Lattice QCD

A short introduction

Nikolai Husung

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Review of the path integral

> Remember:

$$Z = \int \mathcal{D}\phi \, \exp\left(iS[\phi]\right) \, , \; S[\phi] = \int \mathrm{d}^4 x \, \mathscr{L}[\phi] \, . \tag{1}$$

> Perform a Wick rotation in time to obtain positive weight (unnormalised probability distribution)

$$\mathcal{D}P[\phi] = \mathcal{D}\phi \, \exp\left(-S^{\mathrm{eucl}}[\phi]\right) \, , \tag{2}$$

this corresponds to $x_0 \to i x_0^{\mathrm{eucl}}$, i.e. switching to Euclidean space with Euclidean time x_0^{eucl}.

Lattice regularisation

Two ways to interpret the lattice spacing a:

1. Numerics: Estimate occurring derivatives

$$\partial_\mu f(x) = \frac{f(x + a\hat{\mu}) - f(x - a\hat{\mu})}{2a} + \mathrm{O}(a^2)\,. \tag{3}$$

2. Field theory: Introduce a UV regulator in form of the lattice spacing limiting momenta via $|p_\mu| \leq \frac{\pi}{a} \forall \mu$.
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Remark: Finite volume introduces an additional IR cut-off. Assume large volume, i.e. box size $L \simeq 2\,\mathrm{fm}$ for pure gauge, with negligible finite volume effects.

Gauge fields on the lattice

Starting from the (Euclidean) Lagrangian of free fermions one finds

$$
\begin{aligned}
S_{\mathrm{ferm}}^{\mathrm{eucl}}[\psi] &= \int \mathrm{d}^4 x \, \bar{\psi}(x)(\gamma_\mu \partial_\mu + M)\psi(x) \qquad (4)\\
&= a^4 \sum_{n \in \mathbb{Z}^4} \bar{\psi}(n) \left(\gamma_\mu \frac{\psi(n + a\hat{\mu}) - \psi(n - a\hat{\mu})}{2a} + M\psi(n) + \mathrm{O}(a^3) \right) .
\end{aligned}
$$

Gauge fields on the lattice

Performing the usual SU(N) transformation

$$\psi(n) \to \Omega(n)\psi(n)\,,\ \bar{\psi}(n) \to \bar{\psi}(n)\Omega^\dagger(n)\,,\ \Omega(n) \in \mathrm{SU}(N)\,, \tag{5}$$

yields

$$\begin{aligned} \bar{\psi}(n)\psi(n) &\to \bar{\psi}(n)\psi(n)\,, \\ \bar{\psi}(n)\psi(n \pm a\hat{\mu}) &\to \bar{\psi}(n)\Omega^\dagger(n)\Omega(n \pm a\hat{\mu})\psi(n \pm a\hat{\mu})\,. \end{aligned} \tag{6}$$

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$\Rightarrow$ Introduce multiplicatively a gauge transporter between the next neighbouring ψ

$$\begin{aligned}\mathrm{SU}(N) \ni U_\mu(n) =& \mathcal{P}\exp\left(\int_n^{n+a\hat{\mu}} \mathrm{d}x_\mu A_\mu(x)\right) = \exp\left[aA_\mu(n) + \mathrm{O}(a^2)\right]\,,\\ U_\mu(n) \to& \Omega(n)U_\mu(n)\Omega^\dagger(n+a\hat{\mu})\,.\end{aligned} \tag{7}$$

Gauge fields on the lattice

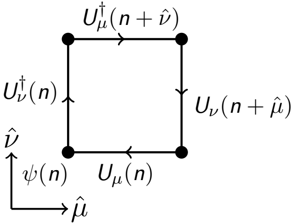

Figure: 2D-schematic of the lattice.

This yields the "naive" fermion action [Wilson, 1974],

$$S_{\text{ferm}}^{\text{latt}} = \frac{a^3}{2} \sum_{\substack{m,n \in \mathbb{Z}^4 \\ \mu}} \bar{\psi}(m) \Big[\gamma_\mu U_\mu(m) \delta_{m+a\hat{\mu},n} + \gamma_\mu U_\mu^\dagger(n) \delta_{m,n+a\hat{\mu}} + 2aM\delta_{m,n} \Big] \psi(n) \,. \tag{8}$$

Gauge fields on the lattice

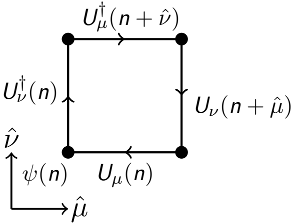

Figure: 2D-schematic of the lattice.

For a general class of the lattice gauge actions, see e.g. [Weisz, 1983], one simple example [Wilson, 1974]

$$S^{\mathrm{latt}}_{\mathrm{gauge}} = \frac{\hat\beta(g_0^2)}{N} \sum_{\substack{n \\ \mu<\nu}} \mathrm{Re}\,\mathrm{tr}\left(\mathbb{1} - U_{\mu\nu}(n)\right) \,, \; \hat\beta(g_0^2) = \frac{2N}{g_0^2} \,. \tag{9}$$

Observables, renormalisation etc.

