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Impact of heavy sterile neutrinos on the triple Higgs coupling

HEFT 2018 Workshop

[based on PRD 94 (2016) 013002; JHEP 1704 (2017) 038; arXiv:1712.07621 [hep-ph]]

18.04.2018, Julien Baglio

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Outline

1. Introduction

2. Neutrino effects on the triple Higgs coupling: 3+1 model

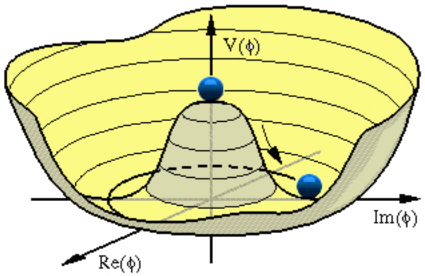
3. Neutrino effects on the triple Higgs coupling: Inverse seesaw

4. Outlook

The SM ultimate test: probing the scalar potential

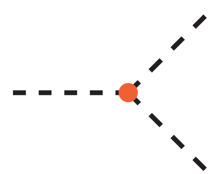
From the scalar potential before EWSB (ϕ as the Higgs field):

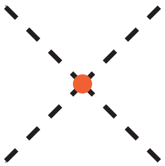
$$V(\phi) = -m^2|\phi|^2 + \lambda|\phi|^4$$



To $V(\phi)$ after EWSB, with $M_H^2 = 2m^2$, $v^2 = m^2/\lambda$:

$$\phi = \begin{pmatrix} 0 \\ \frac{v + H(x)}{\sqrt{2}} \end{pmatrix} \Rightarrow V(H) = \frac{1}{2}M_H^2 H^2 + \frac{1}{2}\frac{M_H^2}{v}H^3 + \frac{1}{8}\frac{M_H^2}{v^2}H^4 + \text{constant}$$





Neutrino properties

- **Neutrino oscillations:** observed experimentally in 1998

  [Super-Kamiokande, PRL 81 (1998) 1562]

  $\Rightarrow$ neutrinos are massive! $\Rightarrow$ new physics required to account for their mass

  Different mixing pattern from CKM, ν lightness; Majorana or Dirac ν?

- **No information through oscillations about:**

  Absolute mass scale:
  cosmology $\Sigma m_{\nu_i} < 0.23$ eV [Planck, A& A 594 (2016) A13]
  β decays $m_{\nu_e} < 2.05$ eV [Mainz experiment, EPJC 40 (2005) 447; Troitsk, PRD 84 (2011) 112003]

  Neutrino nature (Dirac or Majorana):
  Neutrinoless double β decays $m_{2\beta} < 0.061 - 0.165$ eV

  [KamLAND-ZEN, PRL 117 (2016) 082503]

Massive neutrinos and New Physics

- **Standard Model:** $L = \binom{\nu_L}{\ell_L}, \tilde{\phi} = \binom{H^{0*}}{H^-}$

  No right-handed neutrino $\nu_R \Rightarrow$ No Dirac mass term

$$\mathcal{L}_{\text{mass}} = -Y_\nu \bar{L}\tilde{\phi}\nu_R + \text{h.c.}$$

  No Higgs triplet $T \Rightarrow$ No Majorana mass term

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} m \bar{L} T L^c + \text{h.c.}$$

- **Necessary to go beyond the Standard Model for ν mass**

  Radiative models?

  R-parity violation in supersymmetry?

  Seesaw mechanisms? $\to$ ν mass at tree-level

  $\to$ **heavy sterile fermions**

  $\Rightarrow$ **neutrino portal for Dark Matter?**

Dirac neutrinos?

Add gauge singlet (sterile), right-handed neutrinos

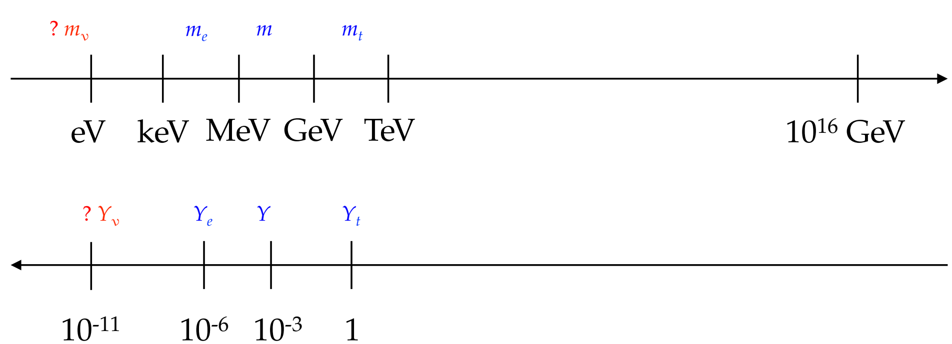
$\nu_R \Rightarrow \nu = \nu_L + \nu_R$

$$\mathcal{L}_{\mathrm{mass}}^{\mathrm{leptons}} = -Y_\ell \bar{L} \phi \ell_R - Y_\nu \bar{L} \tilde{\phi} \nu_R + \mathrm{h.c.}$$

$\Rightarrow$ After electroweak symmetry breaking:

$$\mathcal{L}_{\mathrm{mass}}^{\mathrm{leptons}} = -m_\ell \bar{\ell}_L \ell_R - m_D \bar{\nu}_L \nu_R + \mathrm{h.c.}$$

$\Rightarrow$ **3** light active neutrinos: $m_\nu \lesssim 1\mathrm{eV} \Rightarrow Y^\nu \lesssim 10^{-11}$



Majorana neutrinos?

- Add gauge singlet (sterile), right-handed neutrinos ν_R

$$\mathcal{L}^{\text{leptons}}_{\text{mass}} = -Y_\ell \bar{L}\phi\ell_R - Y_\nu \bar{L}\tilde{\phi}\nu_R - \frac{1}{2}M_R\overline{\nu_R}\nu_R^c + \text{h.c.}$$

$\Rightarrow$ After electroweak symmetry breaking:

$$\mathcal{L}^{\text{leptons}}_{\text{mass}} = -m_\ell \bar{\ell}_L \ell_R - m_D \bar{\nu}_L \nu_R - \frac{1}{2}M_R\overline{\nu_R}\nu_R^c + \text{h.c.}$$

3 $\nu_R \Rightarrow$ **6** mass eigenstates: $\boldsymbol{\nu = \nu^c}$

- ν_R gauge singlets
$\Rightarrow M_R$ not related to SM dynamics, not protected by symmetries
$\Rightarrow M_R$ between 0 and M_P
- $M_R\overline{\nu_R}\nu_R^c$ violates lepton number conservation $\Delta L = 2$

Linking the Higgs sector and neutrinos

How to search for heavy neutrino with $m_\nu > \mathcal{O}(1\,\text{TeV})$?

