Supplemental Assignments for Collider Phenomenology

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Lecture 1: Introduction and Basic Formalism

`Exercise 1.1:` A B-factor (e. g. KEKB) is designed for asymmetric head-on collisions between a positron beam of energy 3.5 GeV and an electron beam of energy 8 GeV. Find the center-of-mass energy for the B-factory. Do you understand why to adopt this design for the energy and for the asymmetry?

`Exercise 1.2:` The dominant decay channel of the top quark is $t \to W^+ b$. The partial decay width given in terms of the known mass parameters at the leading order is

$$\Gamma_t = \frac{G_F m_t^3}{8\pi\sqrt{2}}(1 - \frac{m_W^2}{m_t^2})^2(1 + 2\frac{m_W^2}{m_t^2}).$$

Assuming this formula gives its total decay width, estimate the top-quark life-time in units of yocto-second.

If the QCD scale is $\Lambda_{\rm QCD} \approx 200$ MeV, compare the top-quark life-time with the time scale at which the QCD strong interaction sets in.

Also compare with the b-quark life-time, and try to understand the differences between the decays of the two quarks.

(Use the PDG review for the parameters needed.)

`Exercise 1.3:` (**challenging problem**) In the "Standard Model" of elementary particle physics, the amplitude for the scattering of the (longitudinally polarized) weak gauge bosons (the force mediator for the nuclear β decay) $W^+W^+ \to W^+W^+$ is calculated at high energies to be

$$f(k,\ \theta) = \frac{1}{16\pi k}\left(\frac{-M_H^2}{v^2}\right)\left(\frac{t}{t - M_H^2} + \frac{u}{u - M_H^2}\right)$$

where k is the W^+ momentum in the Center-of-Momentum frame, M_H is the mass of the Higgs boson, and $v \approx 250$ GeV is the Higgs vacuum expectation value. The angular-dependent kinematical variables are

$$t = -2k^2(1 - \cos\theta) \quad \text{and} \quad u = -2k^2(1 + \cos\theta).$$

Note that the amplitude is give in the "natural units" where $c = \hbar = 1$, and everything is expressed in terms of the energy units electron-volts: 1 GeV $= 10^9$ eV.

(a). Take the high-energy limit $2k \gg M_H$, compute the partial wave amplitude a_ℓ. Note that for final state identical particles W^+W^+, the angular integration should be $1/2 \int_{-1}^{1} d\cos\theta$.

(b). Impose the partial wave unitarity condition on a_ℓ for s-wave, determine the bound on the mass of the Higgs boson M_H (in units of GeV).

(c). If the Higgs boson did not exist in Nature, then the amplitude for the weak gauge boson scattering for $W^+W^+ \to W^+W^+$ would be expressed by taking the limit $2k \ll M_H \to \infty$. Using the same procedure above, determine at what energy scale $2k$ the Standard Model theory would break down to violate the partial wave unitarity.

(Remark: The "Large Hadron Collider" (LHC) at CERN, Geneva, provides proton-proton collisions at a c.m. energy of 13,000 GeV, which was designed based on the above physics argument. Consequently, we have witnessed the historical discovery of the Higgs boson!)

`Exercise 1.4:` A 125 GeV Higgs boson will have a production cross section of 20 pb at the LHC. How many events per year do you expect to produce for the Higgs boson with a designed LHC luminosity $10^{33}/\text{cm}^2/\text{s}$? With the expected events, why is the Higgs boson so difficult to observe?

Lecture 2: Relativistic Kinematics and Phase Space, Collider Detectors

`Exercise 2.1:` Show that the phase space element $d\vec{p}/2p^0$ is Lorentz invariant.

`Exercise 2.2:` (**challenging problem**) A particle of mass M decays to two particles isotropically in its rest frame. What does the momentum distribution look like in a frame in which the particle is moving with a speed β_z? Compare the result with your expectation for the shape change for a basket ball.

`Exercise 2.3:` Consider a $2 \to 2$ scattering process $p_a + p_b \to p_1 + p_2$. Assume that $m_a = m_1$ and $m_b = m_2$. Show that

$$\begin{aligned} t &= -2p_{cm}^2(1-\cos\theta_{a1}^*), \\ u &= -2p_{cm}^2(1+\cos\theta_{a1}^*) + \frac{(m_1^2-m_2^2)^2}{s}, \end{aligned}$$

$p_{cm} = \lambda^{1/2}(s, m_1^2, m_2^2)/2\sqrt{s}$ is the momentum magnitude in the c.m. frame.
Note: t is negative definite; $t \to 0$ in the collinear limit, that could be singular for massless-exchange. Comment on the u-channel.