Scale setting

Idea: Translate lattice units into physical units.

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E.g. by use of a force between static quarks using a phenomenological potential [Sommer, 1994]

$$r^2 F(r) = c \Rightarrow r_0 = r|_{c=1.65} \approx 0.5\,\mathrm{fm}\,. \tag{10}$$

Thus we know r_0/a in lattice units and more importantly the lattice spacing a/r_0 multiplied by a constant.

Continuum limit

From the scale setting we can infer the ratio of lattice spacings corresponding to different choices of $\hat{\beta}$. Make sure to keep the physical volume fixed.

Not rigorously proven: $g_0 \to 0$ and thus $\hat{\beta} \to \infty$ correspond to the continuum limit. Educated guess due to asymptotic freedom.

Continuum limit

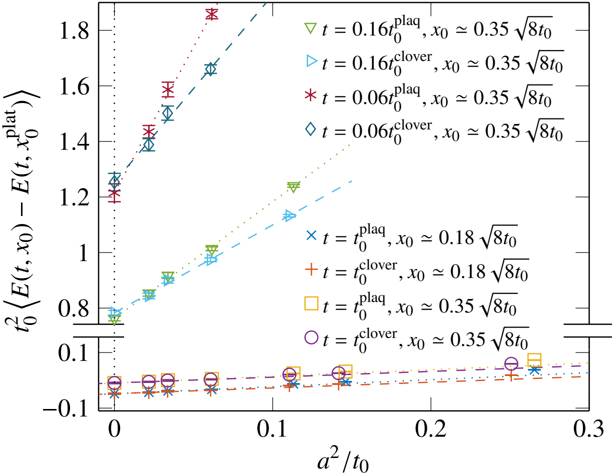

Figure: Example of continuum limits in pure gauge [Husung et al., 2018].

Symanzik effective theory of lattice artifacts

Consider an arbitrary Renormalisation Group Invariant observable X_{RGI} on the lattice

$$\langle X_{\mathrm{RGI}}\rangle_{\mathrm{latt}(a),\mathrm{R}} = \langle X_{\mathrm{RGI}}\rangle_{\mathrm{cont},\mathrm{R}} + \mathrm{O}(a^n)\,. \tag{11}$$

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Choose a basis $\mathcal{O}$ of mass dimension 6 operators diagonal under renormalisation to write down an effective Lagrangian [Symanzik, 1980, 1982]

$$\mathscr{L}_{\mathrm{eff}} = \mathscr{L}_{\mathrm{QCD}} + a^2 \sum_i b_i(a\mu, \alpha)\mathcal{O}_i\,. \tag{12}$$

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For the observable $X_{\mathrm{RGI}}(a) = X_{\mathrm{RGI}}(0) + a^2\delta X + \mathrm{O}(a^3)$ one finds

$$\begin{aligned}\langle X_{\mathrm{RGI}}\rangle_{\mathrm{latt}(a),\mathrm{R}} =& \langle X_{\mathrm{RGI}}\rangle_{\mathrm{cont,R}} + \\ & a^2\left[\langle\delta X\rangle_{\mathrm{cont,R}} + \sum_i b_i(a\mu,\alpha)\int \mathrm{d}^4y\,\langle X_{\mathrm{RGI}}\mathcal{O}_i(y)\rangle_{\mathrm{cont,R}}\right].\end{aligned} \tag{13}$$

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$$\langle X_{\mathrm{RGI}}\rangle_{\mathrm{latt}(a),\mathrm{R}} \overset{\mu=\frac{1}{a}}{=} \langle X_{\mathrm{RGI}}\rangle_{\mathrm{cont,R}} + a^2\Big[\langle\delta X\rangle_{\mathrm{cont,R}} + \sum_i b_i(1,\alpha)[-\ln(a\Lambda)]^{\bar{\gamma}_i/b_0}\int \mathrm{d}^4y\,\langle X_{\mathrm{RGI}}\mathcal{O}_{i,\mathrm{RGI}}(y)\rangle_{\mathrm{cont,R}}\Big] \tag{13}$$

where

$$\mu^2\frac{\mathrm{d}\mathcal{O}_{i,\mathrm{R}}}{\mathrm{d}\mu^2} = \gamma_i(\alpha)\mathcal{O}_{i,\mathrm{R}} = [\bar{\gamma}_i\alpha + \mathrm{O}(\alpha^2)]\mathcal{O}_{i,\mathrm{R}}\,, \tag{14}$$

$$\mathcal{O}_{i,\mathrm{R}}(1/a) = [-\ln(a\Lambda)]^{\bar{\gamma}_i/b_0}\mathcal{O}_{i,\mathrm{RGI}} \times [1 + \mathrm{O}(\alpha)]\,. \tag{15}$$

Conclusion and outlook

Pros and cons of lattice regularisation:

+ Non-perturbative description of QCD.
+ Regularisation, renormalisation etc. get a descriptive method on the lattice.
− Breaks Lorentz-/O(4)-invariance explicitly for $a > 0$, since only discrete translations and rotations are allowed.
− Lattice PT: Each loop order yields additional vertices and finite sums may be more complicated.
− Can be hard to extract observables like PDFs.
± Currently only $N_\mathrm{f} = 2+1, 2+1+1$ accessible, see e.g. FLAG [Aoki et al., 2017].

