Fermi Theory & RG improved PT

Consider decay $\overset{b\bar{d}}{\bar{B}_d^{(0)}} \longrightarrow \overset{c\bar{d}}{D^+} \; \overset{\bar{u}d}{\pi^-}$

mediated by $\quad b \longrightarrow c \, d \, \bar{u}$

on the quark level. Scales: $M_W, m_b, m_c, \Lambda_{QCD}$

Lowest-dim. EFT operators take the form

$$\left.\begin{array}{ll} \bar{d}\,\Gamma\, u & \bar{c}\,\Gamma\, b \\ \bar{d}\,\tilde{\Gamma}\, b & \bar{c}\,\tilde{\Gamma}\, u \end{array}\right\} \text{Fierz}$$

$\Gamma, \tilde{\Gamma} = 1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}$

Using that the W only couples to left-handed fields we only need

color

$$O_1 = \bar{d}_L^i \gamma^\mu u_L^i \; \bar{c}_L^j \gamma_\mu b_L^j$$

$$O_2 = \bar{d}_L^i \gamma^\mu u_L^j \; \bar{c}_L^j \gamma_\mu b_L^i$$

Note: $(t^a)_{ij} \otimes (t^a)_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta_{kl}$

"Color-Fierz"

$$\mathcal{L}_{eff} = \overset{\swarrow \mathcal{L}^{(4)}}{\mathcal{L}_{QCD}} - \underbrace{\frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cb}}_{\hat{G}} \overset{\swarrow \mathcal{L}^{(6)}}{\left[C_1 O_1 + C_2 O_2\right]}$$

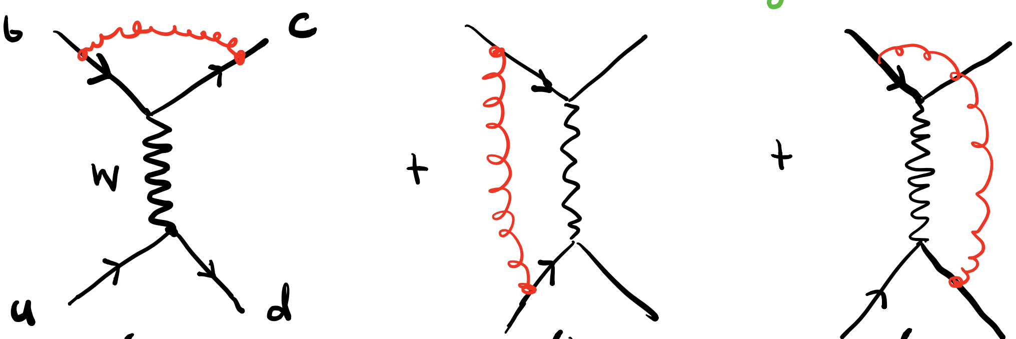
$\frac{G_F}{\sqrt{2}} = g_2^2/8M_W^2$

← we will not include higher-order weak interactions; so this prefactor will remain the same

Tree level: $C_1 = 1$; $C_2 = 0$

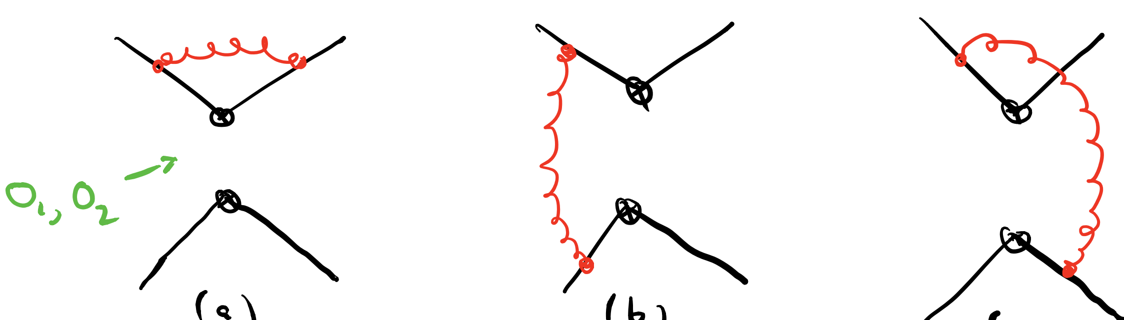
One-loop matching:

SM (full theory) (we will use amputated Green's functions for the matching.)



