Modern Effective Field Theories

In the examples discussed so far the EFT was a regular relativistic field theory, except that we needed to include higher-dim. operators. It was also trivial to see what the relevant degrees of freedom were: these were simply the light particles, while the heavy ones were 'integrated out'.

There are many QFT problems with scale hierarchies which are more complicated, e.g.

non-relativistic systems

$$\lambda = \frac{|\vec{p}_e|}{m_e} \ll 1$$

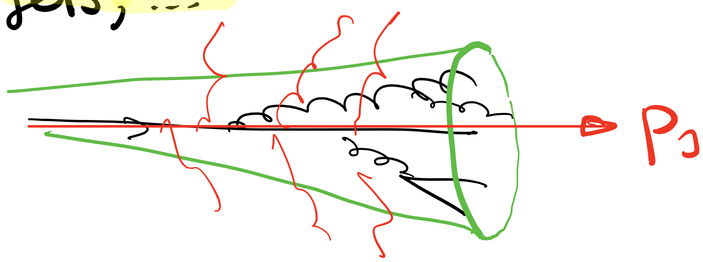
but $E \sim m_e$

EFT: NRQED ← non ← relativistic

= QM + higher-dim operators

+ Euler Heisenberg

jets, ...



EFT : SCET

$\leftarrow$ soft $\leftarrow$ coll

$$\lambda = \frac{|\vec{P}_\perp|}{E_J} \quad \text{or} \quad \lambda^2 = \frac{P_J^2}{E_J^2}$$

Complications:

- Different momentum components scale differently

  $\rightsquigarrow$ Derivative expansion is more complicated

- cannot simply integrate out particles

  $\rightsquigarrow$ Split field into "modes", corresponding to different momentum regions

  $$\text{e.g.} \quad \phi = \underbrace{\phi_h}_{\text{integrate out}} + \phi_c + \phi_s$$

  $\rightsquigarrow$ Need sometimes several fields to describe single particle

- Often need reference vectors

  $\rightsquigarrow$ $v^\mu \sim (1,0,0,0)$ in NRQED

  $\rightsquigarrow$ $n^\mu \sim (1,0,0,1)$ along jet direction in SCET

  EFTs are taylored to the problem at hand...

- Encounter non-localities associated with directions of large momentum flow.

In this lecture, we will use EFT methods to analyze soft photons in e^+e^- scattering. This will allow us to

a.) deal with the aforementioned complications in a simple setting

b.) derive a classical QED factorization theorem by Yennie, Frautschi & Suura '61 using EFT methods.

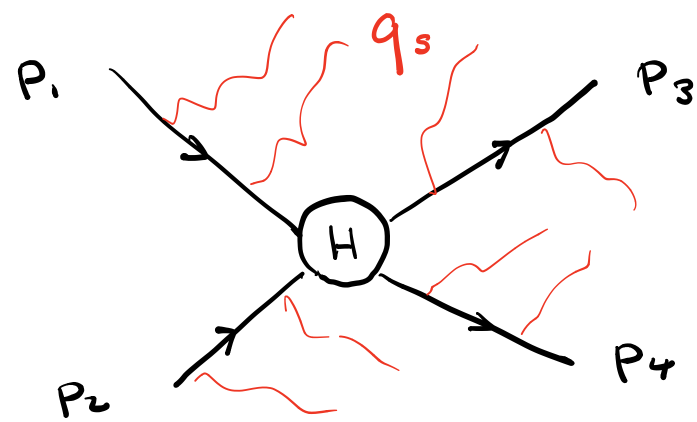
c.) to discuss the "method of regions", an important tool to expand loop integrals and identify momentum modes in an EFT.

The relevant EFT could be called SET (Soft Effective Theory) and is a simpler version of SCET (Soft-Collinear Effective Theory).

We will cover the second chapter of my lecture notes 1803.04310. The reader interested also in the collinear part of SCET and its applications can read the remaining chapters, or read the book "Introduction to SCET" LNP 896 (2015) [1410.1892] (with A. Broggio & A. Ferroglia).

Soft Effective Theory

As was nicely illustrated by Holmfridur in her talk on Wednesday, one needs to include soft photons to get finite results when considering $e^- e^-$-scattering in QED



$E_s \ll m_e$

$\lambda = E_s / m_e$

How much soft radiation is included depends on the definition of the observable, but given finite detector resolution, one cannot avoid having <u>some</u> radiation.