Outlook

Not covered in this talk:

- **Quantities:** Λ_{QCD}, structure constants, glueball and meson spectra, quasi-PDFs etc.
- Scale setting, see e.g. [Sommer, 1994; Lüscher, 2010; Sommer, 2014].
- Symanzik-improvement, see e.g. [Sheikholeslami and Wohlert, 1985; Lüscher et al., 1996, 1997].
- HQET.
- Gradient-flow scheme and small flow-time expansion, see e.g. [Lüscher, 2010; Lüscher and Weisz, 2011; Suzuki, 2013; Harlander and Neumann, 2016; Ejiri et al., 2017].
- Numerical aspects.
- ...

Contact

DESY. Deutsches Elektronen-Synchrotron

Nikolai Husung

John von Neumann-Institute for Computing (NIC)
nikolai.husung@desy.de

www.desy.de

References I

Kenneth G. Wilson. Confinement of quarks. Phys. Rev. D, 10: 2445–2459, Oct 1974.

P. Weisz. Continuum limit improved lattice action for pure Yang-Mills theory (I). Nuclear Physics B, 212(1):1 – 17, 1983.

Martin Lüscher, Peter Weisz, and Ulli Wolff. A numerical method to compute the running coupling in asymptotically free theories. Nuclear Physics B, 359(1):221 – 243, 1991.

Christof Gattringer and Christian B. Lang. Quantum Chromodynamics on the lattice. Number 788 in Lecture Notes in Physics. Springer-Verlag, 2010.

Nikolai Husung, Mateusz Koren, Philipp Krah, and Rainer Sommer. SU(3) Yang Mills theory at small distances and fine lattices. EPJ Web Conf., 175:14024, 2018.

References II

K. Symanzik. CUTOFF DEPENDENCE IN LATTICE ϕ**4 in four-dimensions THEORY. NATO Sci. Ser. B, 59:313–330, 1980.

K. Symanzik. Some topics in quantum field theory, pages 47–58. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982.

S. Aoki, Y. Aoki, D. Bečirević, C. Bernard, T. Blum, G. Colangelo, M. Della Morte, P. Dimopoulos, S. Dürr, H. Fukaya, M. Golterman, Steven Gottlieb, S. Hashimoto, U. M. Heller, R. Horsley, A. Jüttner, T. Kaneko, L. Lellouch, H. Leutwyler, C.-J. D. Lin, V. Lubicz, E. Lunghi, R. Mawhinney, T. Onogi, C. Pena, C. T. Sachrajda, S. R. Sharpe, S. Simula, R. Sommer, A. Vladikas, U. Wenger, and H. Wittig. Review of lattice results concerning low-energy particle physics. The European Physical Journal C, 77(2):112, Feb 2017. ISSN 1434-6052. doi: 10.1140/epjc/s10052-016-4509-7. URL `https://doi.org/10.1140/epjc/s10052-016-4509-7`.

References III

Rainer Sommer. A new way to set the energy scale in lattice gauge theories and its applications to the static force and alpha-s in SU(2) Yang-Mills theory. Nucl. Phys., B411:839–854, 1994.

Martin Lüscher. Properties and uses of the Wilson flow in lattice QCD. JHEP, 08:071, 2010. [Erratum: JHEP03,092(2014)].

Rainer Sommer. Scale setting in lattice QCD. PoS, LATTICE2013:015, 2014.

B. Sheikholeslami and R. Wohlert. Improved Continuum Limit Lattice Action for QCD with Wilson Fermions. Nucl. Phys., B259:572, 1985.

Martin Lüscher, Stefan Sint, Rainer Sommer, and Peter Weisz. Chiral symmetry and O(a) improvement in lattice QCD. Nucl. Phys., B478: 365–400, 1996.

References IV

Martin Lüscher, Stefan Sint, Rainer Sommer, Peter Weisz, and Ulli Wolff. Nonperturbative O(a) improvement of lattice QCD. Nucl. Phys., B491:323–343, 1997.

Martin Lüscher and Peter Weisz. Perturbative analysis of the gradient flow in non-abelian gauge theories. JHEP, 02:051, 2011.

Hiroshi Suzuki. Energy–momentum tensor from the Yang–Mills gradient flow. PTEP, 2013:083B03, 2013. [Erratum: PTEP2015,079201(2015)].

Robert V. Harlander and Tobias Neumann. The perturbative QCD gradient flow to three loops. JHEP, 06:161, 2016.

References V

Shinji Ejiri, Ryo Iwami, Mizuki Shirogane, Naoki Wakabayashi, Kazuyuki Kanaya, Masakiyo Kitazawa, Hiroshi Suzuki, Yusuke Taniguchi, and Takashi Umeda. Determination of latent heat at the finite temperature phase transition of SU(3) gauge theory. In Proceedings, 34th International Symposium on Lattice Field Theory (Lattice 2016): Southampton, UK, July 24-30, 2016, 2017.