0^{(0)} + \gamma_0^{(1)}S_1\cdot S_2\right)$$

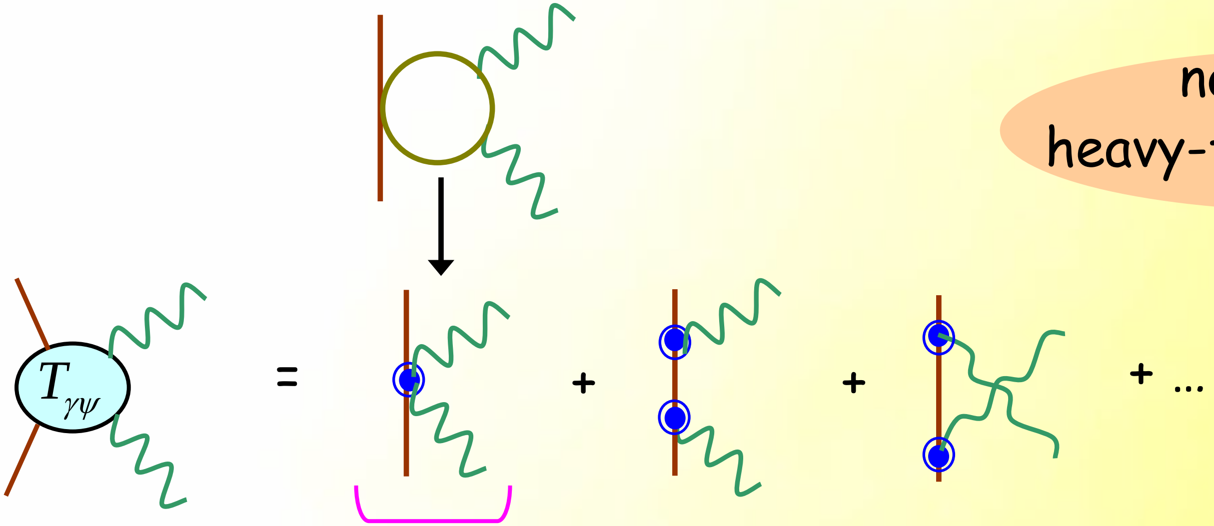
etc.

Various processes at low energies: *e.g.*



light-by-light scattering

no explicit
fermion-antifermion pair creation!

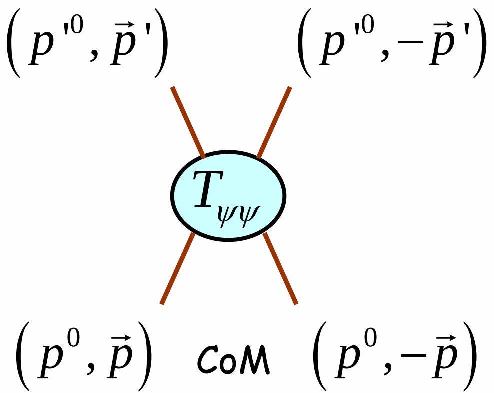


Thompson
limit

no change in
heavy-fermion number!

Compton scattering

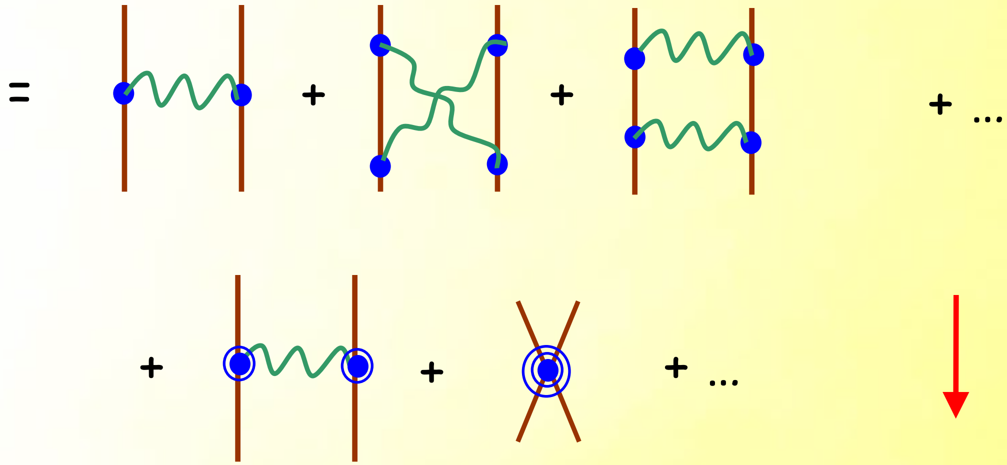
Back to atomic bound states: the NRQED perspective