Use the Higgs sector to probe neutrino mass models

- TeV-scale neutrinos + Large Yukawa couplings
  $\Rightarrow$ Possibly large deviations from SM properties in the Higgs sector
- ***HH* production:** one of the main motivation for high-luminosity LHC and future colliders $\Rightarrow$ need to study the impact of BSM on λ_{HHH} $\Rightarrow$ impact of heavy neutrino(s) on λ_{HHH}?
  - Sizeable SM 1-loop corrections ($\mathcal{O}(10\%)$) $\Rightarrow$ Quantum corrections cannot be neglected
  - Sensitive to diagonal Yukawa couplings Y_ν

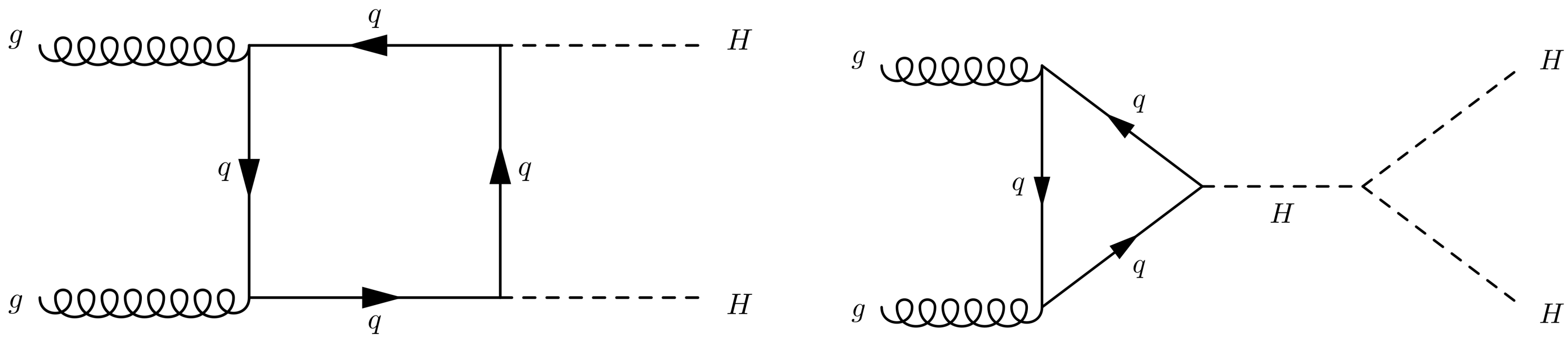
Neutrino effects on the triple Higgs coupling

1. A 3+1 model

[J.B., Weiland, PRD 94 (2016) 013002]

Sensitivity to λ_{HHH}

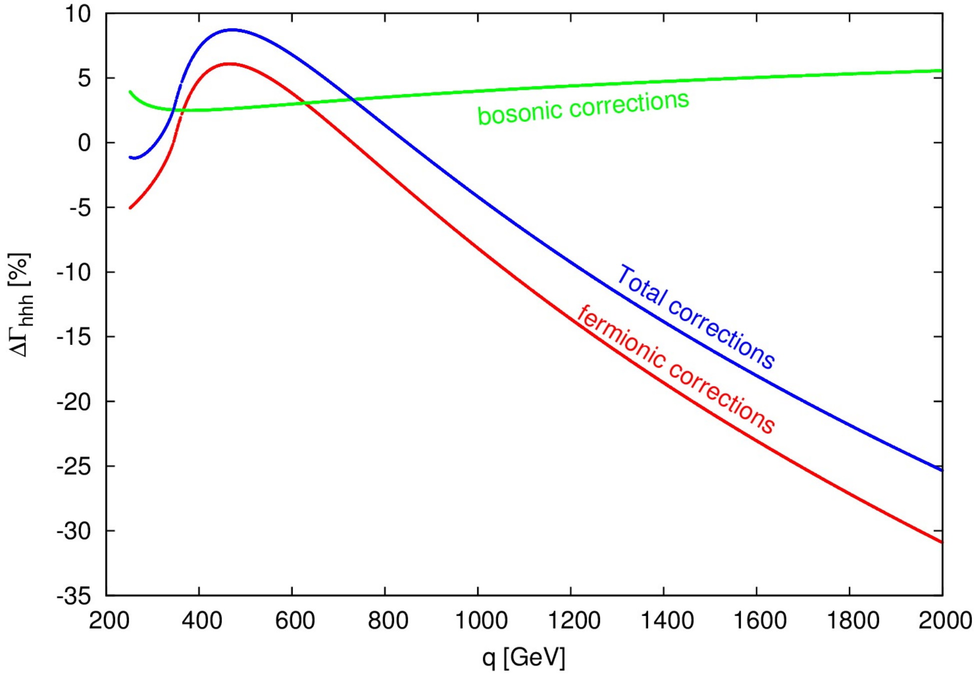
λ_{HHH} extracted from *HH* production



Experimental prospects for the sensitivity to λ_{HHH}:

- HL-LHC: only (optimistic) bounds $\lambda_{HHH}/\lambda_{HHH}^{\rm SM} \in [0.04; 2.7] \cup [5.5; 5.6]$ at 68% CL [Kim, Sakaki, Son, arXiv:1801.06093] $\Rightarrow$ very difficult
- ILC: 27% at 500 GeV with 4 ab^{-1} [Fujii *et al*, arXiv:1506.05992]
  10% at 1 TeV with 5 ab^{-1}
- FCC-hh: 8% / exp with 3 ab^{-1} using only $b\bar{b}\gamma\gamma$ [Je, Ren, Yao, PRD 93 (2016) 015003]
  $\sim$ 5% combining all channels

SM 1-loop corrections



taken from [Arhrib *et al*, JHEP 12 (2015) 007]

tree-level: $\lambda^0_{HHH} = -\frac{3M_H^2}{v}$

Dominant contribution from top-quark loops

[Kanemura *et al*, PRD 70 (2004) 115002]

$$\lambda_{HHH}(q^2, m_H^2, m_H^2) = -\frac{3m_H^2}{v}\left[1 - \frac{1}{16\pi^2}\frac{16m_t^4}{v^2 m_H^2} \times \left\{1 + \mathcal{O}\left(\frac{m_H^2}{m_t^2}, \frac{q^2}{m_t^2}\right)\right\}\right]$$