(a) (b) (c)

EFT



(a) (b) (c)

+ "mirrored" diagrams

Matching: adjust C_1, C_2 so that EFT reproduces full result, expanded in $1/M_W^2$

Since C_1, C_2 are independent of external momenta and masses, we could set $p_i = 0$, $m_i = 0$.

$\rightsquigarrow$ All EFT loop diagrams become scaleless & vanish.

$$\int d^dk \, \frac{1}{k^2} \frac{1}{\not k} \Gamma \frac{1}{\not k} \Gamma = 0 = \text{"} \frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}} \text{"}$$

also present in full theory drop out in matching.

This is very efficient, but we will set $p_i^\mu = p^\mu$, with $p^2 \neq 0$.

Result for amputated Green's function

$$\Gamma_{full} = \hat{G} \Big\{ \Big[1 + 2 C_F \frac{\alpha_s}{4\pi} \Big(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{p^2} \Big) \Big] \langle O_1 \rangle_{tree} \quad (a)$$
$$+ \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{-p^2} \Big[\frac{3}{N_c} \langle O_1 \rangle_{tree} - 3 \langle O_2 \rangle_{tree} \Big] \Big\} \quad (b)+(c)$$

Color factor $C_F = \frac{N_c^2 - 1}{2N_c}$, $N_c = 3$, $\alpha_s = \frac{g^2}{4\pi}$

$$\Gamma_{eff} = \hat{G} \Big\{ C_1^{bare} \Big[\Big(1 + 2C_F \frac{\alpha_s}{4\pi} \Big(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{p^2} \Big) \Big) \langle O_1 \rangle_{tree}$$

$$+ \frac{\alpha_s}{4\pi} \Big(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \Big) \Big(\frac{3}{N_c} \langle O_1 \rangle_{tree} - 3 \langle O_2 \rangle_{tree} \Big) \Big]$$

$$+ \text{"} 1 \leftrightarrow 2 \text{"} \Big\}$$

Wave-function renormalization $\psi_q^{(0)} = Z_q^{1/2} \psi$

$Z_q = 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon}$. Cures divergences from diagram (a), but (b)+(c) are still divergent in EFT. Write

$$\mathcal{L}_{eff} = - \hat{G}\, C_i^{bare} O_i(\psi_q^{(0)})$$

$$= - \hat{G}\, C_i Z_{ij} O_j(\psi_q) \cdot Z_q^2$$

Expand $Z_{ij} = \delta_{ij} + \frac{\alpha_s}{4\pi} Z_{ij}^{(1)} + O(\alpha_s^2)$

$$\rightsquigarrow \quad Z_{ij}^{(1)} = \frac{1}{\varepsilon} \begin{pmatrix} -3/N_c & 3 \\ 3 & -3/N_c \end{pmatrix}$$

$$C_i = C_i^{(0)} + \frac{\alpha_s}{4\pi} C_i^{(1)}$$

From the condition $\Gamma_{full} - \Gamma_{eff} = 0$

one obtains:

above, we have omitted "$\alpha_s \cdot$ const"
(this number is for NDR scheme!)

$$C_1(\mu) = 1 + \frac{3}{N_c}\left(\frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2} - \frac{11}{6}\right)$$

$$C_2(\mu) = 0 - 3\left(\frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2} - \frac{11}{6}\right)$$

* The dependence on p^2 has dropped out. This has to be the case!

* Note that C_i have $\alpha_s^n \ln^n\left(\frac{M_W^2}{\mu^2}\right)$ contributions

PT breaks down for $\mu \ll M_W$! The low-energy matrix elements, on the other hand have $\alpha_s^n \ln\left(\frac{m_b^2}{\mu^2}\right)$, etc.

$\leadsto$ large perturbative corrections, irrespective of μ!

Can use renormalization group (RG) to solve this problem.