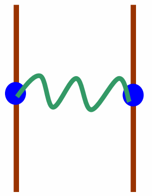
CoM frame

$$|\vec{p}| \sim |\vec{p}\,'| = \mathcal{O}(Q)$$

$$p^0 \sim p'^0 = \mathcal{O}\left(\frac{Q^2}{M}\right)$$

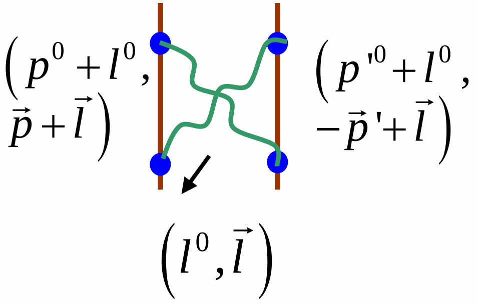


higher powers of $\frac{Q}{M}$



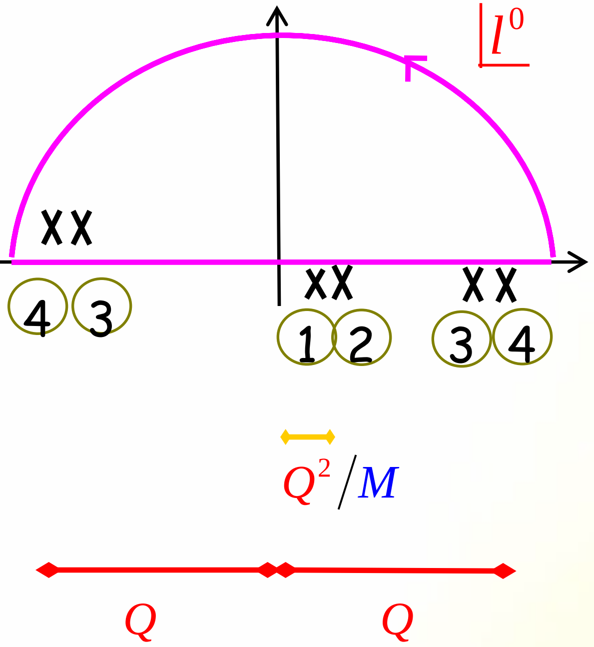
$$= \frac{i\,e^2}{(p-p')^2 + i\varepsilon} = \frac{-i\,e^2}{\left(p^0 - p'^0\right)^2 - \left(\vec{p} - \vec{p}\,'\right)^2 + i\varepsilon} \simeq \frac{i\,e^2}{\left(\vec{p} - \vec{p}\,'\right)^2 - i\varepsilon} \sim \frac{4\pi\alpha}{Q^2}$$

$$\to V(r) = \frac{\alpha}{r}$$



$$= e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^0 + p^0 - \frac{(\vec l + \vec p)^2}{2M} + i\varepsilon} \textcircled{1} \quad \frac{1}{l^0 + p'^0 - \frac{(\vec l - \vec p')^2}{2M} + i\varepsilon} \textcircled{2}$$

$$\frac{1}{\left(p^0 - p'^0 + l^0\right)^2 - \left(\vec p - \vec p' + \vec l\right)^2 + i\varepsilon} \textcircled{3} \quad \frac{1}{l^{02} - \vec l^{\,2} + i\varepsilon} \textcircled{4}$$



