# Precision QCD calculations and geometry

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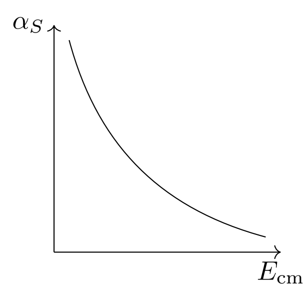
From the lecture of Tao Han we heard that the scattering matrix can be written in the form $S = 1 + iT$, where $T$ is the scattering part. As we do perturbative calculations, we want to expand $T$ around small values of the strong coupling constant $\alpha_S$, so

$$T = \sum_l \alpha_S^{n(l)} T_{(l)}. \tag{1}$$

Since QCD is asymptotically free, the value of $\alpha_S$ decreases with rising center-of-mass energy




and since run two of the LHC produced some results, $\alpha_S$ is actually so small, that we can see NNLO contributions (which correspond to $l = 2$ in (1))!

When we talk about NNLO we distinguish between real corrections, so if we have more particles in the final state, and virtual corrections, which means loop Feynman diagrams. I will stick to virtual corrections. To calculate the virtual part we could use the standard Feynman-diagrammatic approach, however there are some problems with that.

1. At one-loop the number of diagrams is still manageable, but at two-loop there are simply to many diagrams to perform efficient calculations.

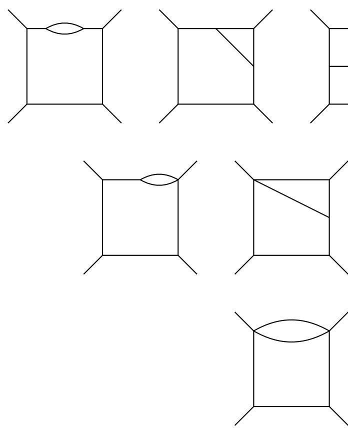
2. We have to deal with complicated tensor integrals.

**Numerical unitarity** Luckily, in Tao Han's lecture we also got to know the starting point for a solution to the above problems, namely the unitarity of the $S$-matrix. We find that $1 = S^2 = (1 + iT)(1 - iT^\dagger)$ so that we have the relation

$i(T-T^\dagger) = 2\mathrm{im}T = TT^\dagger$. Employing the expansion (1), we thus have a relation between loop diagrams and tree diagrams! This relation is made manifest in the Cutkosky cutting rules [1], which tell us to set propagators on-shell, that is, if $\ell$ is the loop momentum and $p_{ij}$ a sum of external momenta, to replace

$$\int [d\ell] \frac{1}{(\ell - p_{ij})^2 - m^2} \to \int [d\ell] \delta\big((\ell - p_{ij})^2 - m^2\big) = \int_X [d\ell] \tag{2}$$

where $X = \{(\ell - p_{ij})^2 = m^2\}$ is the space of solutions of the on-shell conditions for the loop momenta. Diagrammatically we can represent this using cut diagrams, which are hierarchically ordered, according to the number of propagators being set on-shell. In the following we present the application of numerical unitarity to two-loop calculations, where you can read more about in [2]. That is for example for a $2 \to 2$ process the hierarchy of the bubble-box cut is given by



In the first row we have diagrams, that is our scattering amplitude $\mathcal{A}$, so that $T \sim |\mathcal{A}|^2$ where every propagator has been set on-shell or cut. The cuts from the first row we call the maximum cuts and label them by $\Gamma^{\mathrm{M}}$. The second line represents diagrams, where when applying the cutting procedure (2) one propagator has been left out. These are the next-to-maximal cut diagrams and we label them $\Gamma^{\mathrm{NM}}$. Now we make an ansatz for the amplitude

$$\mathcal{A} = \sum\nolimits_\Gamma \sum\nolimits_{i \in \text{Basis}} c_{\Gamma i} I_{\Gamma i} \tag{3}$$

where $c_{\Gamma i}$ are some coefficients and

$$I_{\Gamma i} = \int [d\ell] \frac{m_{\Gamma i}(\ell)}{\rho_1 \cdots \rho_k} \tag{4}$$

are so-called master integrals. Here $m_{\Gamma i}(\ell)$ is the numerator depending on kinematics and $\rho_i$ are the propagators, which are supposed to be cut. For simplicity we define the numerator of the integrand $N(\Gamma, \ell) = \sum_{i \in \text{Basis}} c_{\Gamma i} m_{\Gamma i}(\ell)$. So if every propagator is cut the numerator of the integrand $N(\Gamma, \ell)$ is given by a product of the tree diagrams corresponding to the vertices of the cut diagrams $N(\Gamma^{\mathrm{M}},\ell) = \sum_{\text{states}} \prod_j \mathcal{A}_j^{\text{tree}} \equiv R(\Gamma^{\mathrm{M}},\ell)$, where by states we mean the helicity states of the loop momenta $\ell$. It is easy to compute the tree diagrams using for example Berends–Giele recursion relations. Doing so and evaluating the chosen integrands $m_{\Gamma^{\mathrm{M}}i}(\ell)$ for enough values of $\ell$ gives us a linear system of equations, where the coefficients $c_{\Gamma^{\mathrm{M}}i}$ are the unknown values for which we can solve.

