QED at very low energies

As a first example EFT, we consider QED at very low energies $E_\gamma \ll m_e$.

We end up with non-relativistic charged particles and low-energy photons. To make our life simple, let's treat the charges as a classical source j_μ and focus on the effective theory of photons, which is called Euler-Heisenberg Theory

For $E_\gamma \ll m_e$, we cannot produce $e^- e^+$ pairs: $\mathcal{L}_{eff} = \mathcal{L}_{eff}(A_\mu, j_\mu)$

Construction of $\mathcal{L}_{eff}$:

a.) write down most general $\mathcal{L}_{eff}$, compatible with symm. of QED.

- Building blocks : D_μ, j^μ, $g_{\mu\nu}$, $\varepsilon_{\mu\nu\rho\sigma}$
- Symm, e.g. charge conjugation $e \to -e$, $A_\mu \to -A_\mu \Rightarrow F_{\mu\nu} \to -F_{\mu\nu}$, parity

b.) order terms by derivatives D_μ.

- Contribution of higher derivative terms is suppressed by $\lambda \sim \frac{p_\gamma^\mu}{m_e} \sim \frac{E_\gamma}{m_e}$

Expansion parameter

$$\mathcal{L}_{eff} = \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \mathcal{L}^{(8)} + \dots$$

c.) Each operator comes with a coefficient. These coupling constants are also called Wilson coefficients.

dimension = 3
count like D^3

Lowest order Lagrangian:

$\partial_\mu j^\mu = 0.$

$$\mathcal{L}^{(4)} = -\frac{Z}{4} F_{\mu\nu}F^{\mu\nu} - \overbrace{e A_\mu}^{iD_\mu} j^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}\left[iD_\mu , iD_\nu \right]$$

$$iD_\mu = i\partial_\mu - e A_\mu .$$

Terms of $d=6$:

- $F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\mu = 0$. More generally terms with odd number of F's are forbidden by C-invariance.
  "Furry's theorem"
- only other possibilities are $\partial^2 F^2$ terms

$$\partial_\mu F_{\nu\rho}\, \partial^\mu F^{\nu\rho} , \quad F_{\nu\rho} \Box F^{\nu\rho} , \ldots$$

- Can use integration by part to simplify action:

$$\int d^4x \; \partial_\mu F_{\nu\rho} \partial^\mu F^{\nu\rho}$$

$$\hat{=} -\int d^4x \; F_{\nu\rho} \Box F^{\nu\rho}$$

No need to include both terms, since they are equivalent.

This leaves two terms

$$O_1 = F^{\mu\nu} \Box F_{\mu\nu}$$

$$O_2 = (\partial^\rho F^{\mu\nu})(\partial_\mu F_{\rho\nu})$$

since we can arrange for the derivatives not to be contracted with the field strength they act on.

- using Bianchi identity one can show that $2\,O_2 \hat{=} O_1$ (exercise)

Furthermore, we must include source terms

$$O_3 = j_\mu j^\mu \quad ; \quad O_4 = \partial_\mu F^{\mu\nu} j^\nu$$

and $O_2 \hat{=} \partial_\mu F^{\mu\nu} \partial^\rho F_{\rho\nu}$

There is a final simplification. The classical equation of motion (EOM) of A_μ reads

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \left(\text{EOM from } \mathcal{L}^{(4)}!\right)$$

we will demonstrate shortly that terms

$$O = \hat{O}_\nu \cdot (\partial_\mu F^{\mu\nu} - j^\nu) \qquad \text{"EOM operator"}$$

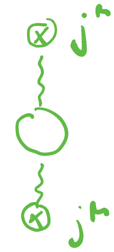
do not contribute to physical quantities, up to higher orders in λ.

For this reason $O_2 \hat{=} O_4 \hat{=} O_3$

and

$$\mathcal{L}^{(6)} = \frac{\overset{\text{dimensionless}}{\downarrow}\, C_0}{m_e^2} \overbrace{j^\mu j_\mu}^{d=6} .$$

full Theory:



This is a contact interaction between the source and does not contribute to photon propagation or scattering.

EOM terms do not contribute because they can be eliminated using field redefintions. Proof: Consider $\mathcal{L}_{eff}(\phi)$ and redefine

$$\phi = \phi' + \underbrace{f(\phi')}_{\delta\phi'}$$

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi)$$

$\frac{\delta S(\phi')}{\delta \phi'} = 0$ is EOM

$$= S[\phi'] + \int d^4x \underbrace{\frac{\delta S(\phi')}{\delta \phi'} f(\phi')}_{\text{EOM operator in } \mathcal{L}} + O(f^2)$$

If f is higher order in λ, we can drop $O(f^2)$ and replace $\frac{\delta S}{\delta \phi'}$ by leading-power EOM. In our case, we would use

$$A^\mu \rightarrow \underbrace{A'_\mu}_{O(\lambda)} + \underbrace{\frac{\alpha}{m_e^2} e j_\mu}_{O(\lambda^3)} + \underbrace{\frac{\beta}{m_e^2}\partial^\nu F_{\nu\mu}}_{O(\lambda^3)}$$

to eliminate O_2 & O_4.