Opposite sign for the threshold ($\sqrt{q^2} = 2m_t$) and m_t^4 contributions

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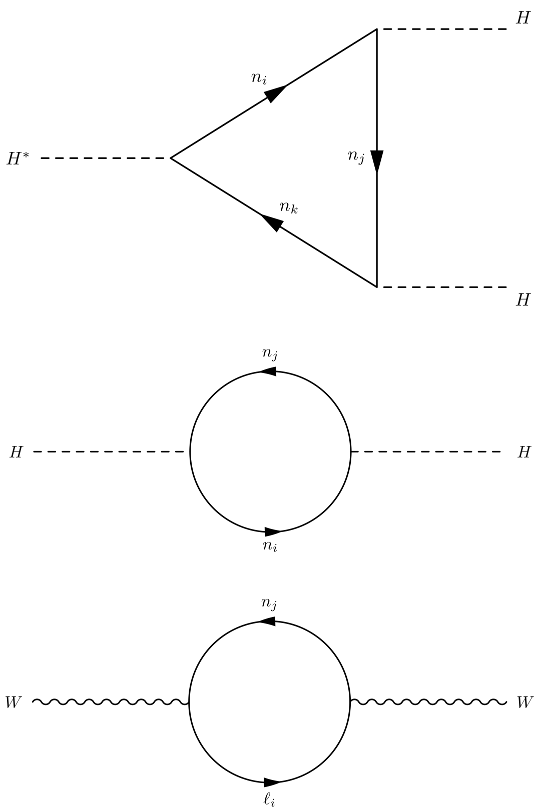
Simplified 3+1 Dirac model

- Simplified models for:
  $\rightarrow$ Simplify study of neutrino (ν) mass models
  $\rightarrow$ Effects of new fermionic coupling through neutrino portal
- Simplified model: 3 light ν ($m_n = 1$ eV) and 1 heavy sterile ν (m_4) parametrized by ν masses and active-sterile mixing B_{ij}

$$
\begin{aligned}
\mathcal{L} \ni & -\frac{g_2}{\sqrt{2}}\bar{\ell}_i\gamma^\mu W_\mu^- B_{ij} P_L n_j + \mathrm{h.c.} \\
& -\frac{g_2}{2M_W}\bar{n}_i (B^\dagger B)_{ij} H(m_{n_i}P_L + m_{n_j}P_R)n_j \\
& -\frac{g_2}{2\cos\theta_W}\bar{n}_i\gamma^\mu Z_\mu (B^\dagger B)_{ij} P_L n_j
\end{aligned}
\qquad
B_{3\times 4} = \begin{pmatrix} B_{e1} & B_{e2} & B_{e3} & \boldsymbol{B_{e4}} \\ B_{\mu 1} & B_{\mu 2} & B_{\mu 3} & \boldsymbol{B_{\mu 4}} \\ B_{\tau 1} & B_{\tau 2} & B_{\tau 3} & \boldsymbol{B_{\tau 4}} \end{pmatrix}
$$

Active-sterile mixing matrix B constructed from the PMNS matrix

New contributions to the triple Higgs coupling



Heavy ν generates new 1-loop diagrams and new counterterms

Counterterm to the triple Higgs coupling:

$$\frac{\delta\lambda_{HHH}}{\lambda^0_{HHH}} = \frac{3}{2}\delta Z_H + \delta t_H \frac{e}{2M_W \sin\theta_W M_H^2} + \delta Z_e + \frac{\delta M_H^2}{M_H^2} - \frac{1}{2}\frac{\delta M_W^2}{M_W^2} + \frac{1}{2}\cot^2\theta_W \left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right)$$

Tools for the calculation:

→ `FeynArts/FormCalc/LoopTools`

→ New Model File for ν interactions

Constraints on the model

Theoretical constraints

- Loose (tight) perturbativity bound:

$$\left(\frac{\max|C_{i4}|\, g_2\, m_{n_4}}{2M_W}\right)^3 < 16\pi\,(2\pi)$$

- Width limit: $\Gamma_{n_4} \leq 0.6\, m_{n_4}$

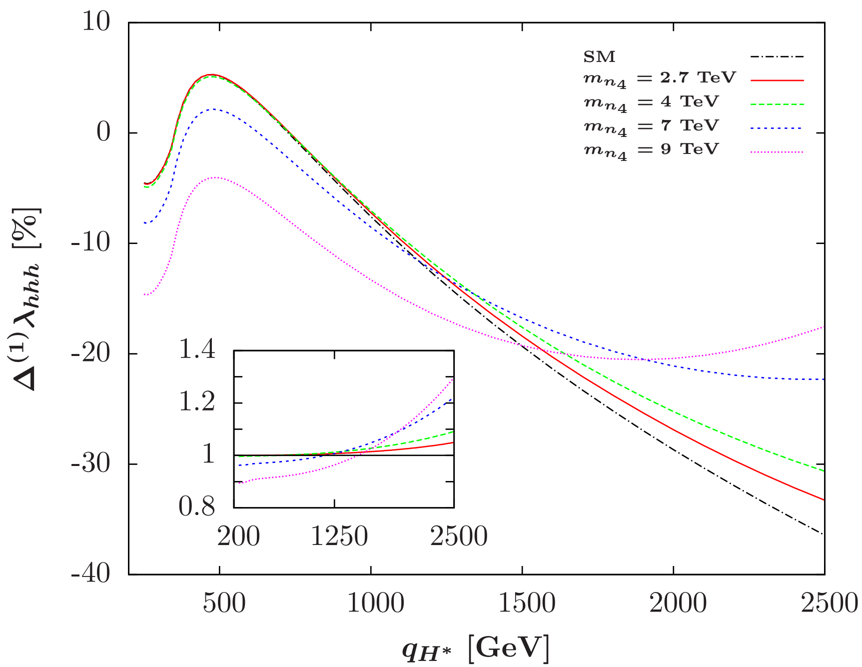
Experimental constraints

- PMNS matrix: best fit of normal hierarchy with no CP-violation [Gonzalez-Garcia, Maltoni, Schwetz, JHEP 11 (2014) 052]

  Lepton flavor violating decays [MEG, EPJC 76 (2016) 434]
- Neutrinoless beta decay: escaped (Dirac ν)
- Strongest experimental constraints on n_4: EW precision observables [del Aguila, de Blas, Pérez-Victoria, PRD 78 (2008) 013010]

$$|B_{e4}| \leq 0.041, \quad |B_{\mu 4}| \leq 0.030, \quad |B_{\tau 4}| \leq 0.087$$

