UV complete me: Positivity Bounds for Particles with Spin

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Higgs Effective Field Theory, 2018

Work with

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Positive Bounds for Scalar Theories 1702.06134
Massive Galileon Positivity Bounds 1702.08577
Positivity Bounds for Spinning Particles 1706.02712
Improved Positivity Bounds and Massive Gravity 1710.09611
Positivity Bounds for Massive Spin 1 and Spin 2 Fields: To appear

Wilsonian Effective Field Theories

Top down

$$e^{\frac{i}{\hbar}S_W[\text{Light}]} = \int D\text{Heavy} \quad e^{\frac{i}{\hbar}S_{UV}[\text{Light},\text{Heavy}]}$$

Wilson: Heavy loops already included,
Light loops not yet included

Bottom Up

Construct $S_W[Light]$ by writing down every local operator consistent with symmetries of low energy theory, suppressed by cutoff scale to the appropriate power

$$\Lambda^4 F\left(\frac{Boson}{\Lambda}, \frac{Fermion}{\Lambda^{3/2}}, \frac{\partial_\mu}{\Lambda}\right)$$

In Cosmology always true because we must have gravity

GR itself should be understood as an EFT with a Planck scale physics - no problem quantizing gravity as a LEEFT,

see e.g. reviews by Donoghue, Burgess

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \cdots + M_P^4\left(\frac{\nabla}{M_P}\right)^{2N}\left(\frac{\text{Riemann}}{M_P^2}\right)^{M}\right]$$

For example, we have no trouble computing loop corrections to scalar and tensor fluctuations produced during inflation

Are all EFTs allowed?

With typical assumption that:
UV completion is Local, Causal, Poincare Invariant and Unitary

Answer: NO! Certain low energy effective theories do not admit well defined UV completions

More precise form of question: Are Wilson coefficients free to take any O(1) value? Answer: NO!

Recent Recognition: Requirement that a given low energy theory admits such a UV completion imposes an (infinite) number of constraints on the form of the low energy effective theory

Positivity Constraints!

Low Energy Criteria

Commonly Imposed Criteria:

Causality $c_s(background) \leq 1$, absence of caustics, **strong** hyperbolicity

Problems: 1. Causality is difficult in gravitational theories since the speed of light is not invariant under field redefinitions of the metric

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \alpha \partial_\mu \phi \partial_\nu \phi$$

2. Caustics are ok - can arise in LEEFT of a UV theory without caustics

3. Strong hyperbolicity is gauge dependent and field redefinition dependent, while desirable for numerics unclear if important

Solution: Remove field redefinition ambiguities by looking at the S-matrix

Look for asymptotic superluminalities:

We cannot send signals faster than what is allowed by asymptotic causal structure of the spacetime Gao and Wald 2000

e.g. Camanho, Edelstein, Maldacena, Zhiboedov, `Causality Constraints on Corrections to the Graviton Three-Point Coupling," arXiv:1407.5597
Massive Spin-2 Scattering and Asymptotic Superluminality
Hinterbichler. Joyce, Rosen arXiv:1708.05716

Amounts to demanding that the Eisenbud-Wigner scattering time delay is positive

$$T \sim \frac{d\delta(E)}{dE} \quad > 0$$

$\delta(E)$ is phase shift between scattered wave and unscattered

Solution: Remove field redefinition ambiguities by looking at the S-matrix

Closely related are the requirements that the S-matrix is

1. Local (Polynomially *(or exponentially)* bounded in momentum space) and
2. Causal (Analytic as a function of Mandelstam variables)

A precise definition of analyticity for the 2-2 scattering amplitude at fixed momentum transfer t was rigourously proven in the 50's and 60's

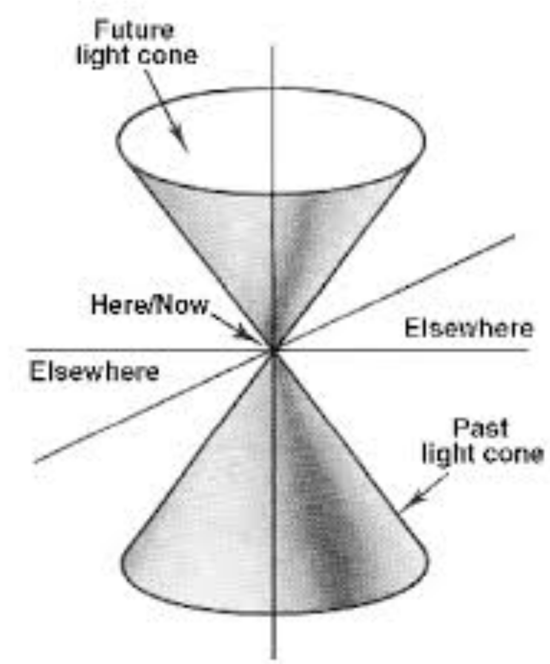
although not too much progress ever made beyond this

Positivity Constraints

Signs of UV completion

Lets Start Simple: Two-point function of a scalar field

Suppose we have a scalar operator $\hat{O}(x)$



Relativistic Locality tells us that ......

$$[\hat{O}(x), \hat{O}(y)] = 0 \qquad \text{if } (x-y)^2 > 0$$

Unitarity (positivity) tells us that

$$\langle\psi|\hat{O}(f)^2|\psi\rangle > 0 \quad \text{where} \qquad \hat{O}(f) = \int d^4x\, f(x)\hat{O}(x)$$

Kallen-Lehmann Spectral Representation

Together with Poincare invariance these imply:

$$\langle 0|\hat{T}\hat{O}(x)\hat{O}(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik.(x-y)} G_O(k)$$

$$G_O(k) = \frac{Z}{k^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{k^2 + \mu - i\epsilon}$$