The redefined fields have exactly the same quantum numbers as the old ones and yield the same scattering amplitudes after LSZ reduction. (but off-shell Green's functions differ)

In principle the redefinition generates a Jacobian

$$\text{"}\delta(x-y) + \frac{\delta f}{\delta \phi'}\text{"}$$

$$\int \mathcal{D}\phi = \int \mathcal{D}\phi' \left| \frac{\delta \phi}{\delta \phi'} \right|$$

but it is trivial in dim. reg. To see this, see the EFT lecture notes on my home page.

The first terms involving photon interactions arise at $d=8$

dimensionless.

$$\mathcal{L}^{(8)} = \frac{C_1}{m_e^4} \left(F^{\mu\nu} F_{\mu\nu} \right)^2$$

$$+ \frac{c_2}{m_e^4} F^{\mu}{}_{\nu} F^{\nu}{}_{\rho} F^{\rho}{}_{\sigma} F^{\sigma}{}_{\mu} .$$

Ex. Is this the complete list? How about the operators with j^μ?

$$\mathcal{L}_j^{(8)} = \frac{c_3}{m_e^4} \partial_\mu j_\nu F^{\mu\rho} F_\rho{}^\nu + \left(\begin{array}{c} \text{are there} \\ \text{others ?} \end{array} \right)$$

[Note that $eA_\mu j^\mu$ is only gauge inv. up to a total derivative. $(eA_\mu j^\mu)^2$ is therefore <u>not</u> gauge invariant!]

Note: In $d=4$, one can write

$$F^{\mu\nu} F_{\nu\rho} F^{\rho\sigma} F_{\sigma\mu} = \frac{1}{4} \big(\underbrace{F^{\mu\nu} \tilde{F}_{\mu\nu}}_{-4\vec{E}\cdot\vec{B}} \big)^2 + \frac{1}{2} \underbrace{F^{\mu\nu} F_{\mu\nu}}_{-2(\vec{E}^2 - \vec{B}^2)}$$

In $d=4$ all higher-order terms are products of these invariants.

The existence of such a relation is not a surprise: $F_{\mu\nu}$ is a 4×4 matrix and the fourth (and higher) powers of such a matrix can be expressed in terms of lower-power invariants, thanks to the Cayley-Hamilton theorem.

Any antisymmetric 4x4 matrix fulfils

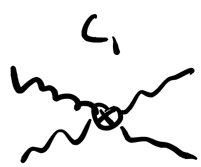
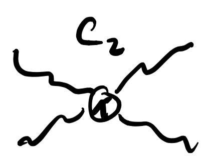
$$F^4 = \frac{1}{2} \underbrace{\overbrace{\langle F^2 \rangle}^{\Phi_1}}_{2(\vec{B}^2-\vec{E}^2)} F^2 - \underbrace{\overbrace{\mathrm{Det}(F)}^{\Phi_2}}_{(\vec{B}\cdot\vec{E})^2} \cdot \mathbb{1}$$

As a consequence, one can express all operators through products of the two invariants Φ_1, Φ_2 and derivatives of them.

There is a lot of interesting recent progress in constructing effective Lagrangians, see 1706.08520 by Henning, Lu, Melia & Murayama.

For a taste of it, see Chapter 9 in 1804.05863 by A. Manohar.

The next step in our construction of the EFT is the matching. But even without performing it, we can already estimate the $\gamma\gamma \to \gamma\gamma$ scattering cross section:

$\sigma \sim \Big|$  $+$  $\Big|^2$

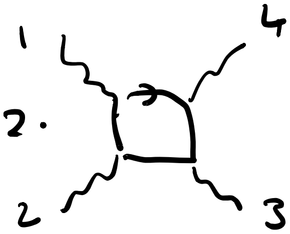
$$\sigma \sim \left(\frac{1}{m_e^4}\right)^2 \cdot E_\gamma^6 \cdot \underbrace{c_{1,2}^2}_{\alpha^4}$$

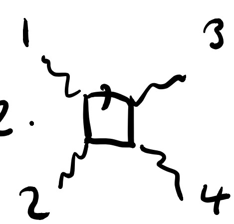
Extremely small and for this reason low-E $\gamma\gamma$ scattering has never been observed, but there are efforts to measure it using intense lasers. It is hard to see anything but free photons at low energies...