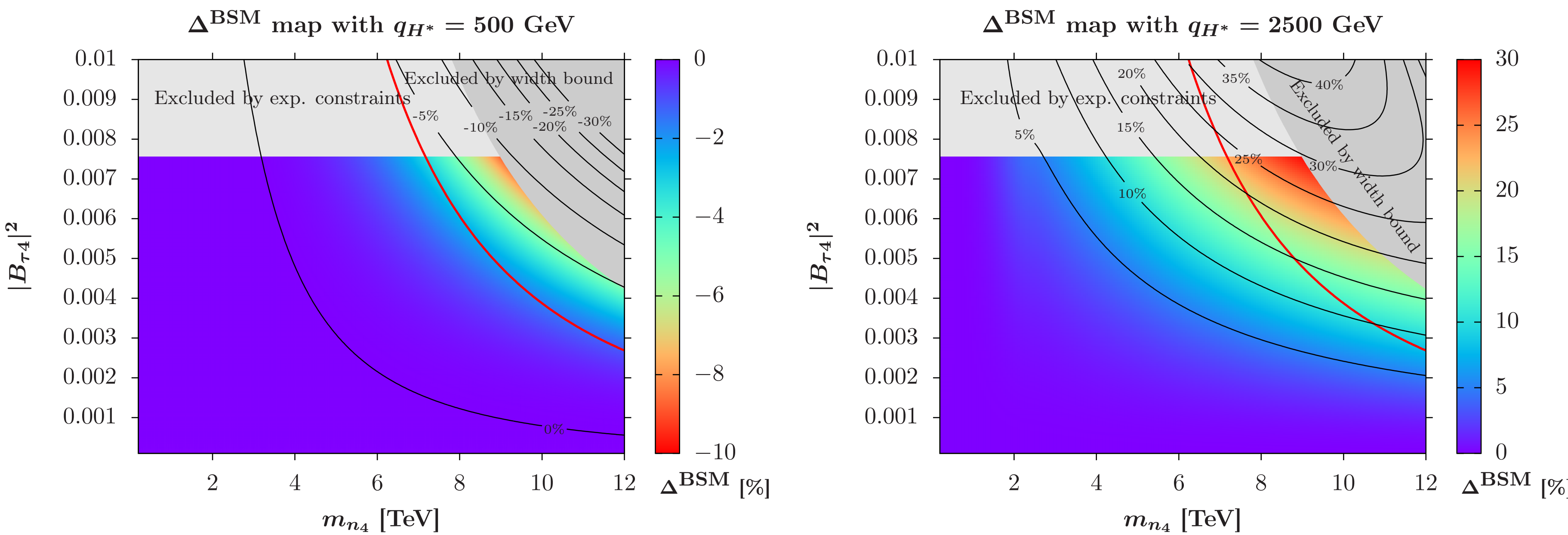
Momentum dependence



- $\Delta^{(1)}\lambda_{HHH} = \frac{1}{\lambda^0}\left(\lambda^{1r}_{HHH} - \lambda^0\right)$
- $B_{\tau 4} = 0.087$, $B_{e4} = B_{\mu 4} = 0$
- Deviation of the BSM correction with respect to the SM correction in the insert
- $C_{44} m_{n_4} = m_t \Rightarrow m_{n_4} = 2.7$ TeV
  tight perturbativity bound: $m_{n_4} = 7$ TeV
  width bound: $m_{n_4} = 9$ TeV

- Largest positive correction at $q^*_H \simeq 500$ GeV, heavy ν decreases it
- Large negative correction at large q^*_H, heavy ν increases it

Contour maps in a 3+1 simplified model



- $\Delta^{\rm BSM} = \left(\lambda^{1r,\rm full}_{HHH} - \lambda^{1r,\rm SM}_{HHH}\right) / \lambda^{1r,\rm SM}_{HHH}$
- Red line: tight perturbativity bound
- Heavy ν effects out of reach of the HL-LHC
- Heavy ν effects visible at the ILC (10%) and FCC-hh (5%)
- Similar plots for B_{e4} and $B_{\mu 4}$

Neutrino effects on the triple Higgs coupling

2. The inverse seesaw

[J.B., Weiland, JHEP 1704 (2017) 038]

From the 3+1 Dirac model to the inverse seesaw

- TeV-scale neutrino induces sizeable corrections to λ_{HHH}
  - $\rightarrow$ Decrease at $q_H^* \simeq 500$ GeV
  - $\rightarrow$ Increase at large q_H^*
- Effects could be used to constrain the active-sterile mixing at the ILC and FCC-hh
- What are the effects in an appealing low-scale seesaw model ?
  - ▶ Inverse seesaw $\rightarrow$ Additional constraints need to be included

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The inverse seesaw (ISS) mechanism

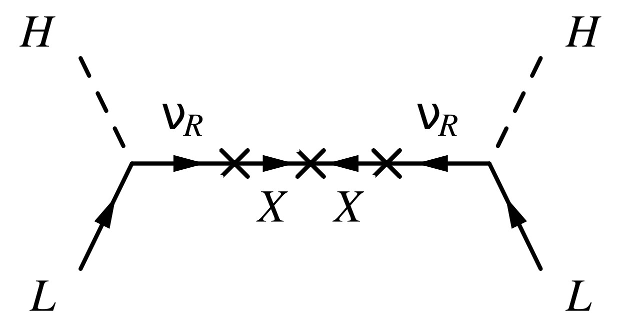
- Lower seesaw scale from (nearly) conserved lepton number
- Add fermionic gauge singlets ν_R ($L = +1$) and X ($L = -1$)

[Mohapatra, PRL 56 (1986) 561; Mohapatra, Valle, PRD 34 (1986) 1642; Bernabéu *et al.* , PLB 187 (1987) 303]

$$\mathcal{L}_{\rm ISS} = -Y_\nu \overline{L}\tilde{\phi}\nu_R - M_R\overline{\nu_R^c}X - \frac{1}{2}\mu_X\overline{X^c}X + \text{h.c.}$$

with $m_D = Y_\nu v/\sqrt{2}\,, M^\nu = \begin{pmatrix} 0 & m_D & 0 \\ m_D^T & 0 & M_R \\ 0 & M_R^T & \mu_X \end{pmatrix}$

$$m_\nu \approx \frac{m_D^2}{M_R^2}\mu_X, \quad m_{N_1,N_2} \approx \mp M_R + \frac{\mu_X}{2}$$