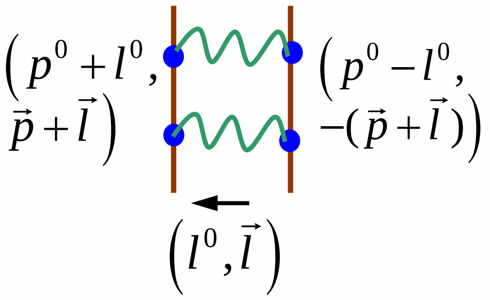
$$= i e^4 \int \frac{d^3 l}{(2\pi)^3} \frac{1}{\left|\vec l\right| - p^0 + \cancel{\frac{(\vec l + \vec p)^2}{2M}} - i\varepsilon} \quad \frac{1}{\left|\vec l\right| - p'^0 + \cancel{\frac{(\vec l - \vec p')^2}{2M}} - i\varepsilon}$$

$$\frac{1}{\left(\cancel{p^0 - p'^0} - \left|\vec l\right|\right)^2 - \left(\vec p - \vec p' + \vec l\right)^2 + i\varepsilon} \quad \frac{1}{2\left|\vec l\right| - i\varepsilon} \quad \textcircled{4}$$

$$+ \dots \quad \textcircled{3}$$

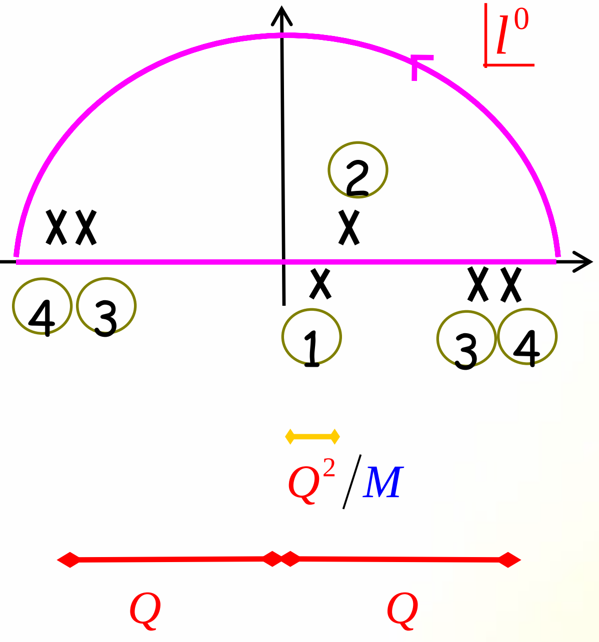
$$\sim e^4 \frac{Q^3}{(4\pi)^2} \frac{1}{Q} \frac{1}{Q} \frac{1}{Q^2} \frac{1}{Q} \sim \underset{\ll 1}{\frac{\alpha}{4\pi}} \frac{4\pi\alpha}{Q^2}$$

just as expected…



$$= e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^0 + p^0 - \frac{(\vec{l}+\vec{p})^2}{2M} + i\varepsilon} \textcircled{1} \quad \frac{1}{-l^0 + p^0 - \frac{(\vec{l}+\vec{p})^2}{2M} + i\varepsilon} \textcircled{2}$$

$$\frac{1}{\left(p^0 - p'^0 + l^0\right)^2 - \left(\vec{p} - \vec{p}' + \vec{l}\right)^2 + i\varepsilon} \textcircled{3} \quad \frac{1}{l^{02} - \vec{l}^{\,2} + i\varepsilon} \textcircled{4}$$