Positive Spectral Density
as a result of Unitarity

$$\rho(\mu) \geq 0$$

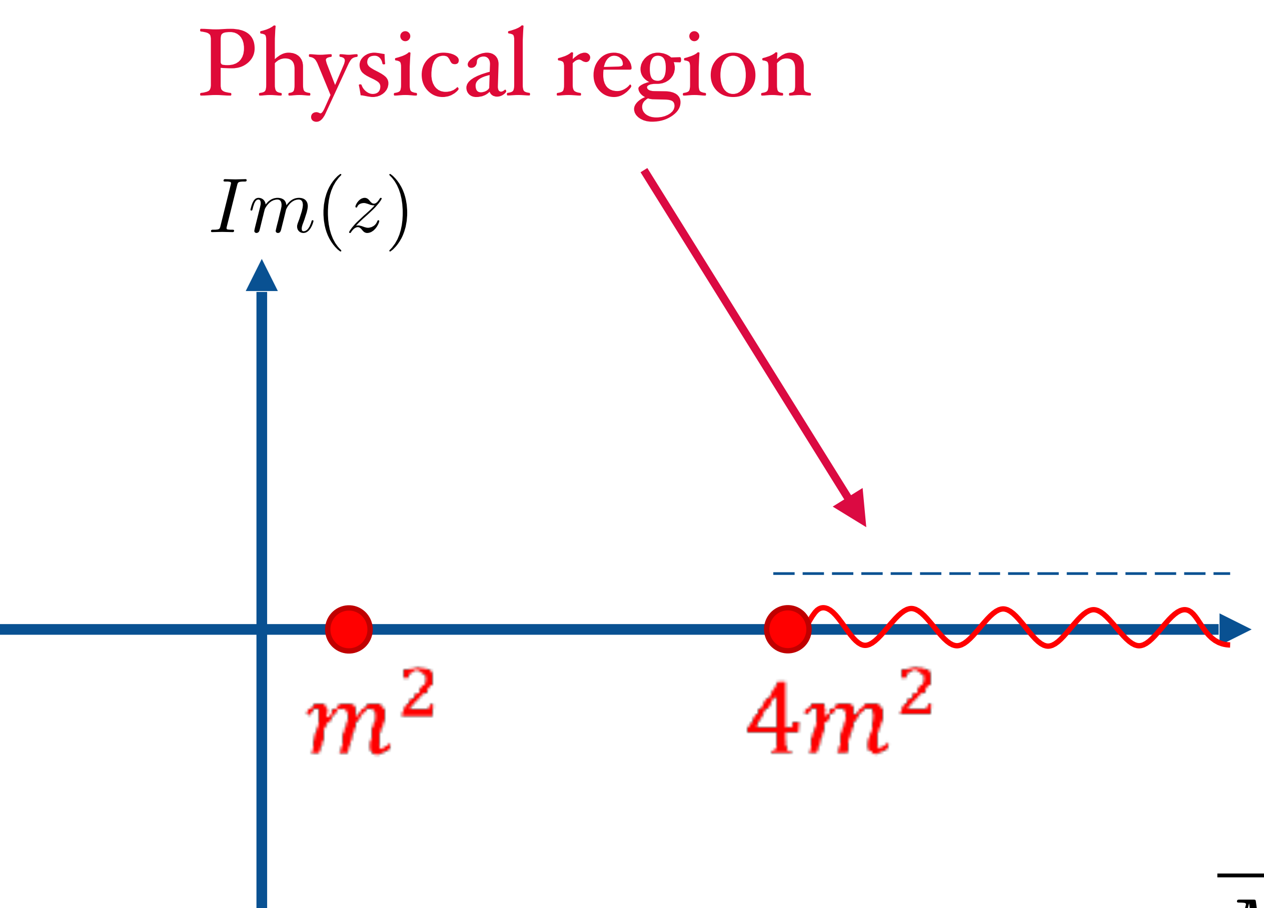
Introduce the Complex Plane

To simplify a problem you should make it complex

Define complex momenta squared $z = -k^2 + i\epsilon$

Pole — Branch cut

$$G_O(z) = \frac{Z}{m^2 - z} + S(z) + z^N \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$



Two point function is an analytic function with a pole and a branch cut

Discontinuity across branch cut is positive definite

$$M \geq N$$

$$\frac{1}{M!} \frac{d^M G'_O(0)}{dz^M} = \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{M+1}} > 0$$

What does this tell us about EFT?

e.g. Suppose scalar field in EFT with tree level action ....

$$S = \int d^4x \hat{O}(x)[\Box + a_1 \frac{\Box^2}{M^2} + a_2 \frac{\Box^3}{M^3} + \ldots]\hat{O}(x)$$

Feynman propagator is

$$G_O = -\frac{1}{z + a_1 \frac{z^2}{M^2} + a_2 \frac{z^3}{M^3} + \ldots}$$

$$G'_O(z) = \frac{a_1}{M^2} + \frac{(a_2 - a_1^2)}{M^4} z + \mathcal{O}(z^2)$$

assuming no subtractions

Positivity Bounds: $a_1 > 0$ and $a_2 > a_1^2$

What about gravity?

If we repeat the same argument for gravity:

spin 2 part　　　　spin 0 part

$$S = \int d^4x \frac{M_P^2}{2}\left[R + \frac{a}{\Lambda^2}(R_{\mu\nu}R^{\mu\nu} - (5/3)R^2) + \frac{b}{\Lambda^2}R^2)\right]$$

$$+\frac{1}{\Lambda^4}R_{....}\nabla_{.}\nabla_{.}R_{....}$$ corrections

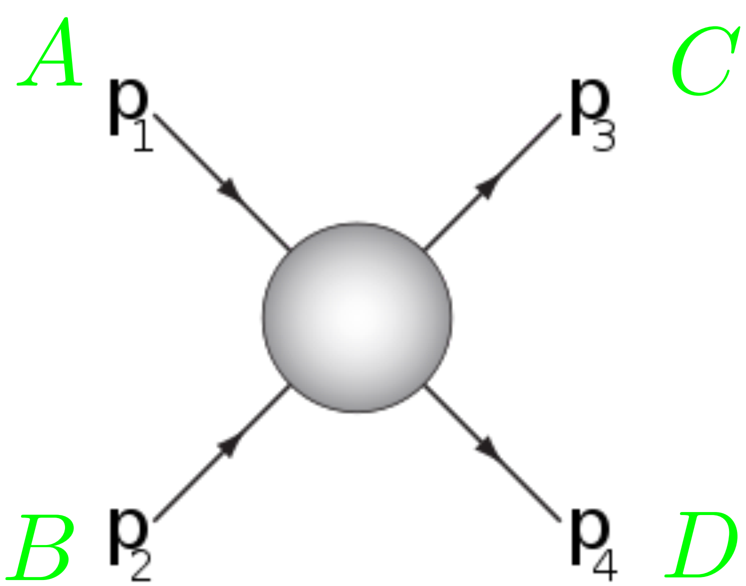
Positivity Bounds: $a \geq 0$ $b \geq 0$

S-matrix Positivity Constraints

Signs of UV completion

Crossing Symmetry

Identical scattering amplitudes for s and u channel interactions (up to analytic continuation)