We discussed the Lagrangian for soft photons in detail in lecture 2. It is the Euler-Heisenberg theory

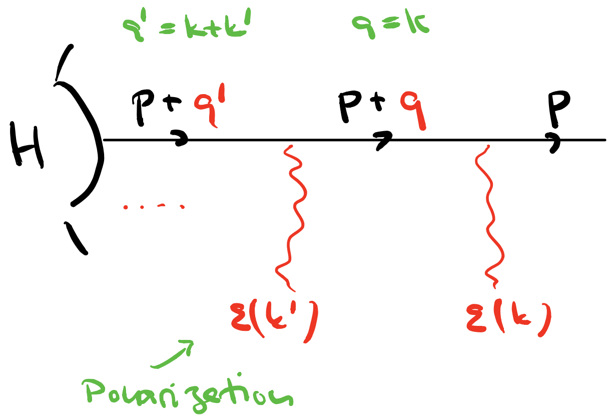
$$\mathcal{L}_\gamma^{eff} = \mathcal{L}_\gamma^{(4)} + \cancel{\frac{1}{m_e^2} \mathcal{L}_\gamma^{(6)}} + \cdots$$

irrelevant for today, only include leading power.

$$\mathcal{L}_\gamma^{(4)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} .$$

This by itself is however not sufficient, since we also need to account for the e^- which radiate the photons. The energy of the radiation is too small to produce e^+e^- pairs, but the electrons which are present remain due to fermion number conservation. So we need a field for electrons but not positrons.

To understand what is needed, let's consider a single fermion line emitting soft photons



Set $p^\mu = m_e v^\mu$ with $v^2 = 1$. Now expand the intermediate propagator in the soft momentum q^μ:

$$\Delta(p+q) = i \frac{\not{p} + \not{q} + m_e}{(p+q)^2 - m_e^2 + i0} = i \frac{\not{p} + m_e}{2p \cdot q + i0}$$

$$= i \underbrace{\frac{\not{v}+1}{2}}_{P_v} \frac{1}{v \cdot q + i\varepsilon}$$

Note that (exercise)

$$\not{v} P_v = P_v \quad ; \quad P_v^2 = P_v \quad ; \quad P_v \not{\varepsilon} P_v = P_v \, \varepsilon \cdot v$$

Expanding the propagators and using these properties, we find that the above diagram simplifies to

$$\bar{u}(p)\, P_v \frac{i}{v\cdot q} (-ie\, \varepsilon \cdot v)\, P_v \frac{i}{v\cdot q'} (-ie\, \varepsilon' \cdot v) \cdot \ldots$$

This form of the soft amplitude is called the eikonal approximation.

Can we obtain the expanded amplitudes directly from an effective Lagrangian? (We already know the Feynman rules!) Consider

$$\mathcal{L}_{eff} = \bar{h}_v\, i v\cdot D\, h_v$$

where h_v is an auxiliary fermion field which fulfils $P_v h_v = h_v$. (Can use $h_v = P_+ \psi$.)

The propagator for h_v is $\dfrac{i}{v\cdot q + i\varepsilon}$

The gluon emission vertex is $-iev^\mu$

To account for the fermions along the four directions $p_i^\mu = m_e v_i^\mu$ we need four auxiliary fields

$$\rightarrow \mathcal{L}_{eff} = \sum_{i=1}^{4} \bar{h}_{v_i} i v_i \cdot D h_{v_i} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Delta \mathcal{L}_{int}$$