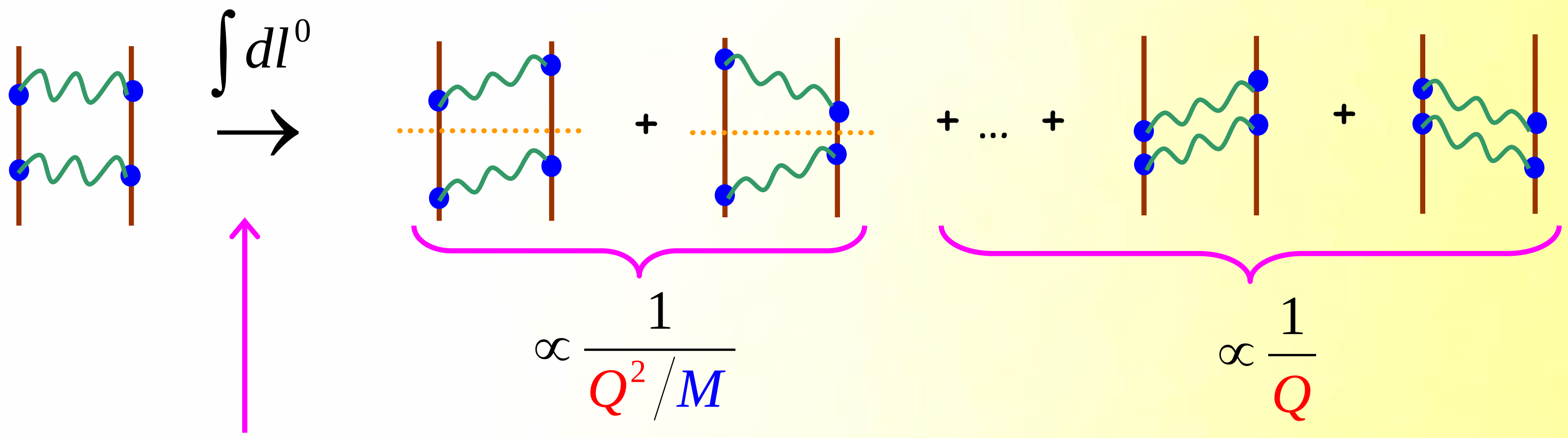
$$= i e^4 \int \frac{d^3 l}{(2\pi)^3} \boxed{\frac{1}{-2p^0 + \frac{(\vec{l}+\vec{p})^2}{M} - i\varepsilon}} \ \frac{1}{\left(p^0 - \cancel{\frac{(\vec{l}+\vec{p})^2}{2M}}\right)^2 - \vec{l}^{\,2} + i\varepsilon}$$

$$\frac{1}{\left(2p^0 \cancel{- p'^0 - \frac{(\vec{l}+\vec{p})^2}{2M}}\right)^2 - \left(\vec{p} - \vec{p}' + \vec{l}\right)^2 + i\varepsilon} \textcircled{2}$$

$$+ \ldots \quad \textcircled{3}\textcircled{4}$$

$$\sim (4\pi\alpha)^2 \frac{Q^3}{4\pi} \frac{M}{Q^2} \frac{1}{Q^2} \frac{1}{Q^2} + \frac{\alpha}{4\pi} \frac{4\pi\alpha}{Q^2} \sim \alpha \frac{4\pi\alpha}{Q^2} \left(\overset{\gg 1}{\boxed{\frac{M}{Q}}} + \frac{1}{4\pi} \right)$$

infrared enhancement!