2 scales: μ_X and M_R

- Decouple neutrino mass generation from active-sterile mixing
- Inverse seesaw: $Y_\nu \sim \mathcal{O}(1)$ and $M_R \sim 1$ TeV
  $\Rightarrow$ within reach of the LHC and low energy experiments

Most relevant constraints for the ISS

- Accommodate low-energy neutrino data using parametrization

$$vY_\nu^T = V^\dagger \mathrm{diag}(\sqrt{M_1}, \sqrt{M_2}, \sqrt{M_3})\, R\, \mathrm{diag}(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}) U_{PMNS}^\dagger$$

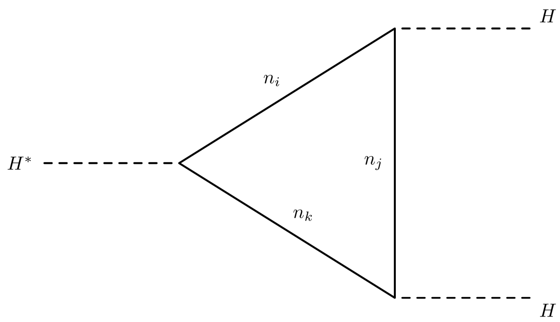
$$M = M_R \mu_X^{-1} M_R^T \text{ (Casas-Ibarra parametrization)}$$

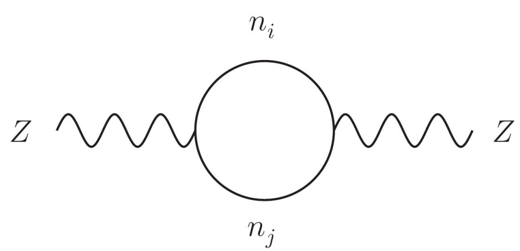
or

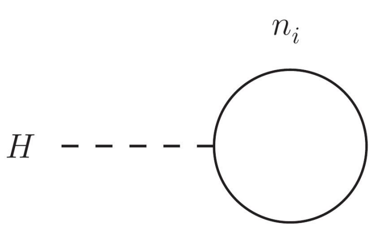
$$\mu_X = M_R^T\, Y_\nu^{-1}\, U_{\mathrm{PMNS}}^* m_\nu U_{\mathrm{PMNS}}^\dagger\, Y_\nu^{T^{-1}} M_R v^2$$ and beyond

- Charged lepton flavor violation
  $\rightarrow$ For example: $\mathrm{Br}(\mu \rightarrow e\gamma) < 4.2 \times 10^{-13}$ [MEG, EPJC 76 (2016) 434]
- Global fit to EWPO and lepton universality tests [Fernandez-Martinez *et al.* , JHEP 1608 (2016) 033]
- Electric dipole moment: 0 with real PMNS and mass matrices
- Invisible Higgs decays: $M_R > m_H$, does not apply
- Yukawa perturbativity: $|\frac{Y_\nu^2}{4\pi}| < 1.5$

Calculation in the ISS







- Similar diagrams to the 3+1 Dirac scenario but with Majorana neutrinos
- Same tools as for the 3+1 model
- More heavy neutrinos
  $\Rightarrow$ effects generically larger than in the 3+1 model

Analytical formulae for both Dirac and Majorana fermions

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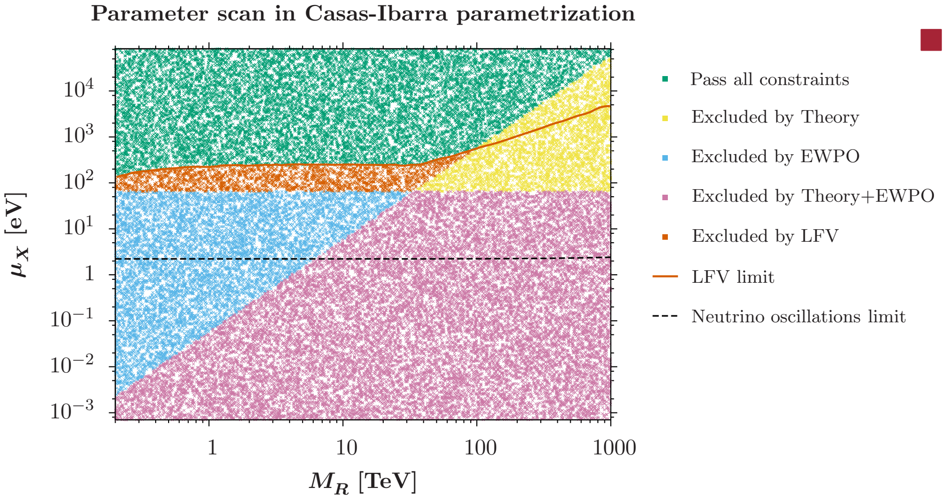
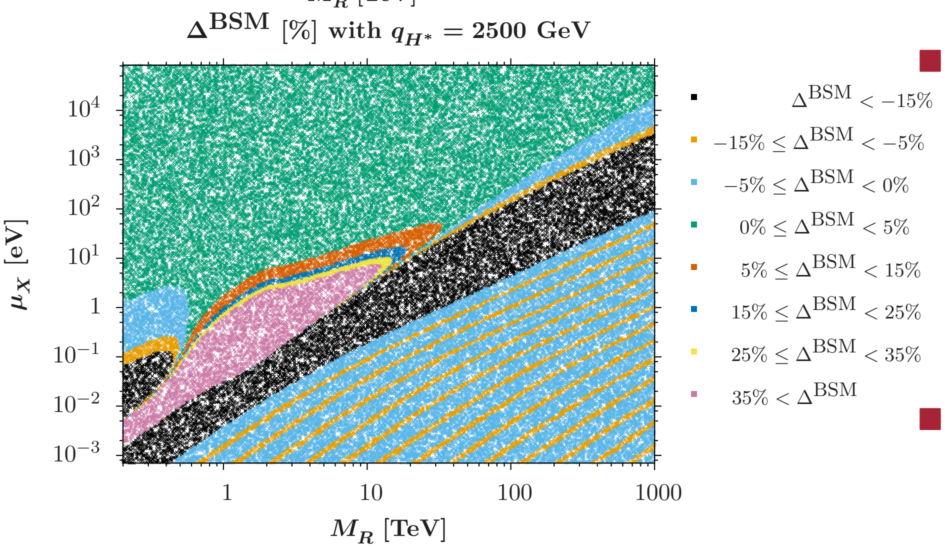
Results using the Casas-Ibarra parametrization





- Random scan: 180000 points with degenerate (diagonal) M_R and μ_X, θ_i angles of the matrix R,

$$
\begin{aligned}
0 &\leq \theta_i \leq 2\pi, \ (i = 1, 2, 3) \\
0.2 \text{ TeV} &\leq M_R \leq 1000 \text{ TeV} \\
7 \times 10^{-4} \text{ eV} &\leq \mu_X \leq 8.26 \times 10^{4} \text{ eV}
\end{aligned}
$$

- Strongest constraints:
  - Lepton flavor violation, mainly $\mu \to e\gamma$
  - Yukawa perturbativity (and neutrino width)
- Large effects necessarily excluded by LFV constraints ?