s-channel

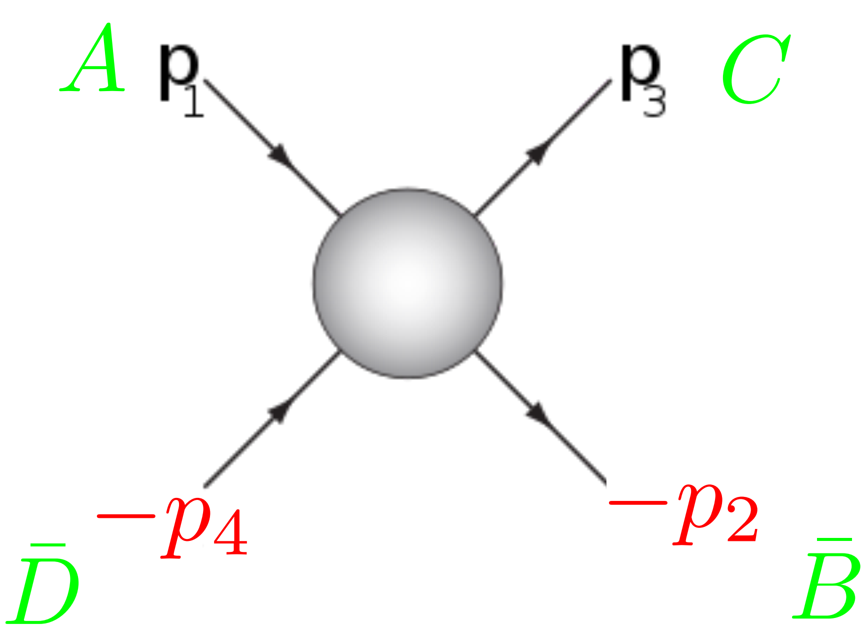
$$A + B \to C + D$$

$$s + t + u = 4m^2$$

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

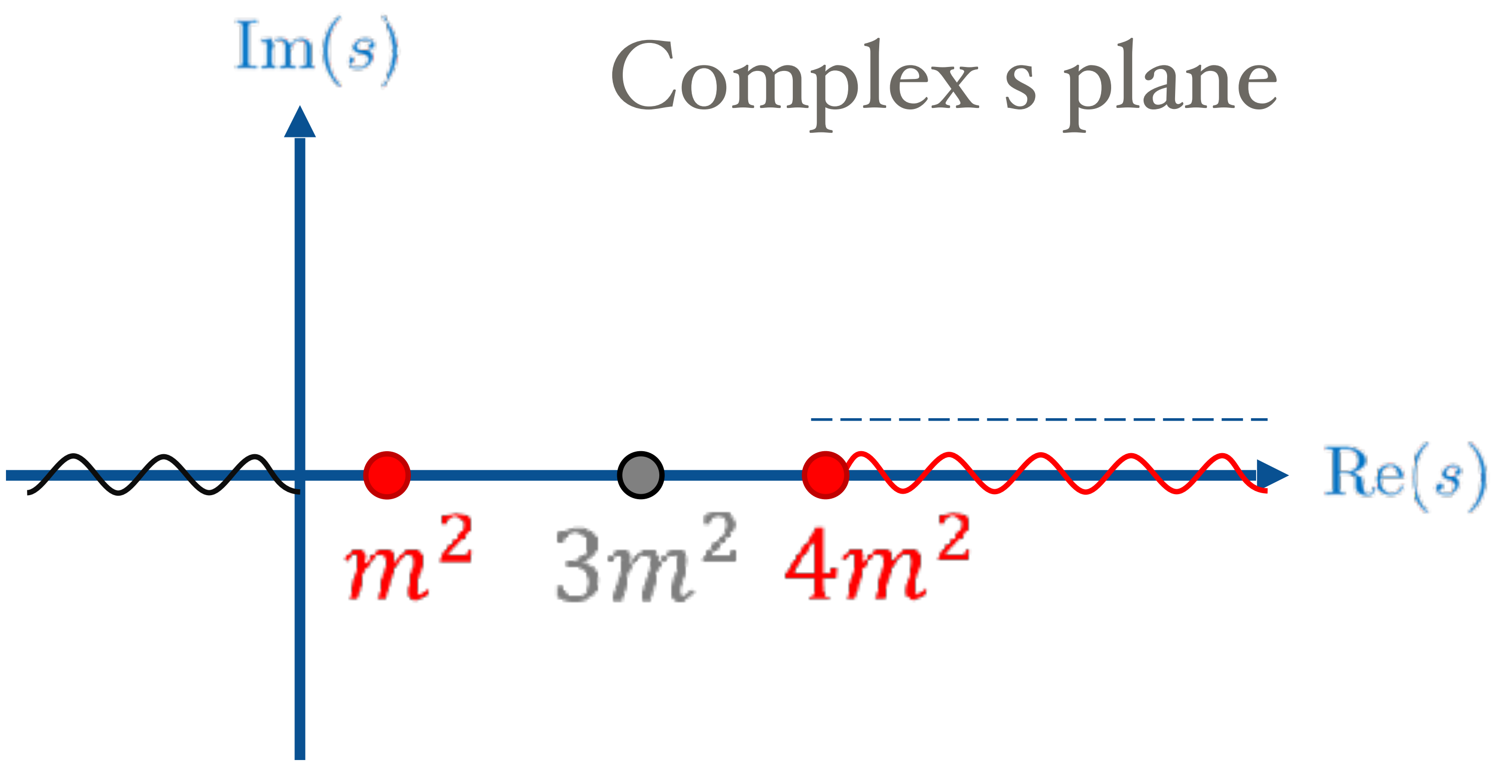


u-channel

$$A + \bar{D} \to C + \bar{B}$$

$$s \leftrightarrow u$$

Forward Scattering Limit Amplitude Analyticity



Complex s plane

Physical scattering region is $s \geq 4m^2$

crossing: $u = 4m^2 - s$

$$\mathcal{A}(s,0) = \frac{\lambda}{m^2 - s} + (a + bs) + s^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^2(\mu - s)} + \frac{\lambda}{m^2 - u} + u^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{u^2(\mu - u)}$$

Unitarity=Positivity

$$\rho(s) = \frac{1}{\pi} Im[A(s,0)] = \frac{\sqrt{s(s-4m^2)}\sigma(s)}{\pi} > 0$$

Number of subtractions determined by Froissart bound:

$$\sigma(s) < \frac{c}{m^2}(\log(s/s_0))^2$$

Why it is crucial external scatterers are massive

Positivity Constraint

Recipe: Subtract pole, differentiate to remove subtraction constants

$$\mathcal{B}(s,0) = \mathcal{A}(s,0) - \frac{\lambda}{m^2 - s} - \frac{\lambda}{m^2 - u}$$

$$= (a + bs) + s^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^2(\mu - s)} + u^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{u^2(\mu - u)}$$

$$\frac{d^M}{ds^M} B(2m^2, 0) = 2M! \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{(\mu - 2m^2)^{M+1}} > 0$$

$$M \geq 2$$

Adams et. al. 2006

E.g. Assume Weakly Coupled UV Completion

Constraints can be applied directly on LEEFT tree level scattering amplitudes

$$\mathcal{B}^{\rm tree}(s,0) = \mathcal{A}^{\rm tree}(s,0) - \frac{\lambda^{\rm tree}}{m^2 - s} - \frac{\lambda^{\rm tree}}{m^2 - u}$$

$$\frac{d^M}{ds^M} B^{\rm tree}(2m^2, 0) = 2M! \int_{\Lambda^2_{\rm th}}^{\infty} d\mu \frac{\rho(\mu)}{\mu - 2m^2} \qquad M \geq 2$$

$\Lambda_{\rm th} =$ threshold for new physics, i.e. cutoff of LEEFT

Adams et. al. 2006

Directly translates into constraints on Wilsonian action

Improved Positivity Bound

What if we can't assume weakly coupled UV completion?

Consider exact amplitude MINUS calculable low energy imaginary part

Calculate in LEEFT

$$\tilde{A}(s,0) = A(s,0) - \frac{s^2}{\pi}\int_{4m^2}^{\epsilon^2\Lambda^2} d\mu \frac{Im[A(\mu,0)]}{\mu^2(\mu-s)} - \frac{u^2}{\pi}\int_{4m^2}^{\epsilon^2\Lambda^2} d\mu \frac{Im[A(\mu,0)]}{\mu^2(\mu-u)}$$

$\tilde{A}(s,0)$ has the same analyticity properties as $A(s,0)$

with a branch cut which starts at $s = \epsilon^2\Lambda^2$

Example: Positivity Bounds for P(X)

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{\Lambda^4}(\partial\phi)^4 + \cdots$$

$$\mathcal{A}^{\text{tree}}_{2\to 2} = \frac{c}{\Lambda^4}\left(s^2 + t^2 + u^2 - 4m^2\right)$$

Positivity bounds requires: $c > 0$

P(X) models with $c \leq 0$ cannot admit a local/Poincare invariant UV completion

Adams et. al. 2006

DBI versus anti-DBI

$$\mathcal{L}_{\text{DBI}} \sim -\sqrt{1+(\partial\phi)^2}$$

Model relevant for inflation

$$(\partial\phi)^2 = -\dot{\phi}^2 \rightarrow -1$$

Model that naturally emerges as probe brane in extra dimension

No obstructions to standard UV completion (known so far)

$$\mathcal{L}_{\overline{\text{DBI}}} \sim \sqrt{1-(\partial\phi)^2}$$

Model relevant for dark energy with screening in dense environments

$$(\partial\phi)^2 = \phi'(r)^2 \rightarrow 1$$

Model that naturally emerges as probe brane in extra *time* dimension…

Known obstructions to standard UV completion

Extension away from forward scattering limit

So far we have only used Optical Theorem in the forward scattering limit, and not the full implications of Unitarity

$$\mathcal{A}(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)a_\ell(s)$$

$$\Downarrow$$

$$\mathrm{Im}\, a_\ell(s) = |a_\ell(s)|^2 + \cdots$$

$$\Downarrow$$

$$0 \leq |a_\ell(s)^2| \leq \mathrm{Im}\, a_\ell(s) \leq 1 \quad \text{for} \quad s \geq 4m^2$$

Extension away from forward scattering limit

$$\mathcal{A}(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)a_\ell(s)$$

$$\mathrm{Im}\, a_l(s) > 0\,, \quad s \geq 4m^2$$

$$\frac{d^n}{dt^n}\mathrm{Im}A(s,t)\Big|_{t=0} > 0$$

using $\frac{d^n}{dx^n}P_l(x)\Big|_{x=1} > 0$

$$\mathrm{Im}\, A(s,t) > 0\,, \quad 0 \leq t < 4m^2\,,\ s \geq 4m^2$$

$$M \geq 2$$

$$\frac{d^M}{ds^M}B(2m^2 - t/2, t) = \frac{2M!}{\pi}\int_{4m^2}^{\infty} d\mu\, \frac{\mathrm{Im}A(\mu,t)}{(\mu - 2m^2 + t/2)^{M+1}} > 0$$

Non-forward analyticity

Theorems!