`Exercise 2.4:` (**challenging problem**) A particle of mass M decays to three particles $M \to abc$. Show that the phase space element can be expressed as

$$dPS_3 = \frac{1}{2^7\pi^3}\, M^2 dx_a dx_b.$$
$$x_i = \frac{2E_i}{M},\ (i = a, b, c,\ \sum_i x_i = 2).$$

where the integration limits for $m_a = m_b = m_c = 0$ are

$$0 \le x_a \le 1, \quad 1 - x_a \le x_b \le 1.$$

Note: For the decay in the M-rest frame, three of the four angular variables can be trivially integrated out (ignoring the spins of the particles).

`Exercise 2.5:` For a π^0, μ^-, or a τ^- respectively, calculate its decay length if the particle has an energy $E = 10$ GeV.

Lecture 3: Lepton Colliders

`Exercise 3.1:` For a resonant production $e^+e^- \to V^*$ with a mass M_V and total width Γ_V, derive the Breit-Wigner formula (**If you find it too challenging for the calculation, you may skip this part and move on to the next line.**)

$$\sigma(e^+e^- \to V^* \to X) = \frac{4\pi(2j+1)\Gamma(V \to e^+e^-)\Gamma(V \to X)}{(s - M_V^2)^2 + \Gamma_V^2 M_V^2}\ \frac{s}{M_V^2},$$

Consider a beam energy spread Δ in Gaussian distribution

$$\frac{dL}{d\sqrt{\hat{s}}} = \frac{1}{\sqrt{2\pi}\ \Delta}\exp[\frac{-(\sqrt{\hat{s}} - \sqrt{s})^2}{2\Delta^2}],$$

obtain the appropriate cross section formulas for (a) $\Delta \ll \Gamma_V$ (resonance line-shape) and (b) $\Delta \gg \Gamma_V$ (narrow-width approximation).

`Exercise 3.2:` An event was identified to have a μ^+ and a μ^- along with some missing energy. What can you say about the kinematics of the system of the missing particles? Consider for both an e^+e^- and a hadron collider.

`Exercise 3.3` **(challenging problem)**: Derive the Weizsäcker-Williams spectrum for a photon with an energy xE off an electron with an energy E

$$P_{\gamma/e}(x) \approx \frac{\alpha}{2\pi}\frac{1 + (1-x)^2}{x}\ln\frac{E^2}{m_e^2}.$$

Note that this procedure is the direct analog to deriving the DGLAP $q \to q'g$ splitting in QCD.

Lecture 4: Hadron Colliders

`Exercise 4.1:` For a four-momentum $p \equiv p^\mu = (E, \vec{p})$, define

$$\begin{aligned} E_T &= \sqrt{p_T^2 + m^2}, \quad p_T^2 = p_x^2 + p_y^2, \quad y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z}, \\ \text{then show } p^\mu &= (E_T \cosh y,\ p_T \cos\phi,\ p_T \sin\phi,\ E_T \sinh y), \\ \text{and, } \frac{d^3\vec{p}}{E} &= p_T dp_T d\phi\ dy = E_T dE_T d\phi\ dy. \end{aligned}$$

Due to the random boost between the Lab-frame (O) and the c.m. frame (O') for every event,

$$y' = \frac{1}{2}\ln\frac{E'+p'_z}{E'-p'_z} = \frac{1}{2}\ln\frac{(1-\beta_{cm})(E+p_z)}{(1+\beta_{cm})(E-p_z)} = y - y_{cm},$$

where β_{cm} and y_{cm} are the speed and rapidity of the c.m. frame w.r.t. the lab frame.

In the massless limit, the rapidity y defines the pseudo-rapidity:

$$y \to \eta = \frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} = \ln\cot\frac{\theta}{2}.$$

`Exercise 4.2:` For a two-body massless final state with an invariant mass squared s, show that

$$\frac{d\hat{\sigma}}{dp_T} = \frac{4p_T}{s\sqrt{1-4p_T^2/s}}\ \frac{d\hat{\sigma}}{d\cos\theta^*}.$$

where $p_T = p\sin\theta^*$ is the transverse momentum and θ^* is the polar angle in the c.m. frame. Comment on the apparent singularity at $p_T^2 = s/4$.

`Exercise 4.3:` What would be needed to construct a CP-odd observable in the hadron collider environment, such as the LHC? Try to construct one (anyone and for any process of your choice).

Reference: *Genuine CP-odd Observables at the LHC*, by Tao Han and Yingchuan Li: `Phys.Lett. B683 (2010) 278-281`, e-Print: arXiv:0911.2933.