Wilson coefficients fulfil RG equation, follows from μ-indep. of bare coefficients

$$\frac{d}{d\ln\mu} C_j^{\text{bare}} = 0 = \frac{d}{d\ln\mu} C_i Z_{ij}$$

$$\to \left[\frac{d}{d\ln\mu} C_i(\mu)\right] Z_{ij} + C_i \frac{d}{d\ln\mu} Z_{ij} = 0$$

$$\to \boxed{\begin{aligned} \frac{d}{d\ln\mu} C_i(\mu) &= C_j(\mu)\gamma_{ji} \\ \hat{\gamma} &= -\left(\frac{d}{d\ln\mu}\hat{Z}\right)\hat{Z}^{-1} \end{aligned}}$$

RG-equation

In $\overline{MS}$ scheme $\hat{Z}$ is a sum of poles:

$$\hat{Z} = \mathbb{1} + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} \hat{Z}_k(\alpha_s)$$

Finiteness of $\hat{\gamma}$ implies "magic" relation

$$\boxed{\hat{\gamma} = +2\alpha_s \frac{\partial \hat{Z}_1}{\partial \alpha_s}} = \frac{\alpha_s}{4\pi}\begin{pmatrix} -6/N_c & 6 \\ 6 & -6/N_c \end{pmatrix}$$

Derivation:

$$-\hat{\gamma}\,\hat{z} = \frac{d}{d\ln\mu}\hat{z} = \frac{\partial\hat{z}}{\partial\alpha_s}\frac{d\alpha_s}{d\ln\mu}$$

$$= \frac{\partial\hat{z}}{\partial\alpha_s}\beta(\alpha_s,\varepsilon) = \frac{\partial\hat{z}}{\partial\alpha_s}\left[\beta(\alpha_s) - 2\alpha_s\varepsilon\right]$$

$\alpha_s^{bare} = \mu^{2\varepsilon} Z_\alpha \alpha_s(\mu)$

Now take $O(\varepsilon^0)$ term

$$\longrightarrow -\hat{\gamma} = -2\alpha_s\frac{\partial\hat{z}_1}{\partial\alpha_s}$$

One can write a formal solution to the RG as a (μ-ordered) matrix exponential.

To solve the equation in practice, it is easiest to change to a basis in which the lowest order $\hat{\gamma}$ is diagonal.

For us, this is $C_\pm = C_1 \pm C_2$ for which

$$\frac{d}{d\ln\mu}C_\pm(\mu) = \gamma_\pm C_\pm(\mu) = \frac{\alpha_s}{4\pi}6\left(\pm1-\frac{1}{N_c}\right)C_\pm(\mu)$$

$$= \frac{\alpha_s}{4\pi}\gamma_0^\pm C_\pm$$

To solve this equation perturbatively, we

use
$$\frac{d\alpha_s}{d\ln\mu} = \beta(\alpha_s) = -2\alpha_s\left[\frac{\alpha_s}{4\pi}\beta_0 + \dots\right]$$

$$\rightsquigarrow \quad \frac{dC_\pm}{C_\pm} = d\ln\mu \cdot \gamma_\pm = \frac{d\alpha_s}{\beta(\alpha_s)}\gamma(\alpha_s)$$

$$= -\frac{d\alpha_s}{\alpha_s}\left(\frac{\gamma_0^\pm}{2\beta_0} + \dots\right)$$

$$\ln\left(\frac{C_\pm(\mu)}{C_\pm(M_W)}\right) = -\frac{\gamma_0^\pm}{2\beta_0}\ln\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right) + O(\alpha_s)$$

$$\rightarrow \quad C_\pm(\mu) = C_\pm(M_W)\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\gamma_0^\pm/2\beta_0} + O(\alpha_s)$$

$$\uparrow$$

$$C_\pm(M_W) = 1 + O(\alpha_s)$$

... and can then transform back to $O_{1,2}$

basis. Numerically, for $\mu \approx m_b$

$$C_1(\mu) = 1{,}1 \qquad ; \qquad C_2(\mu) = -0{,}3$$

Remarks:

* Can use $\alpha_s(\mu) = \dfrac{\alpha_s(M_W)}{1 + \beta_0 \frac{\alpha_s(M_W)}{4\pi} \ln\left(\frac{\mu^2}{M^2}\right)}$

$$\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\gamma_0}{2\beta_0}} = \left(1 + \beta_0 \frac{\alpha_s(M_W)}{4\pi} \ln\left(\frac{\mu^2}{M^2}\right)\right)^{\frac{\gamma_0}{2\beta_0}}$$

$$= 1 + \frac{\alpha_s(M_W)}{4\pi} \gamma_0 \ln\left(\frac{\mu}{M}\right) + \frac{1}{2}\left[\frac{\alpha_s(M_W)}{4\pi} \gamma_0 \ln\left(\frac{\mu}{M}\right)\right]^2$$

$$+ \dots$$

leading-log resummation

RG resums tower of logs $\alpha^n L^n$

* After RG improvement, we get expansion in $\alpha_s(M_Z)$ and $\alpha_s(\mu)$, which are counted as of the same order. Expansion works as long as $\alpha_s(\mu) \ll 1$.

Note that $\log\left(\frac{\mu^2}{M_W^2}\right)$ is counted as $1/\alpha_s$ so that

$\alpha_s \cdot \ln\left(\frac{\mu^2}{M}\right) \sim O(1)$

* Accuracy & ingredients

	C_i, $\langle O_i \rangle$	γ, β	accuracy
LO	tree	1-loop	$O(1)$
NLO	1-loop	2-loop	$O(\alpha_s)$

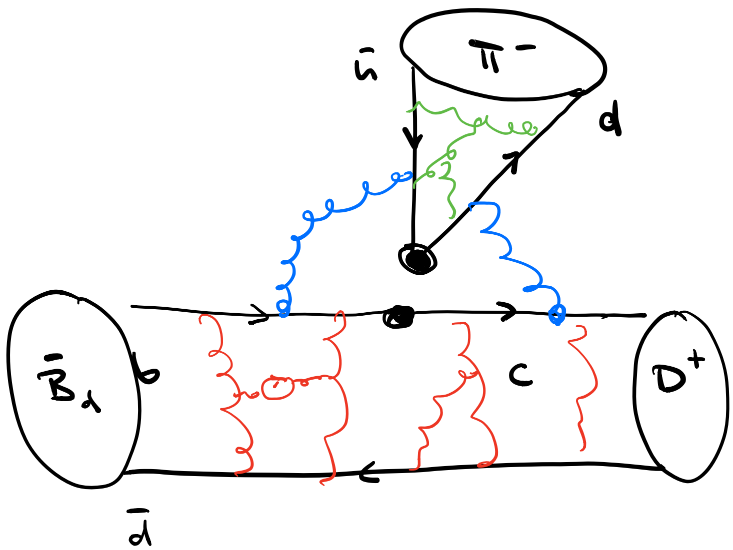
Need anomalous dimensions one order higher than matrix elements & Wilson coefficients.

Finally, to get the decay rate, we need matrix elements

$$\langle \pi^- D^+ | O_i | \bar{B} \rangle$$

This contains two scales : m_b , Λ_{QCD}

One can use Soft-Collinear Effective theory (SCET) to separate the two scales and compute the ones associated with m_b perturbatively



If $m_b \sim m_c \gg \Lambda_{QCD}$, the blue corrections are perturbative and can be computed.

In SCET, there are different types of fields for the different momentum modes which contribute

perturbative: "hard-collinear" $p^2 \sim m_b \Lambda$

non-pert.:

- "soft" $p^2 \sim \Lambda^2$
- "collinear" $p^2 \sim \Lambda,$ $E \sim m_b$

Furthermore, the heavy quarks are nonrelativistic and can be described in Heavy-Quark Effective Theory (HQET)

This leads to a factorization theorem

$$\langle D^+\pi^- | O_i | \bar{B}_d \rangle = F_{B\to D^-}(0) \cdot \int_0^1 dx\, T_i(x, m_b, m_c)\, \phi_\pi(x)$$

soft, NP

hard-coll, perturbative

coll, NP

see hep-ph/0006124
hep-ph/0107002