Scattering amplitude $A(s,t)$

is an analytic function of s for fixed t in the range

$$0 \leq t < 4m^2$$

and has an imaginary part bounded by $Im(A(s,t)) < as^2$

which continues to imply a dispersion relation
with 2 subtractions!!

Non-forward analyticity

Scattering amplitude $A(s,t)$

is an analytic function of s for fixed t in the range $0 \leq t < 4m^2$

$$\tilde{A}(s,t) = \text{poles} + \frac{s^2}{\pi}\int_{\epsilon^2\Lambda^2}^{\infty} \frac{Im(A(\mu,t)}{\mu^2(\mu - s)} + \frac{u^2}{\pi}\int_{\epsilon^2\Lambda^2}^{\infty} \frac{Im(A(\mu,t)}{\mu^2(\mu - u)}$$

$$\frac{d^n}{dt^n}\mathrm{Im}A(s,t)\Big|_{t=0} > 0 \qquad 0 \leq t < 4m^2$$

Non-forward analyticity

Defining

$$f(s,t) = \frac{1}{2}\frac{\partial^2}{\partial s^2}(\tilde{A}(s,t) - \text{poles})$$

We derive

s between cuts

$0 \leq t < 4m^2$

$$f(s,t) > 0$$

$$\frac{\partial f(s,t)}{\partial t} + \frac{3}{2\mathcal{M}^2} f(s,t) > 0$$

$$\frac{1}{2}\frac{\partial^2 f(s,t)}{\partial t^2} + \frac{3}{2\mathcal{M}^2}\left(\frac{\partial f(s,t)}{\partial t} + \frac{3}{2\mathcal{M}^2} f(s,t)\right) > 0$$

$\mathcal{M} \sim \epsilon\Lambda$ or Λ_{th}

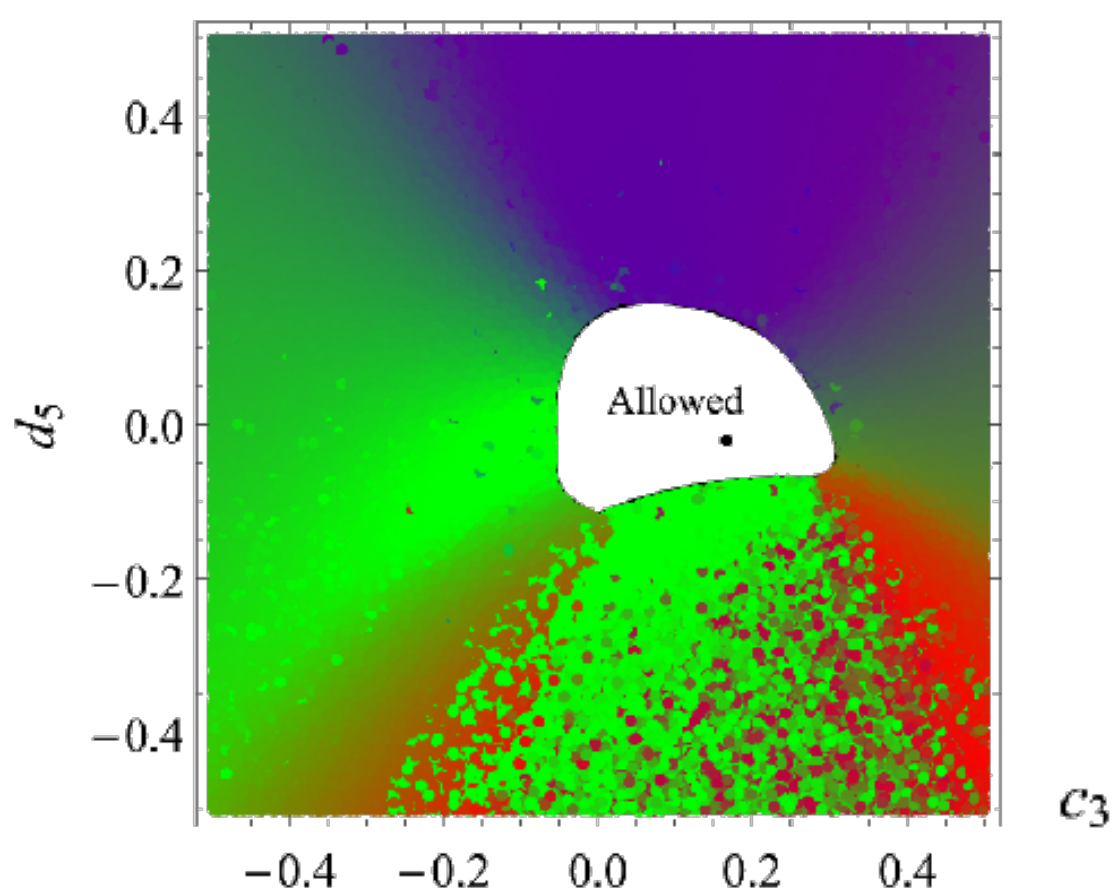
Helicity: $\frac{\mathbf{J}\cdot\mathbf{p}}{|\mathbf{p}|}|\mathbf{p},S,\lambda\rangle = \lambda|\mathbf{p},S,\lambda\rangle$

What about general spins?

In forward limit, dispersion relation holds for helicity amplitudes

$A_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,0)$ has dispersion relation with 2 subtractions

Also applies to INDEFINITE helicity



Constrains on the mass parameters in massive gravity

Eg. see Cheung & Remmen, JHEP 1604 (2016) for massive gravity

And for spin -1 Proca field, see Bonifacio, Hinterbichler & Rosen PRD94 (2016)

both in the forward scattering limit

Can we extend these results away from the forward scattering limit?

Very non-trivial because of 2 things

1. Crossing Symmetry is very complicated for general spin scattering
2. Discontinuity along left hand branch cut is **no longer positive** definite
3. Scattering amplitude for general spin have a significantly more complicated analytic structure

Crossing Symmetry for Spins

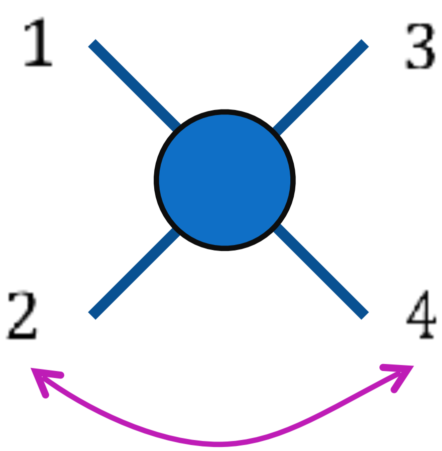
s-channel

$$A + B \to C + D$$

u-channel

$$A + \bar{D} \to C + \bar{B}$$

A definite helicity mode transforms non-trivially under crossing



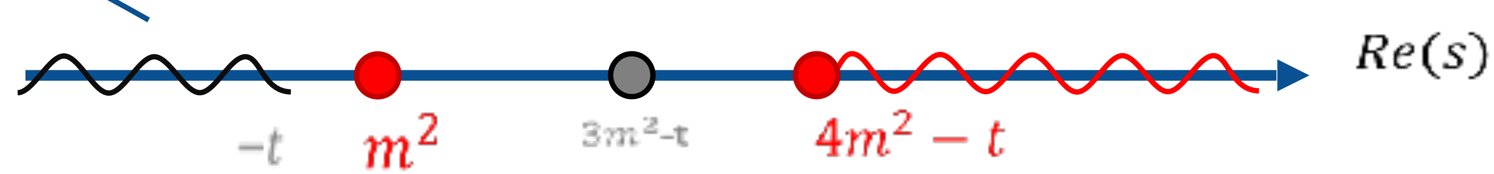
$$\mathcal{H}^{s}_{\lambda_1\lambda_2\mu_1\mu_2}(s,t) = (-1)^{\sigma} e^{i\pi(\mu_1-\lambda_1)} \cdot \sum_{\lambda_1'\lambda_2'\mu_1'\mu_2'} d^{S_1}_{\lambda_1'\lambda_1}(\pi-\chi)\, d^{S_2}_{\lambda_2'\lambda_2}(\chi)\, d^{S_1}_{\mu_1'\mu_1}(\chi-\pi)\, d^{S_2}_{\mu_2'\mu_2}(-\chi)\, \mathcal{H}^{u}_{\lambda_1\mu_2,\mu_1\lambda_2}(u,t)$$