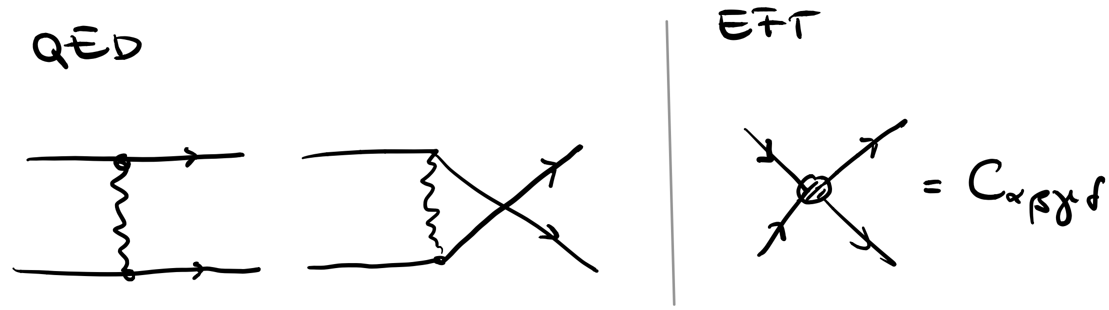
So we have split the e^--field into four fields, which describe an e^- along v_i^μ with momentum $m v_i^\mu + q^\mu$. The final ingredients are interactions, which take the form

$\propto \alpha/m_e^2$ Dirac index

$$\Delta \mathcal{L}_{int} = C_{\alpha\beta\gamma\delta}(v_1, v_2, v_3, v_4, m_e)\, h_{v_1}^\alpha\, h_{v_2}^\beta\, \bar{h}_{v_3}^\gamma\, h_{v_4}^\delta$$

at leading power in λ. (Interactions with two fields are forbidden: an e^- cannot change velocity when emitting soft radiation)

To determine the Wilson coefficients we do on-shell matching and compute $e^- e^- \to e^- e^-$ w/o soft radiation



The Wilson coefficient is simply the $e^- e^-$ amplitude w/o external spinors! This works also at loop level: then both QED and the EFT have IR div's which cancel. Since the EFT diagrams vanish as $\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}$, the IR divergences in QED are in one-to-one correspondence to UV divergences of the EFT!

$\leadsto$ IR divergences can be discussed as UV divergences in an EFT and can be renormalized.

Now introduce the Wilson line

$$S_i(x) = \exp\left[-ie\int_{-\infty}^{0} ds\, v_i \cdot A(x+sv_i)\right]$$

which fulfils

$$v \cdot D_i\, S_i(x) = 0$$

and redefine

$$h_{v_i}(x) = S_i(x)\, h_{v_i}^{(0)}(x)$$

The fermion Lagrangian takes the form

$$\bar{h}_{v_i}\, i v_i \cdot D\, h_{v_i} = \dots = \bar{h}_{v_i}^{(0)}\, i v \cdot \partial\, h_{v_i}^{(0)} .$$

The fermion no longer interacts with the soft photons! Instead one finds Wilson lines in $\mathcal{L}_{int}$:

$$\mathcal{L}_{int} = C_{\alpha\beta\gamma\delta}\, h_{v_1}^{(0)\alpha}\, h_{v_2}^{(0)\beta}\, \bar{h}_{v_3}^{(0)\gamma}\, h_{v_4}^{(0)\delta}$$

$$\cdot\, S_{v_1} S_{v_2} S_{v_3}^{\dagger} S_{v_4}^{\dagger}$$

state with n soft photons

Now lets compute $\mathcal{M}(e^+e^- \to e^+e^- + X_s)$.

Since there are no intractions, the amplitude factorizes

$$\mathcal{M} = \underbrace{u_{v_1}^{\alpha}\, u_{v_2}^{\beta}\, \bar{u}_{v_3}^{\gamma}\, \bar{u}_{v_4}^{\delta}\, C_{\alpha\beta\gamma\delta}}^{\mathcal{M}(e^-e^- \to e^-e^-)} \cdot$$

$$* \langle X_s | S_{v_1} S_{v_2} S_{v_3}^{\dagger} S_{v_4}^{\dagger} | 0 \rangle$$

Squaring the amplitude then gives a factorized cross section:

$$\boxed{\sigma = H(m_e, \{\underline{v}\}) \cdot \mathcal{S}(E_s, \{\underline{v}\})}$$

where

$$\mathcal{S} = \sum_{X_S} |\langle X_S | S_3^\dagger S_1 S_4^\dagger S_2 | 0 \rangle|^2$$
$$\Theta (E_S - E_{X_S})$$