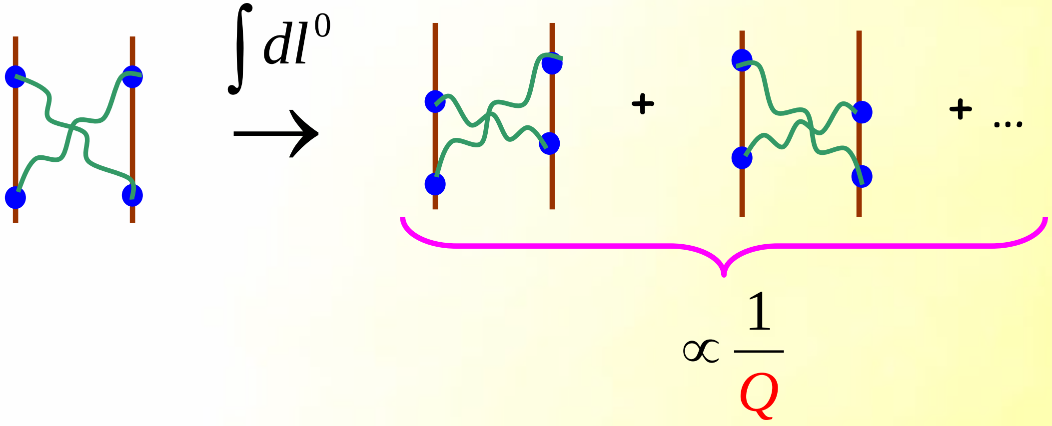
$$\int dl^0 \rightarrow$$

$$\propto \frac{1}{Q^2/M}$$

$$\propto \frac{1}{Q}$$

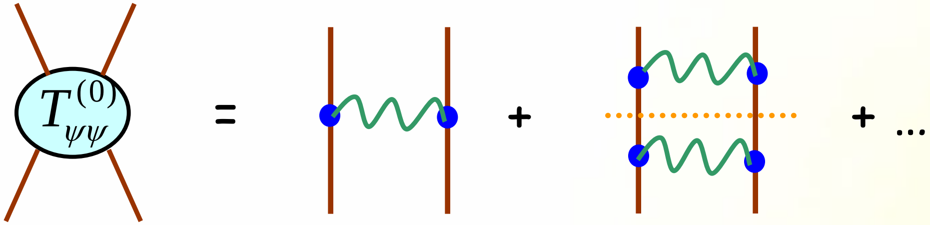
$$\frac{1}{-l^0-\frac{\vec{l}^{\,2}}{2M}+i\varepsilon} = 2\pi\delta\left(l^0+\frac{\vec{l}^{\,2}}{2M}\right) - \frac{1}{l^0+\frac{\vec{l}^{\,2}}{2M}+i\varepsilon}$$

"time-ordered perturbation theory"



$$\int dl^0 \rightarrow$$

$$\propto \frac{1}{Q}$$

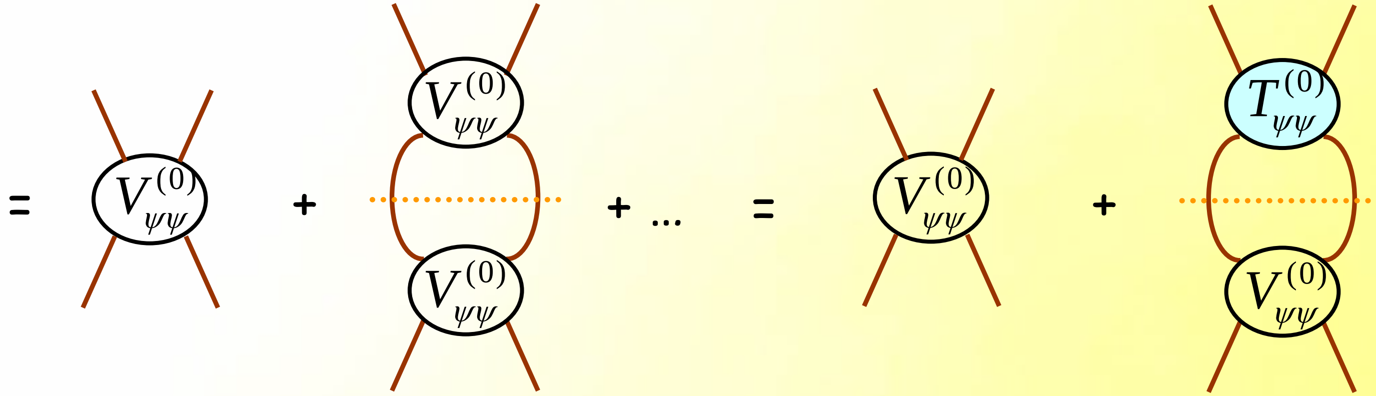


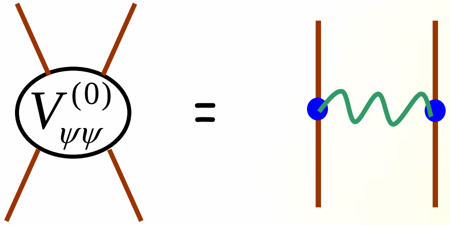
$$\sim \frac{e^2}{Q^2}\left\{1+\mathcal{O}\left(\alpha\frac{M}{Q}\right)+\ldots\right\}\sim\frac{e^2}{Q^2}\frac{1}{1-\mathcal{O}\left(\alpha\frac{M}{Q}\right)}$$