d: Wigner matrices

$$\sin\chi = \frac{-2m\sqrt{t}}{\sqrt{(s-4m^2)(u-4m^2)}}$$

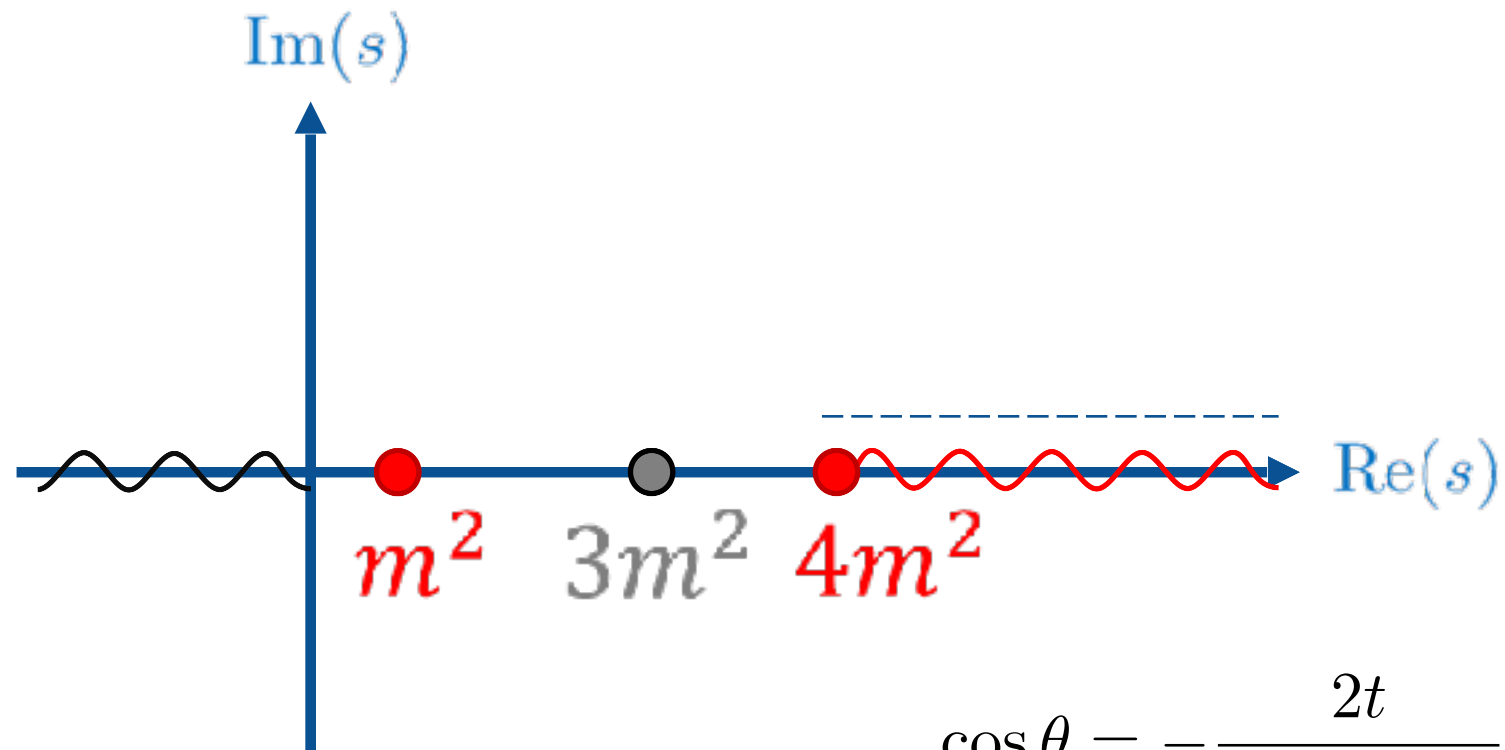
Results from change of c.o.m. frame

No obvious positivity properties in the 2nd branch cut in helicity formalism



Analyticity for Spins

In addition to usual scalar poles and branch cuts we have ....



$$\cos\theta = -\frac{2t}{(s-4m^2)}$$

1. Kinematic (unphysical) poles at $s = 4m^2$
2. $\sqrt{stu}$ branch cuts
3. For Boson-Fermion scattering $\sqrt{-su}$ branch cuts

Origin: non-analyticities of polarization vectors/spinors

Both Problems Solvable!

Problem 2 Solution

1. Kinematic (unphysical) poles at $s = 4m^2$
2. $\sqrt{stu}$ branch cuts
3. For Boson-Fermion scattering $\sqrt{-su}$ branch cuts

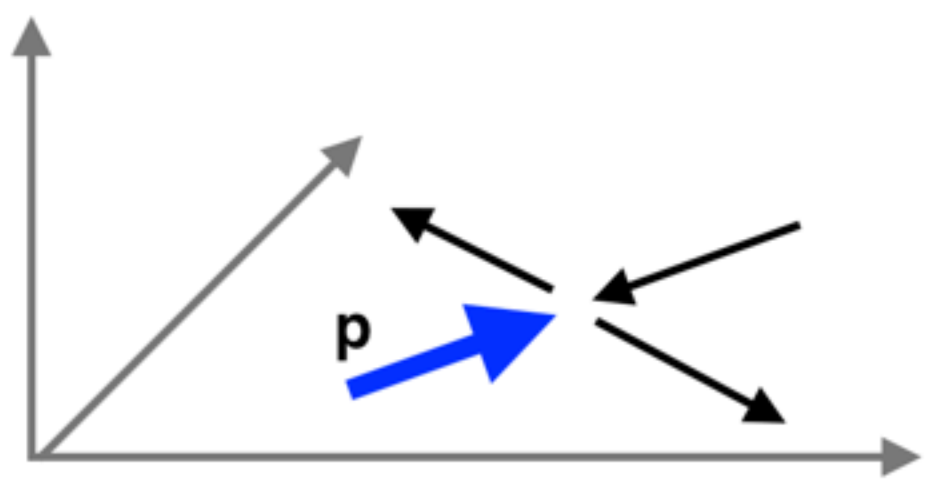
All kinematic singularities are factorizable or removable by taking special linear combinations of helicity amplitudes (known historically as regularized helicity amplitudes)