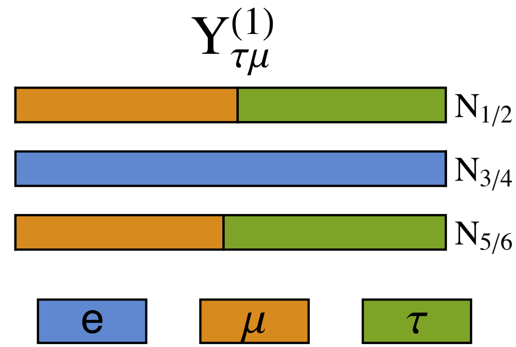
Suppressing LFV constraints

- How to evade the LFV constraints ?
- Approximate formulas for large Y_ν [Arganda *et al.* , PRD 91 (2015) 015001]:

$$\mathrm{Br}^{\mathrm{approx}}_{\mu\to e\gamma} = 8\times 10^{-17}\mathrm{GeV}^{-4}\frac{m_\mu^5}{\Gamma_\mu}\left|\frac{\mathrm{v}^2}{2M_R^2}(Y_\nu Y_\nu^\dagger)_{12}\right|^2$$

- Solution: Textures with $(Y_\nu Y_\nu^\dagger)_{12} = 0$

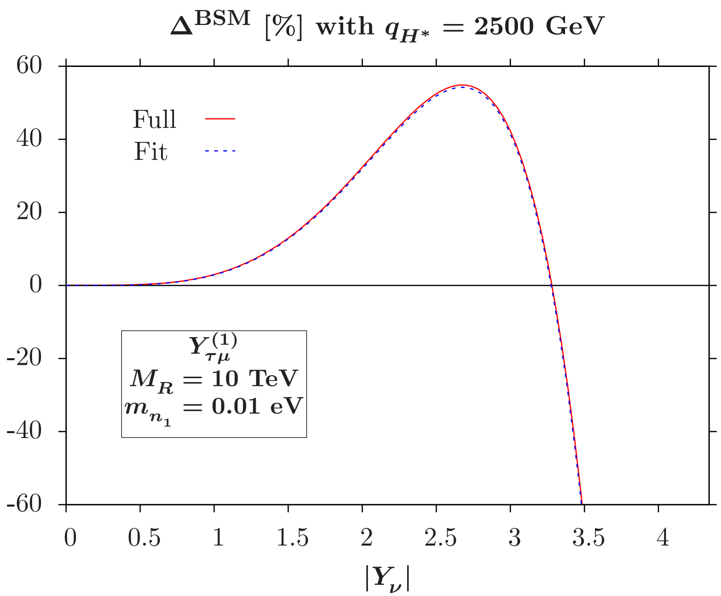
$$Y^{(1)}_{\tau\mu} = |Y_\nu|\begin{pmatrix} 0 & 1 & -1 \\ 0.9 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

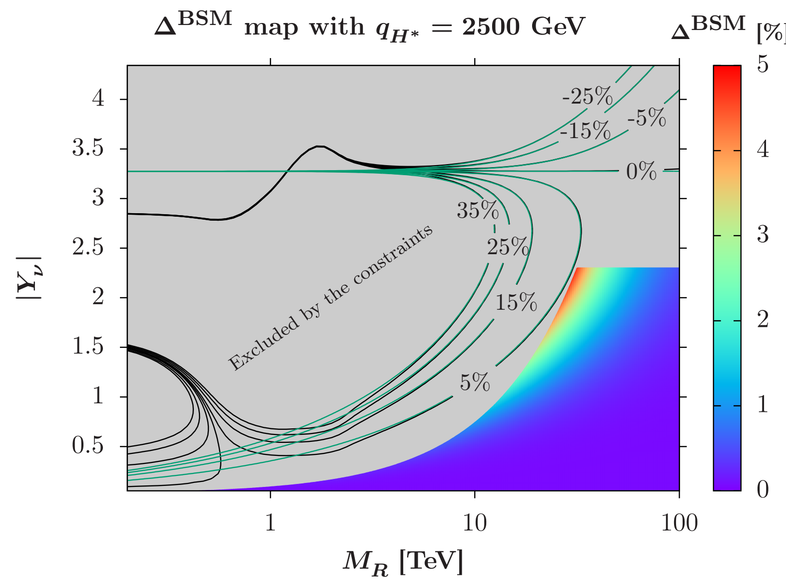


[taken from Arganda *et al.* , PLB 752 (2016) 46]

- Or even take Y_ν diagonal

Results for $Y^{(1)}_{\tau\mu}$





- $\Delta^{\text{BSM}} = \left(\lambda^{1r,\text{full}}_{HHH} - \lambda^{1r,\text{SM}}_{HHH}\right) / \lambda^{1r,\text{SM}}_{HHH}$
- Right: Full calculation in black, approximate formula in green
- Well described at $M_R > 3$ TeV by approximate formula

$$\Delta^{\text{BSM}}_{\text{approx}} = \frac{(1\text{ TeV})^2}{M_R^2}\left(8.45\,\text{Tr}(Y_\nu Y_\nu^\dagger Y_\nu Y_\nu^\dagger) - 0.145\,\text{Tr}(Y_\nu Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu Y_\nu^\dagger)\right)$$