bound state at

$$Q\sim\alpha M$$

$$-E\sim\frac{Q^2}{M}\sim\alpha^2 M$$





$$=\mathcal{O}\left(\frac{e^2}{Q^2}\right)$$

Coulomb potential

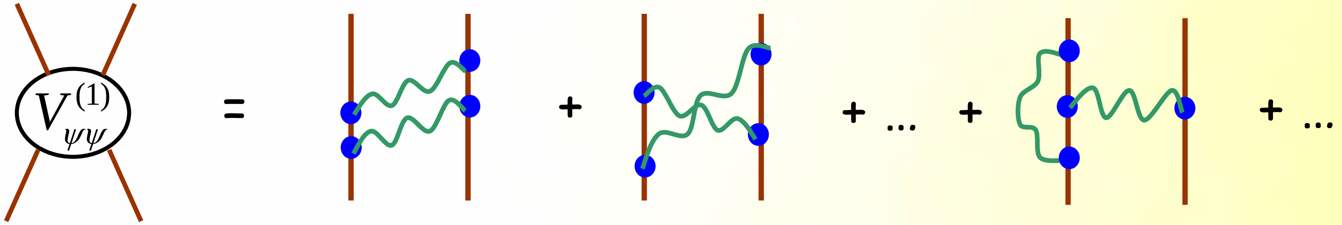
Lippmann-Schwinger eq.

= Schrödinger eq.

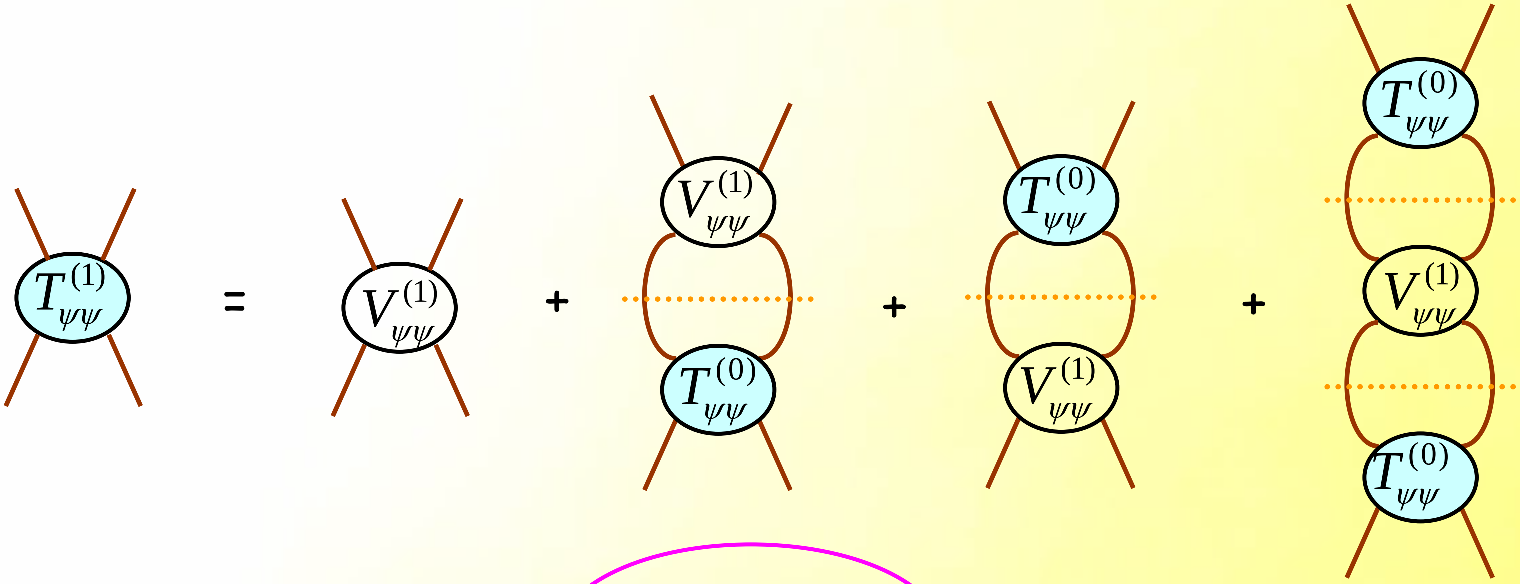
$$\left(\frac{\hat{p}^2}{2M}+V^{(0)}_{\psi\psi}\right)\left|\psi^{(0)}\right\rangle=E^{(0)}\left|\psi^{(0)}\right\rangle$$

known results…

But more:

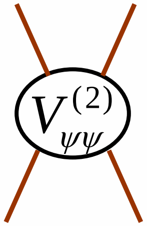


$$= \mathcal{O}\left(\frac{\alpha}{4\pi}\,\frac{4\pi\alpha}{Q^2}\right)$$



➡ $$E^{(0+1)} = E^{(0)} + \boxed{\left\langle \psi^{(0)} \middle| V^{(1)}_{\psi\psi} \middle| \psi^{(0)} \right\rangle} = \mathcal{O}\left(\frac{\alpha}{4\pi} E^{(0)}\right)$$

"First-order distorted-wave
Born approximation"



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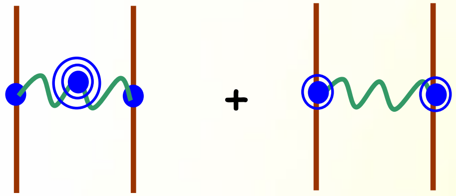
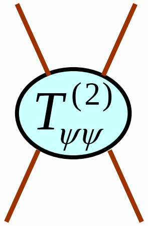


+ …

$$= \mathcal{O}\left(\frac{Q^2}{M^2}\frac{4\pi\alpha}{Q^2}\right)$$



= …

"Second-order distorted-wave Born approximation"

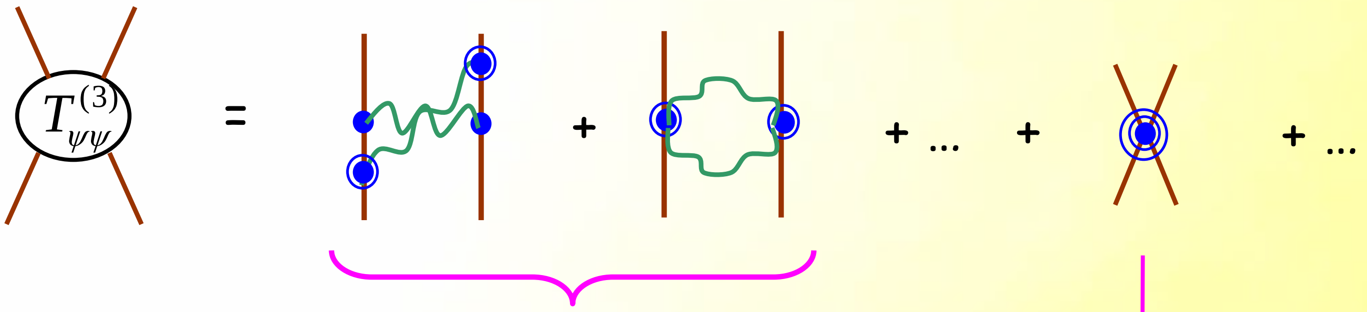
$$\Rightarrow \quad E^{(0+1+2)} = E^{(0+1)} + \underbrace{\left\langle \psi^{(0)} \middle| V_{\psi\psi}^{(2)} \middle| \psi^{(0)} \right\rangle}_{} + \ldots \qquad = \mathcal{O}\left(\frac{Q^2}{M^2}E^{(0)}\right)$$

piece $\propto \vec{\mu}_1 \cdot \vec{\mu}_2 \int d^3\vec{r}\, \psi^{(0)*}(\vec{r})\,\delta^{(3)}(\vec{r})\,\psi^{(0)}(\vec{r}) = \vec{\mu}_1 \cdot \vec{\mu}_2 \left|\psi^{(0)}(0)\right|^2$