Transversity Formalism

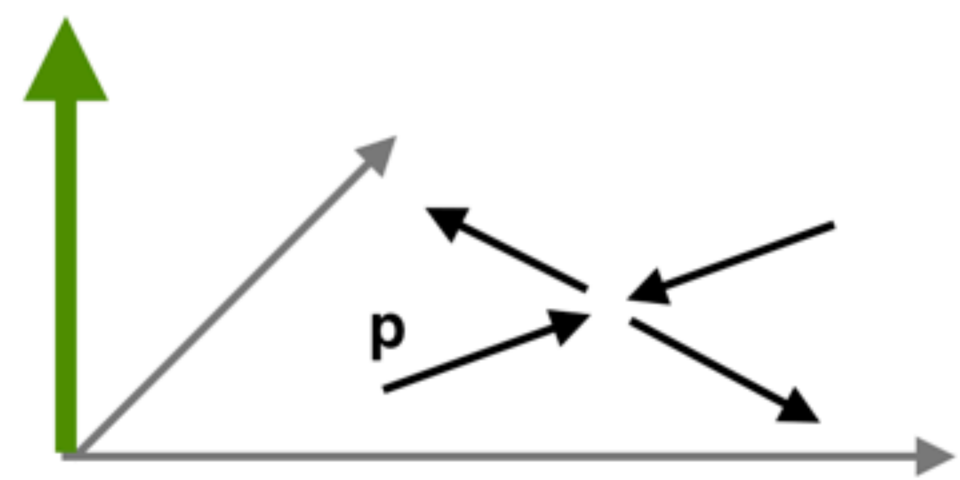
Problem 1 Solution

Change of Basis

Helicity



Transversity



$$\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4} = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} u^{S_1}_{\lambda_1\tau_1} u^{S_2}_{\lambda_2\tau_2} u^{S_1*}_{\tau_3\lambda_3} u^{S_2*}_{\tau_4\lambda_4} \mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}$$

Crossing now 1-1 between s and u channel:

$$\mathcal{H}_{\lambda_1\lambda_2\mu_1\mu_2}(s,t) = (-1)^{\sigma} e^{i\pi(\mu_1-\lambda_1)} \cdot \sum_{\lambda_1'\lambda_2'\mu_1'\mu_2'} d^{S_1}_{\lambda_1'\lambda_1}(\pi-\chi) d^{S_2}_{\lambda_2'\lambda_2}(\chi) d^{S_1}_{\mu_1'\mu_1}(\chi-\pi) d^{S_2}_{\mu_2'\mu_2}(-\chi) \mathcal{H}_{\lambda_1\mu_2,\mu_1\lambda_2}(u,t)$$

Nasty :(

$$T^{s}_{\tau_1\tau_2\tau_3\tau_4}(s,t,u) = e^{-i\sum_i \tau_i \chi} T^{u}_{-\tau_1-\tau_4-\tau_3-\tau_2}(u,t,s)$$

Nice :)

Problem 2 Solution

Transversity Formalism

Work with regularized transversity amplitudes

$$\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = \left(\sqrt{-su}\right)^{\xi}\mathcal{S}^{S_1+S_2}\left(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta)\right)$$

$$\mathcal{S} = s(s-4m^2)$$

Example:

Tree level scalar fermion scattering:

$$\mathcal{L}_{\text{int}} = \lambda\bar{\psi}_C\psi_A\phi_B\phi^{\dagger}_D$$

$$\mathcal{T}^{s}_{\tau_1 0\tau_3 0}(s,t,u) = \mathcal{T}^{\psi\phi\to\psi\phi}_{\tau_1 0\tau_3 0} = \lambda\bar{u}_{\tau_3}(\theta_3)u_{\tau_1}(\theta_1) = \frac{\lambda s}{\sqrt{-su}\sqrt{\mathcal{S}}}\left(-u + i\tau_1\frac{\sqrt{stu}}{m}\right)\delta_{\tau_1\tau_3}$$

Nasty :(

$$\mathcal{T}^{+}_{\tau_1 0\tau_3 0}(s,t,u) = -2\lambda su\,\delta_{\tau_1\tau_3}$$

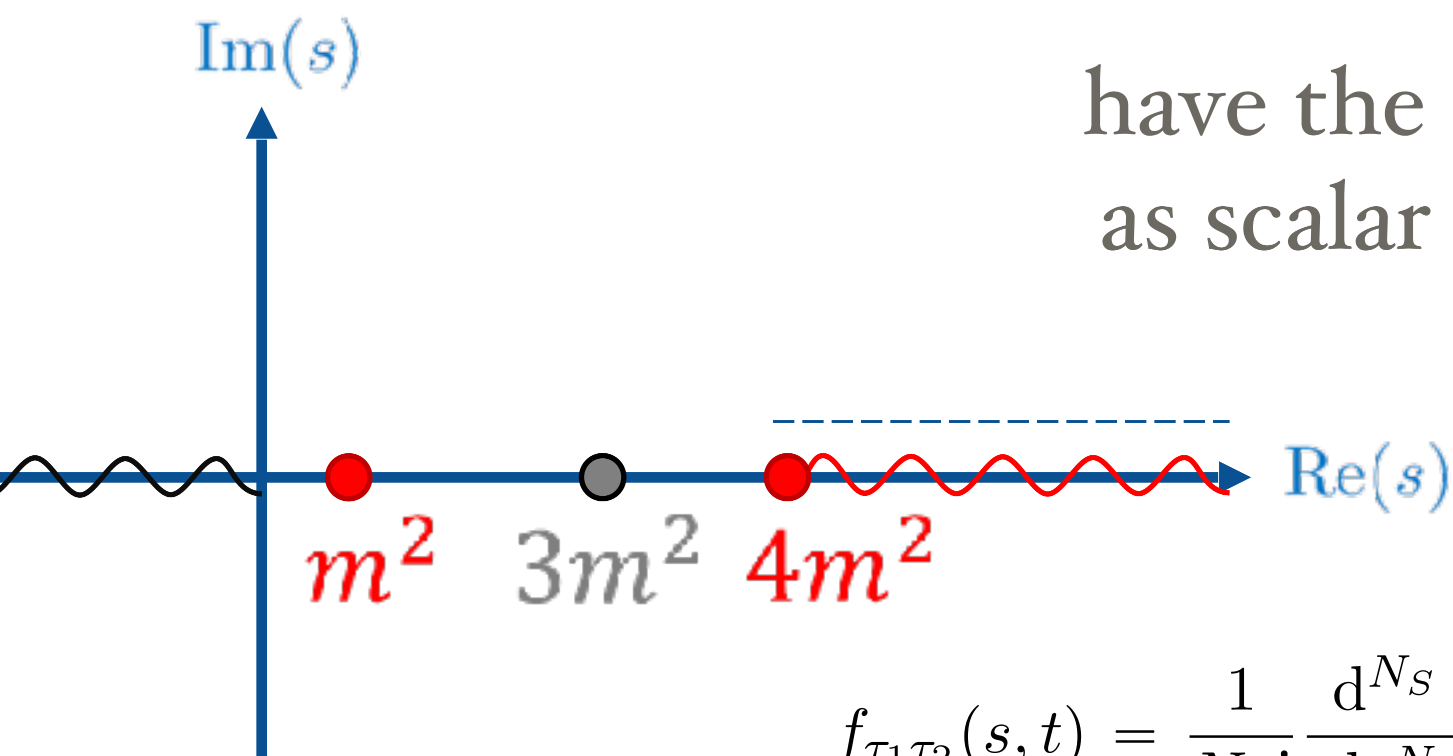
Nice :)

Analyticity

Punch line: The specific combinations:

$$\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = \left(\sqrt{-su}\right)^{\xi}\mathcal{S}^{S_1+S_2}\left(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta)\right)$$

have the same analyticity structure as scalar scattering amplitudes!!!!!!!