Having found the coefficients for maximal cuts, we can proceed to the next-to-maximal diagrams. Here the numerators of the integrands $N(\Gamma^{\mathrm{NM}},\ell)$ can also be calculated using the corresponding products of trees $R(\Gamma^{\mathrm{NM}},\ell)$, however when looking at the ansatz for the amplitude (3) we see that the cut propagators will also appear in the integrands corresponding to the maximum cuts $\Gamma^{\mathrm{M}}$. That is why we have to subtract these contributions from the tree calculation result $R(\Gamma^{\mathrm{NM}},\ell)$ in order to obtain only $N(\Gamma^{\mathrm{NM}},\ell)$. Writing $\rho_k$ for the propagator which is cut in the maximum cuts but uncut in the next-to-maximum this means

$$N(\Gamma^{\mathrm{NM}},\ell) = R(\Gamma^{\mathrm{NM}},\ell) - \sum_k N(\Gamma^{\mathrm{M},k},\ell)/\rho_k$$

Again, because $N(\Gamma^{\mathrm{NM}},\ell) = \sum_i c_{\Gamma^{\mathrm{NM}}i} m_{\Gamma^{\mathrm{NM}}i}(\ell)$, when we sample about enough values of $\ell$ we can solve for the coefficients $c_{\Gamma^{\mathrm{NM}}i}$. This procedure has to be iterated to obtain all the coefficients needed in the amplitude ansatz (3).

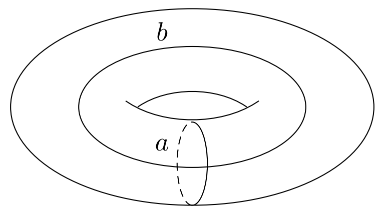
**Geometry** Now that we talked about the coefficients of the master integrals, how about the actual master integrals $I_{\Gamma i}$? In order not to have to integrate to much it is a good thing to ask how many integrals are needed. This can be answered using the beautiful underlying structure of their spaces. You can read further on that in [3] and also in [4].

So if we want to ask the dimension of the integral space on-shell, we equivalently could ask, how many differential $k$-forms there are, where $k$ is the dimension of the uncut loop momentum space $X$ we had in equation (2) and the forms are the numerators $[d\ell]m_{\Gamma i}(\ell)$. These turn out to be closed forms. For a form $\omega \in \Omega^k(X)$ that means $d\omega = 0$, where $d$ is the exterior derivative. However we actually do not want to count the forms $[d\ell]m_{\Gamma i}(\ell)$, where the integral vanishes, because these cannot contribute to the master integral basis. We call these terms spurious terms. If the integral of a form vanishes it is an exact form. Exact forms are the $\omega \in \Omega^k(X)$, so that there is an $\alpha \in \Omega^{k-1}(X)$ so that $\omega = d\alpha$. That means, in order to calculate the on-shell dimension of the master integrals, we calculate the dimension of the closed form space, where we factor out the exact form space. If we write for short $\omega = [d\ell]m_{\Gamma i}(\ell)$ In formulas

$$\begin{aligned}\dim\{I_{\Gamma i}\} &= \dim\{\omega \in \Omega^k(X) \mid d\omega = 0\}/\{\omega \in \Omega^k(X) \mid \exists \alpha \in \Omega^{k-1}(X)\colon \omega = d\alpha\} \\ &= \dim H^k(X) = h^k(X)\end{aligned}$$

which is basically a definition of the de Rham cohomology groups $H^k(X)$ of the space $X$. By de Rham theorem we know that $h^k(X) = h_k(X)$ meaning we can get the dimensions of the cohomology groups also from the homology $H_k(X)$, which one can imagine as non-contractible $k$-dimensional subspaces disregarding their position within the space.

**Example** So assume we want to compute the dimension of the space of 1-forms on the two dimensional torus $T^2$




we first figure out what the elements of the homology groups $H_1(T^2)$ are. These are the cycles $a$ and $b$, which you cannot transform into one-another by translating. This means $\dim H^1(T^2) = \dim H_1(T^2) = \dim\langle a, b\rangle = 2$.

So this counting of cycles or the dimension of homology groups gives us the dimension of the form spaces on the on-shell spaces $X$ of the loop momenta. The picture becomes more complicated when less legs are cut, because the dimension of $X$ rises. Luckily there are techniques like Morse theory helping to calculate the homology groups then, when you cannot draw the space and count cycles as in the example above.

# References

[1] J. S. Rozowsky, "Feynman diagrams and cutting rules," hep-ph/9709423.

[2] S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier and B. Page, Phys. Rev. D **95**, no. 9, 096011 (2017) doi:10.1103/PhysRevD.95.096011 [arXiv:1703.05255 [hep-ph]].

[3] H. Ita, Phys. Rev. D **94**, no. 11, 116015 (2016) doi:10.1103/PhysRevD.94.116015 [arXiv:1510.05626 [hep-th]].

[4] S. Abreu, R. Britto, C. Duhr and E. Gardi, JHEP **1706**, 114 (2017) doi:10.1007/JHEP06(2017)114 [arXiv:1702.03163 [hep-th]].