magnetic interaction

N
O
T
E

starting at $T_{\psi\psi}^{(3)}$, sufficiently many derivatives appear at vertices that loops bring positive powers of Λ , which need to be compensated by $\gamma_0^{(i)}(\Lambda)$ and higher-order "counterterms"



$$= \mathcal{O}\left(\frac{\alpha}{4\pi}\frac{Q^2}{M^2}\frac{4\pi\alpha}{Q^2}\right)$$

$$\propto \frac{\alpha^2}{M^2}\ln\Lambda \qquad \leftrightarrow \qquad \gamma_0^{(i)} \propto \frac{\alpha^2}{M^2}\left(-\ln\Lambda + \text{constant}\right)$$

renormalization

to be determined by "matching" to QED (and/or from data)

etc.

Example: g factor for electron bound in H-like atoms

$$\boxed{g = 2(1+\kappa)}$$

electron
Larmor frequency

known ion mass

measured

measured

$$\frac{\omega_{\mathrm{L}}}{\omega_{\mathrm{c}}} = \frac{g}{2}\frac{|e|}{q}\frac{m_{\mathrm{ion}}}{m}$$

trapped-ion
cyclotron frequency

ion charge

electron
mass

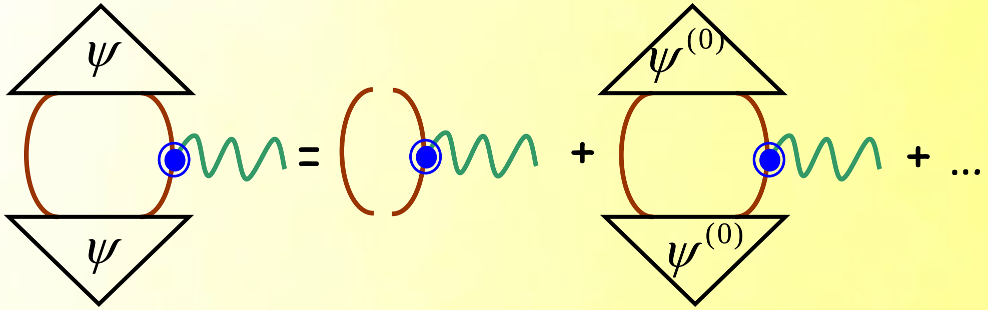


TABLE II. Individual contributions to the 1s bound-electron g factor, $1/\alpha$ from [12] is 137.035 999 11(46).

	$12C^{5+}$	$16O^{7+}$
Dirac value (point)	1.998 721 354 39(1)	1.997 726 003 06(2)
Finite nuclear size	0.000 000 000 41	0.000 000 001 55
Free QED, $\sim(\alpha/\pi)$	0.002 322 819 47(1)	0.002 322 819 47(1)
Binding SE, $\sim(\alpha/\pi)$	0.000 000 852 97	0.000 001 622 67(1)
Binding VP, $\sim(\alpha/\pi)$	−0.000 000 008 51	−0.000 000 026 37(1)
Free QED, $\sim(\alpha/\pi)^2 \cdots (\alpha/\pi)^4$	−0.000 003 515 10	−0.000 003 515 10
Binding QED, $\sim(\alpha/\pi)^2(Z\alpha)^2$	−0.000 000 001 13	−0.000 000 002 01
Binding QED, $\sim(\alpha/\pi)^2(Z\alpha)^4$	0.000 000 000 41(11)	0.000 000 001 06(35)
Recoil	0.000 000 087 63	0.000 000 116 97
Total	2.001 041 590 52(11)	2.000 047 021 28(35)

Pachucki, Jentschura + Yerokhin '04

$$\left(u = \frac{m_{12C(gs)}}{12}\right)$$

$$m(^{12}\mathrm{C}^{5+}) = 0.000\,548\,579\,909\,41\,(29)(3)\ \mathrm{u},$$

$$m(^{16}\mathrm{O}^{7+}) = 0.000\,548\,579\,909\,87(41)(10)\ \mathrm{u},$$

Most precise determination of electron mass (expt)(th)