Implies Dispersion Relation

$$f_{\tau_1\tau_2}(s,t) = \frac{1}{N_S!}\frac{\mathrm{d}^{N_S}}{\mathrm{d}s^{N_S}}\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t)$$

$$f_{\tau_1\tau_2}(v,t) = \frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}\mu\frac{\mathrm{Abs}_s\mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-2m^2+t/2-v)^{N_S+1}} + \frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}\mu\frac{\mathrm{Abs}_u\mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-2m^2+t/2+v)^{N_S+1}}$$

General Spin Positivity Bounds

$$\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = \left(\sqrt{-su}\right)^{\xi} \mathcal{S}^{S_1+S_2} \left(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta)\right)$$

$$f_{\tau_1\tau_2}(s,t) = \frac{1}{N_S!}\frac{\mathrm{d}^{N_S}}{\mathrm{d}s^{N_S}}\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t)$$

Following Identical Steps to the Scalar Case:

$$N_S = 2 + 2(S_1+S_2) + \xi$$

$$f_{\tau_1\tau_2}(v,t) > 0,$$

$$\frac{\partial}{\partial t} f_{\tau_1\tau_2}(v,t) + \frac{N_S+1}{2\mathcal{M}^2} f_{\tau_1\tau_2}(v,t) > 0,$$

$$\frac{1}{2}\frac{\partial^2}{\partial t^2} f_{\tau_1\tau_2}(v,t) + \frac{N_S+1}{2\mathcal{M}^2}\left(\frac{\partial}{\partial t} f_{\tau_1\tau_2}(v,t) + \frac{N_S+1}{2\mathcal{M}^2} f_{\tau_1\tau_2}(v,t)\right) > 0,$$

Application Massive Spin 1

LEEFT of Heavy Higgs Mechanism

$$\mathcal{L}_{\text{Proca}} \supset -\frac{1}{4}F^{\nu}_{\mu}F^{\mu}_{\nu} - \frac{1}{2}\phi^2_{\mu} + \frac{a_0}{\Lambda^4_{\phi}}\phi^4_{\mu} + \frac{a_1}{\Lambda^3_{\phi}}\partial_{\mu}\phi_{\nu}\phi^{\mu}\phi^{\nu}$$
$$+ \frac{1}{\Lambda^6_{\phi}}\left(a_3(\phi_{\mu}\partial^{\mu}\phi_{\nu})^2 + a_4(\partial_{\mu}\phi_{\nu}\phi^{\nu})^2 + a_5\phi^2_{\mu}\partial_{\alpha}\phi_{\beta}\partial^{\beta}\phi^{\alpha}\right)$$
$$+ \frac{m^2}{\Lambda^6_{\phi}}\left(c_1 F^{\mu}_{\nu}F^{\nu}_{\rho}F^{\rho}_{\sigma}F^{\sigma}_{\mu} + c_2(F^{\nu}_{\mu}F^{\mu}_{\nu})^2\right)$$
$$+ \frac{m^2}{\Lambda^6_{\phi}}\left(C_1\phi_{\mu}\phi^{\nu}F^{\alpha\mu}F_{\alpha\nu} + C_2\phi^2_{\mu}F^2_{\alpha\beta}\right) ,$$

natural suppresion due to decoupling limit

$$\mathcal{L}_{\phi} = \Lambda^4_{\phi}\,\mathcal{F}\left[\frac{\partial}{\Lambda_{\phi}}, \frac{\phi_{\mu}}{\Lambda^2_{\phi}}\right]$$

ϕ Stueckelberg Field

$$\phi_{\mu} = D_{\mu}\phi = \partial_{\mu}\phi + mA_{\mu}$$

$$\Delta\mathcal{L}_{\phi} \supset \Lambda^4_{\phi}\left(\frac{\partial}{\Lambda_{\phi}}\right)^n\left(\frac{mA}{\Lambda^2_{\phi}}\right)^n \sim \left(\frac{m}{\Lambda_{\phi}}\right)^n \frac{F^n}{\Lambda^{2n-4}_{\phi}}$$

Application Massive Spin 1

Same action in Unitary Gauge

$$\phi_\mu = D_\mu \phi = \partial_\mu \phi + m A_\mu$$

$$\mathcal{L}^{\text{unitary}}_{\text{Proca}} \supset -\frac{1}{4} F^\nu_\mu F^\mu_\nu - \frac{1}{2} m^2 A_\mu A^\mu + \frac{m^4 a_0}{\Lambda^4_\phi} (A_\mu A^\mu)^2$$
$$+ \frac{m^4}{\Lambda^6_\phi} \Big(a_3 A_\mu A_\nu \partial^\mu A_\rho \partial^\nu A^\rho + a_4 A_\mu A_\nu \partial_\rho A^\mu \partial^\rho A^\nu + a_5 A_\mu A^\mu \partial_\alpha A_\beta \partial^\beta A^\alpha \Big)$$
$$+ \frac{m^2}{\Lambda^6_\phi} \big(c_1 F^\mu_\nu F^\nu_\rho F^\rho_\sigma F^\sigma_\mu + c_2 (F^2_{\mu\nu})^2 \big) + \frac{m^4}{\Lambda^6_\phi} \big(C_1 A_\mu A^\nu F^{\alpha\mu} F_{\alpha\nu} + C_2 F^2_{\mu\nu} A_\alpha A^\alpha \big)$$

Transversity Amplitudes

Transversity Polarizations:
$$\epsilon^{\mu}_{\tau=\pm1} = \frac{i}{\sqrt{2}m}(p, E\sin\theta \pm im\cos\theta, 0, E\cos\theta \mp im\sin\theta)$$
$$\epsilon^{\mu}_{\tau=0} = (0,0,1,0),$$

$$\tilde{s} = s - 4m^2$$

$$\mathcal{T}^{+}_{0000} = 2s^2\tilde{s}^2\left(24\frac{m^4}{\Lambda_\phi^4}a_0 - 8\frac{m^6}{\Lambda_\phi^6}(a_4 + C_1 + 2C_2) + 8\frac{m^2\left(6m^4 + x\right)}{\Lambda_\phi^6}(c_1 + 2c_2)\right),$$

$$\mathcal{T}^{+}_{-11-11} = 2s^2\tilde{s}^2\left[\frac{x - 4m^2(t - 4m^2)}{\Lambda_\phi^4}\left(a_0 - \frac{1}{2}\frac{m^2}{\Lambda_\phi^2}(a_4 - 4(c_1 + 2c_2) + C_1)\right)\right.$$
$$\left. + \frac{3}{8}\frac{y}{\Lambda_\phi^6}(a_3 + a_4 - 2a_5) - \frac{m^2 su}{\Lambda_\phi^6}\left(\frac{3}{2}a_3 - a_4 + a_5 + \frac{3}{2}C_1 + 2C_2\right)\right],$$

$$\mathcal{T}^{+}_{0101} = \frac{m^2 s^2\tilde{s}(st - 4m^2u)}{\Lambda_\phi^4}\left[4a_0 - \frac{1}{2}\frac{u}{\Lambda_\phi^2}(a_3 + C_1) + 2\frac{s-t}{\Lambda_\phi^2}c_1\right.$$
$$\left. + \frac{t}{\Lambda_\phi^2}(-a_4 + a_5) - 4\frac{2t - 4m^2}{\Lambda_\phi^2}c_2 + 2\frac{t - 4m^2}{\Lambda_\phi^2}C_2\right] + \frac{m^2 s^2\tilde{s}^3(s-u)}{2\Lambda_\phi^6}(a_3 + 4c_1 + C_1)$$

$$\mathcal{T}^{+}_{1111} = \frac{2s^2}{\Lambda_\phi^4}\left[\tilde{s}^2(t^2 + t\tilde{s} + \tilde{s}^2) + 4m^2 s(8t^2 + 8t\tilde{s} + \tilde{s}^2)\right]\left(a_0 + \frac{2m^2}{\Lambda_\phi^2}(c_1 + 2c_2 - C_2)\right)$$
$$+ \frac{s^2\tilde{s}}{4\Lambda_\phi^6}\left[\tilde{s}^2(4m^2 s - 3tu) + 16m^2 t(t + \tilde{s})(3s - 4m^2)\right](a_3 + a_4 - 2a_5),$$