- Can maximize Δ^{BSM} by taking $Y_\nu \propto \mathrm{I}_3$

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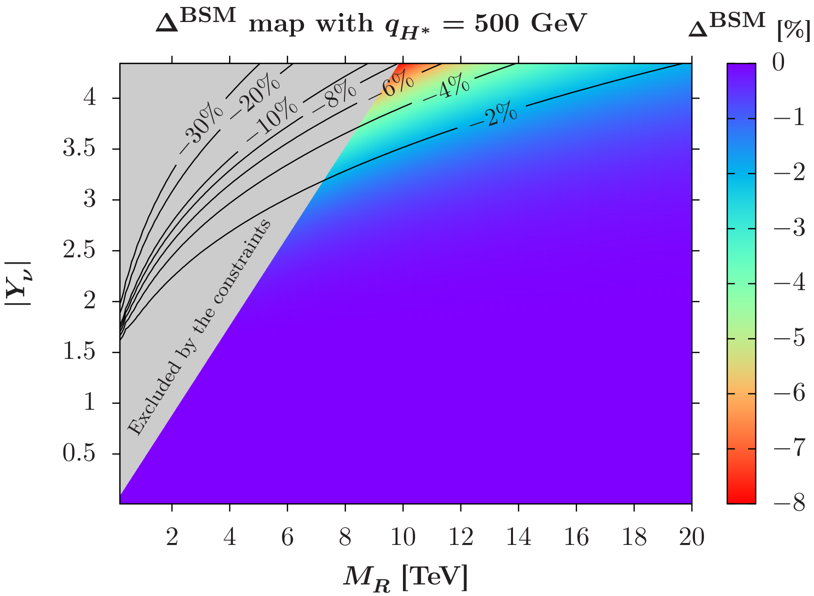
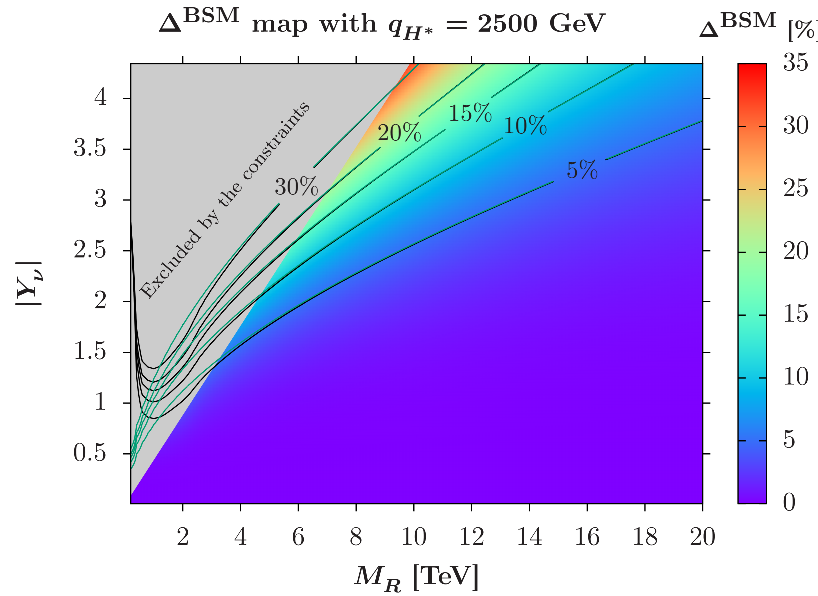
Results in the ISS



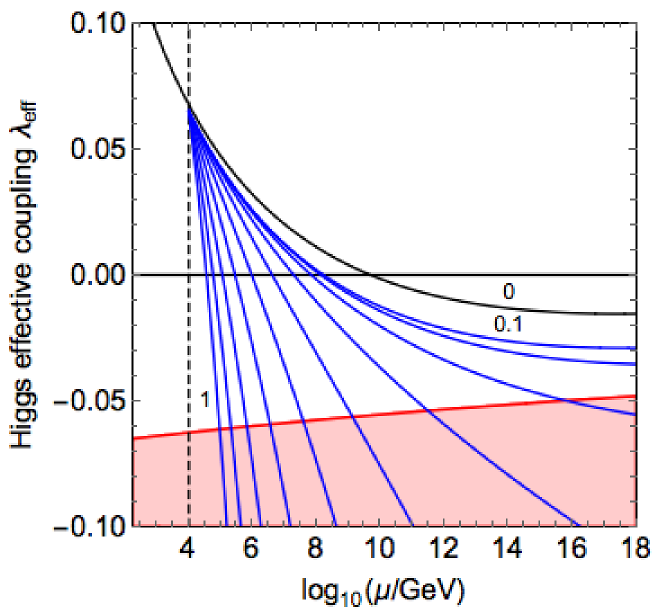


- $\Delta^{\text{BSM}} = \left(\lambda_{HHH}^{1r,\text{full}} - \lambda_{HHH}^{1r,\text{SM}}\right) / \lambda_{HHH}^{1r,\text{SM}}$
- $Y_\nu \propto \mathrm{I}_3$ + diagonal M_R, full calculation in black
  approximate formula in green; Maximize the correction
- Confirm 3+1 Dirac analysis despite stronger constraints
- Effects potentially visible at the 1 TeV ILC (10% sensitivity)
  clearly visible at the FCC-hh (5% sensitivity)

Look beyond the lower scales

- λ_{HHH} complementary to existing observables, **provide a new probe of the $\mathcal{O}(10)$ TeV region of neutrino mass models, especially with diagonal, real Yukawa couplings!**
- **What about the running of λ / the stability of $V(H)$?**

[see Delle Rose, Marzo, Urbano, JHEP 1512 (2015) 050]



With $V_{\rm eff}(H) = \frac{1}{4}\lambda_{\rm eff}(H)H^4$, common heavy neutrino threshold $M_R = 10$ TeV:

[Gröber, Di Luzio, Spannowsky, EPJC 77 (2017) 788]

Large Yukawa couplings could require a cut-off scale at a few tens of TeV

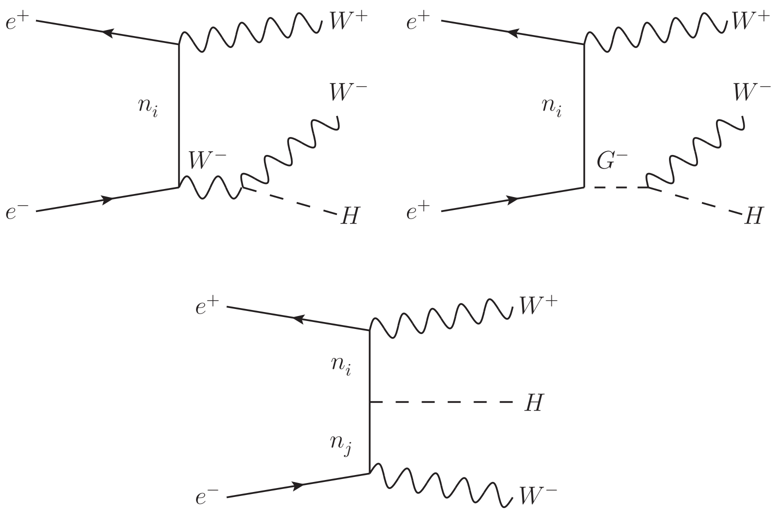
$\Rightarrow$ **A thorough analysis of UV-complete models would be required, safe if $\Lambda_{\rm UV} > \Lambda_{\rm seesaw}$**