The soft function is the low-energy matrix element, while $H = \sigma(e^- e^- \to e^- e^-)$ is the bare Wilson coefficient. We can renormalize to obtain

$$\sigma = H(m_e, \{v\}, \mu) \, \mathcal{S}(E_s, \{v\}, \mu)$$

The soft function has a very interesting property: it exponentiates

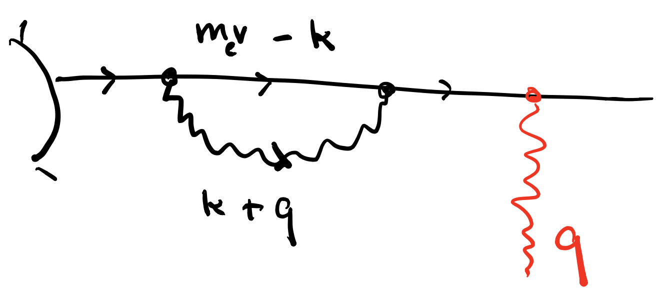
$$S(E_v, \{\underline{v}\}) = \exp\left[\frac{\alpha}{4\pi} \mathcal{S}^{(1)} \right]$$

Scale independence of σ then implies that also the $\ln(\frac{\mu}{m_e})$ terms in H must exponentiate.

These are tied to IR divergences in on-shell amplitudes. We have thus demonstrated that the IR divergences exponentiate.

This of course assumes that our construction, which was based on expanding tree-level diagrams is also valid at the loop level. To show this, we should now discuss the "method of regions"

To discuss the expansion, let's consider the simplest example loop diagram



The associated scalar integral takes the form

$$F = \int d^d k \, \frac{1}{(k+q)^2} \, \frac{1}{(m_e v - k)^2 - m_e^2}$$

In the low-E theory we assume that $k_\mu \sim q_\mu \ll m_e$.

Expanding the integrand yields

$$F_{low} = \int d^d k \, \frac{1}{(k+q)^2} \, \frac{1}{-2 m_e v \cdot k} \left\{ 1 + \frac{k^2}{2 m_e v \cdot k} + \cdots \right\}$$

The expansion yields exactly the $i/v \cdot k$ propagators we encountered at tree level. At large $k^\mu \sim m_e$ the expansion is no longer justified and we encounter UV divergences which are stronger than in the additional integral. To correct for this consider

$$F_{high} = F - F_{low}$$

$$= \int d^dk \, \frac{1}{(k+q)^2} \left\{ \frac{1}{(m_e v - k)^2 - m_e^2} - \frac{1}{-2m_e v\cdot k}\left[1 + \frac{k^2}{2m v\cdot k} + \ldots\right] \right\}$$

By construction, this difference in the integrand only has support for $k^\mu \gg q^\mu$ since the bracket $\{\ldots\}$ vanishes for $k \to 0$. We can therefore expand the integrand around $q^\mu \to 0$. This yields

$$F_{high} = \int d^dk \, \frac{1}{k^2}\left[1 - \frac{2q\cdot k}{k^2} + \ldots\right]\{\ldots\}$$

Next we use that integrals of the form

$$\int d^dk \, (k^2)^\alpha (v\cdot k)^\beta (q\cdot k)^\gamma = 0$$

all vanish because they are scaleless. This leaves

$$F_{high} = \int d^dk \, \frac{1}{k^2}\left[1 - \frac{2q\cdot k}{k^2} + \ldots\right]\frac{1}{(m_e v - k)^2 - m_e^2}.$$

Note that this is simply the expansion of the integrand for $k^\mu \sim m_e \gg q^\mu$.

The upshot is that we recover the full integral by expanding twice. Once for

(i) $k^\mu \sim q^\mu \ll m_e$ "soft region"

$\rightsquigarrow F_{low}$

(ii) $k^\mu \sim m_e \gg q^\mu$ "hard region"

$\rightsquigarrow F_{high}$

The contributions (i) correspond loop integrals in the EFT (so it is OK to expand also the loop momenta!), while the contributions (ii) contribute to the matching (to get them, one can expand the full theory integrals in the small external momenta; as advertised earlier in the lecture.)

This method to obtain the expansion of an integral by expanding in different regions and integrating is very general. Sometimes, one encounters several low energy regions, e.g. "soft" + "collinear" in jet processes. One then introduces a field for each momentum region and constructs a Lagrangian which incorporates the scalings of the momenta in each case. The references given at the beginning discuss how this is done in detail.