Forward Limit Bounds

Indefinite Transversities:

$$\alpha_\pm \equiv \frac{1}{\sqrt{2}}\left(\alpha_{-1} \pm \alpha_{+1}\right), \qquad \beta_\pm \equiv \frac{1}{\sqrt{2}}\left(\beta_{-1} \pm \beta_{+1}\right)$$

$$\begin{aligned} f_{\alpha\beta}\Big|_{t=0} = &\frac{8}{\Lambda_\phi^4}\left(a_0 - \frac{1}{2}\frac{m^2}{\Lambda_\phi^2}\left(a_4 + C_1\right)\right)|\alpha_+|^2|\beta_+|^2 \\ &+ \frac{4m^2}{\Lambda_\phi^6}\left(a_3 - 2a_4 + 2a_5 + C_1 + 4C_2\right)\left(\mathrm{Re}[\alpha_0^*\alpha_+]\mathrm{Re}[\beta_0^*\beta_+] - \mathrm{Re}[\alpha_-^*\alpha_+]\mathrm{Re}[\beta_-^*\beta_+]\right) \\ &+ \frac{2m^2}{\Lambda_\phi^6}\left(a_3 + C_1\right)\left(|\alpha_+|^2|\beta|^2 + |\alpha|^2|\beta_+|^2\right) \\ &+ \frac{8m^2}{\Lambda_\phi^6}c_1\left(|\alpha_0|^2 + |\alpha_-|^2\right)\left(|\beta_0|^2 + |\beta_-|^2\right) \\ &+ \frac{8m^2}{\Lambda_\phi^6}\left(c_1 + 4c_2\right)\left(|\alpha_0\beta_0 - \alpha_-\beta_-|^2 - 2\mathrm{Im}[\alpha_0^*\alpha_-]\mathrm{Im}[\beta_0^*\beta_-]\right). \end{aligned} \qquad (3.14$$

Positivity Bounds:

$$a_0 > 0\,, \qquad c_1 > 0\,, \qquad c_1 + 2c_2 > 0\,, \quad \text{and} \quad a_3 + C_1 > 0$$

Beyond Forward Limit

$$\frac{\partial}{\partial t} f_{\alpha\beta}\Big|_{t=0} = \frac{3}{4}\frac{a_3 + a_4 - 2a_5}{\Lambda_\phi^6}\,|\alpha_+|^2|\beta_+|^2$$

$$\frac{\partial}{\partial t} f_{\tau_1\tau_2} + \frac{N_S + 1}{2\mathcal{M}^2} f_{\tau_1\tau_2} > 0$$

Positivity Bounds:

$$3(a_3 + a_4 - 2a_5) + 112 a_0 \gtrsim 0$$

Application Massive Spin 2

Unitary Gauge Massive Gravity

Einstein-Hilbert — Mass Term

$$\mathcal{L} \supset \frac{M_{\rm Pl}^2}{2}\left(R\,[g] - \frac{m^2}{4}V(g,h)\right)$$

Parameterize mass term as

$$V(g,h) \supset [h^2] - [h]^2 + (c_1-2)[h^3] + (c_2+\frac{5}{2})[h^2][h] + (d_1+3-3c_1)[h^4] + (d_3-\frac{5}{4}-c_2)[h^2]^2 + \dots \,.$$

where $[h] = \eta^{\mu\nu}h_{\mu\nu},\ [h^2] = \eta^{\mu\nu}h_{\mu\alpha}\eta^{\alpha\beta}h_{\beta\nu};$

$$d_3 = -d_1/2 + 3/32 + \Delta d, \quad c_2 = -3c_1/2 + 1/4 + \Delta c$$

Application Massive Spin 2

Forward Limit

$$2M_{\rm Pl}^2 m^6 \frac{\partial^2}{\partial v^2} f_{\alpha\beta}|_{t=0} = \frac{352}{9}|\alpha_S\beta_S|^2\left(\Delta c\left(-6+9c_1-4\Delta c\right)-6\Delta d\right) + \frac{176}{3}\alpha_S^*\beta_S^*(\alpha_{V_1}\beta_{V_1}-\alpha_{V_2}\beta_{V_2})\,\Delta c\left(3-3c_1+4\Delta c\right)$$

Positivity for general helicity implies: $\Delta c = 0$

Beyond forward

$$\frac{\partial}{\partial t} f_{\tau_1\tau_2}(v,t) \;\propto\; \frac{v}{\Lambda_5^{10}}\Delta d + \mathcal{O}\left(\frac{m^2}{\Lambda_5^{10}}\right) \;>\; 0$$

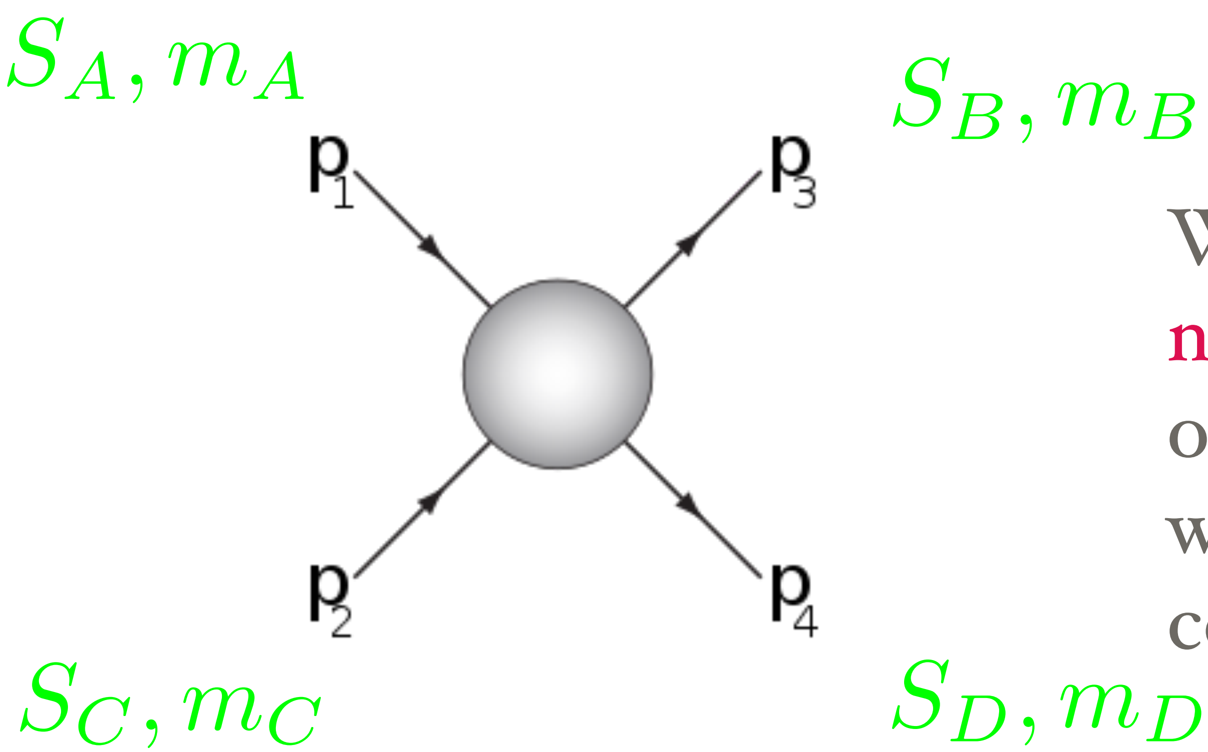
$$\Delta d = 0$$

These are precisely the tunings that raise the cutoff from

$$\Lambda_5 = (m^4 M_{\rm Planck})^{1/5} \qquad \Lambda_3 = (m^2 M_{\rm Planck})^{1/3}$$

Summary

For the 2-2 scattering amplitude for four particles
of different masses and spins
(bosons AND fermions)



We have been able to derive an infinite number of conditions on s and t derivatives of **transversity scattering amplitude** which impose positive properties on combinations of coefficients in the EFT

Largest set of conditions we know that determine whether a given EFT admits a local UV completion

In the case of Massive Spin 2, positivity bounds impose special tunings that raise the cutoff to $\Lambda_3 = (m^2 M_{\rm Planck})^{1/3}$