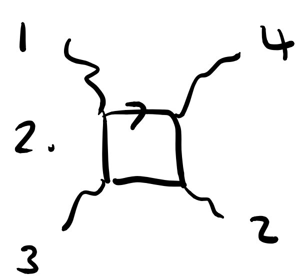
Exact computation of unpolarized cross section in c.m.s yields (painful exercise: derive Feynman rules compute σ)

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\left(48 c_1^2 + 40 c_1 c_2 + 11 c_2^2\right)\frac{E_\gamma^6}{m_e^8} \cdot \left(3 + \cos^2\theta\right)^2 \qquad (*)$$

To perform the matching, we can compute the forward amplitude for two different helicity configurations to get both c_1 & c_2. The QED amplitude is

$\mathcal{A} = 2 \cdot$ 

$+ 2 \cdot$ 

$+ 2 \cdot$ 

Can simplify the computation by expanding in external momenta before loop integration. Then only loop integrals of the form

$$\int d^d k \, \frac{(k^2)^\alpha}{(k^2 - m_e^2)^\beta}$$

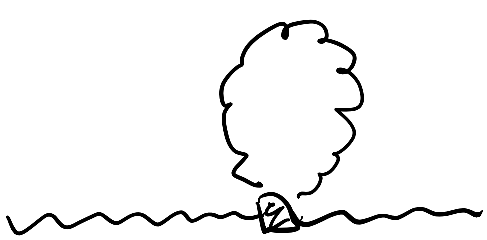
are needed.

(...)

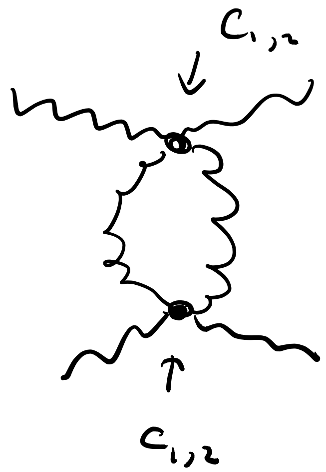
$$\rightsquigarrow \quad C_1 = -\frac{1}{36}\alpha^2 \; ; \; C_2 = \frac{7}{90}\alpha^2$$

We find that the coefficients are finite!

Must be the case: the only loop diagram at $1/m_e^4$ is

 $= 0$

The first divergent loop diagrams are



$$+ \ \dots \ \sim \left(\frac{1}{m_e^4}\right)^2$$

Renormalize $\mathcal{L}^{(12)}$!

Power counting

Using dim. reg. it is trivial to count powers of λ in any diagram. Consider dimensionless observable Q. The contribution of a matrix element with N operators

with $k_i = n_i - d$, $i = 1, \dots, N$ scales as

dim.

$$\Delta Q \sim \lambda^{k} \; ; \; k = \sum_{i=1}^{N} k_i$$

Proof: Count the powers of $\frac{1}{\lambda}$! Since these are prefactors, the counting is trivial, also for loop diagrams.

With a UV cutoff Λ_{UV} one gets contributions of the form $\frac{\Lambda_{UV}}{\Lambda} \sim 1$ and the statement only holds after these power divergences are absorbed into couplings.

Plugging in the values of the couplings c_1 & c_2, one gets the result

$$\frac{d\sigma}{d\Omega} = 139 \left(\frac{\alpha}{180\pi}\right)^2 \left(3 + \cos^2\theta\right)^2 \frac{E_\gamma^6}{m_e^8}$$

What are the dominant corrections?

Three sources

i.) loops from $\mathcal{L}^{(4)}$

not present for E-H

ii) pert. corr's to C_i

from higher-order matching

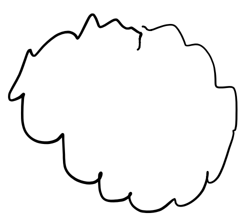
iii) contributions from higher $\mathcal{L}^{(n)}$'s; $n > 4$

suppressed by λ.

Neat application: e^- corrections to photon energy density (hep-ph/9803216)

$$\varepsilon = E/V = \frac{\pi^2}{15} T^4 \quad \text{(Stefan-Boltzmann)}$$

This is obtained from



$\neq 0$, since

$$\int d^4k \longrightarrow T \sum_{\omega_n} \int d^3k$$

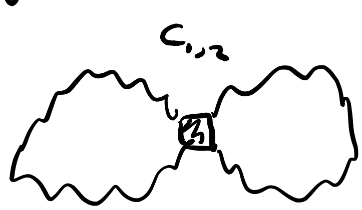
$\omega_n = 2\pi T \cdot n$

(Matsubara Frequencies)

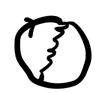
computed at finite T

Euler Heisenberg:

$$\Delta\varepsilon \sim$$



$$\sim \frac{\alpha^2}{m_e^4} \cdot E^8$$

In QED, this is a three-loop diagrams and there are also two-loop diagrams  which do not contribute at the end.

It would be nice to continue a bit and consider QED with e^- and μ^- and then construct the effective theory, where the μ^- is integrated out, i.e.

$$\mathcal{L}_{full} = \mathcal{L}[A, \mu, e]$$

$$\Big\downarrow$$

$$\mathcal{L}_{eff} = \mathcal{L}[A, e]$$

This $\mathcal{L}_{eff}$ will contain higher-dim operators suppressed by $1/m_\mu$. The most important ones are $\bar{\psi}_e \sigma^{\mu\nu} F_{\mu\nu} \psi_e$ contribution to anomalous magnetic moment.

$$\bar{\psi} \Gamma \psi \, \bar{\psi} \Gamma \psi$$

For lack of time, we will not persue this here; a discussion can be found in my EFT lecture notes and also in 0908.4392 by Grozin.