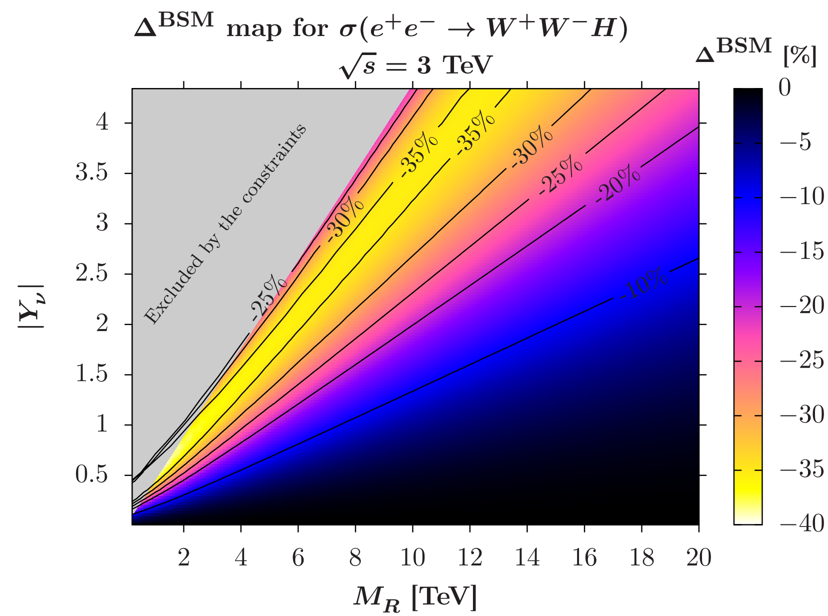
- UV completion natural to explain μ_X, link to $B - L$ symmetry

Probing ISS at tree-level: A lepton collider observable

- **Looking directly at a production cross section: are similar deviations possible?**
- **Case example: W^+W^-H production at a lepton collider**

[J.B., Pascoli, Weiland, arXiv:1712.07621 [hep-ph]]





- **Sizable effects on a larger subset of the parameter space! Motivate a detailed sensitivity analysis** [J.B., Pascoli, Weiland, in preparation]

Conclusion and outlook

- **The big question: Is the observed scalar boson an SM Higgs boson or a first window on BSM physics?**

  $\Rightarrow$ **Study of BSM effects on the triple Higgs coupling**

- **Neutrino oscillations: New physics needed to generate m_ν**
  $\rightarrow$ low-scale seesaw appealing to generate tree-level m_ν
  Inverse seesaw: $Y_\nu \sim \mathcal{O}(1)$ AND $M_R \sim \mathcal{O}(0.1 - 10)$ TeV

- **Neutrino effects on the triple Higgs coupling: Up to 30% correction**
  $\rightarrow$ Measurable at future colliders (ILC, FCC-hh)
  $\rightarrow$ Probe a new part of the parameter space of the mass models
  $\rightarrow$ Generic effect expected in all models with TeV fermions and large Higgs couplings
  $\rightarrow$ Give new constraints on active-sterile neutrino mixing: Impact on astrophysics, cosmology, neutrino physics

- **Future work:** corrections to HH production cross-section at lepton colliders, UV completion to low-scale seesaw models

Backup slides

Renormalization procedure for the HHH coupling I

- No tadpole: $t_H^{(1)} + \delta t_H = 0 \Rightarrow \delta t_H = -t_H^{(1)}$
- Counterterms:

$$M_H^2 \to M_H^2 + \delta M_H^2$$
$$M_{W/Z}^2 \to M_{W/Z}^2 + \delta M_{W/Z}^2$$
$$e \to (1 + \delta Z_e) e$$
$$H \to \sqrt{Z_H} = (1 + \frac{1}{2}\delta Z_H) H$$

- Full renormalized 1–loop triple Higgs coupling:

$\lambda_{HHH}^{1r} = \lambda^0 + \lambda_{HHH}^{(1)} + \delta\lambda_{HHH}$

$$\frac{\delta\lambda_{HHH}}{\lambda^0} = \frac{3}{2}\delta Z_H + \delta t_H \frac{e}{2M_W \sin\theta_W M_H^2} + \delta Z_e + \frac{\delta M_H^2}{M_H^2} - \frac{\delta M_W^2}{2M_W^2} + \frac{1}{2}\frac{\cos^2\theta_W}{\sin^2\theta_W}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right)$$

Renormalization procedure for the HHH coupling II

- OS scheme

$$\begin{aligned} \delta M_W^2 &= Re\Sigma^T_{WW}(M_W^2) \\ \delta M_Z^2 &= Re\Sigma^T_{ZZ}(M_Z^2) \\ \delta M_H^2 &= Re\Sigma_{HH}(M_H^2) \end{aligned} \tag{1}$$

- Electric charge:

$$\delta Z_e = \frac{\sin\theta_W}{\cos\theta_W}\frac{\mathrm{Re}\Sigma^T_{\gamma Z}(0)}{M_Z^2} - \frac{\mathrm{Re}\Sigma^T_{\gamma\gamma}(M_Z^2)}{M_Z^2}$$

- Higgs field renormalization

$$\delta Z_H = -\mathrm{Re}\frac{\partial\Sigma_{HH}(k^2)}{\partial k^2}\bigg|_{k^2=M_H^2}$$

Next-order terms in the μ_X-parametrization

- Weaker constraints on diagonal couplings
  $\rightarrow$ Large active-sterile mixing $m_D M_R^{-1}$ for diagonal terms
- Previous parametrizations built on the 1st term in the $m_D M_R^{-1}$ expansion $\rightarrow$ Parametrizations breaks down
- Solution: Build a parametrization including the next order terms
- The next-order μ_X-parametrization is then

$$\mu_X \simeq \left(\mathbf{1} - \frac{1}{2} M_R^{*-1} m_D^\dagger m_D M_R^{T-1}\right)^{-1} M_R^T m_D^{-1} U_{\mathrm{PMNS}}^* m_\nu U_{\mathrm{PMNS}}^\dagger m_D^{T-1} M_R \times \left(\mathbf{1} - \frac{1}{2} M_R^{-1} m_D^T m_D^* M_R^{\dagger-1}\right)^{-1}$$