Analytical Rational Coefficients for One Loop Standard Model Scattering Amplitudes

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Objective

The aim of this talk is to discuss the reconstruction of analytical one loop amplitudes from numerical results obtained by BlackHat. This allows to avoid large intermediate expressions that traditionally appear in this type of analytical calculations.

To this end, I will explain:

1) how exploit the structure of poles and zeros, arising from single and double collinear limits in complexified momentum space;

2) the importance of spurious singularities and of partial fractioning large expressions in order to respect the pole structure;

3) how to fit numerators by repetedly evaluating generic ansatze in particular collinear limits.

Why Analytical?

There are a number of reasons:

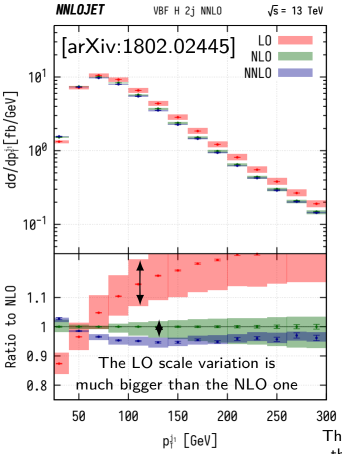
1) physical predictions often rely on repeated evalutations of matrix elements, and numerical computations are much slower than having an analytical formula;

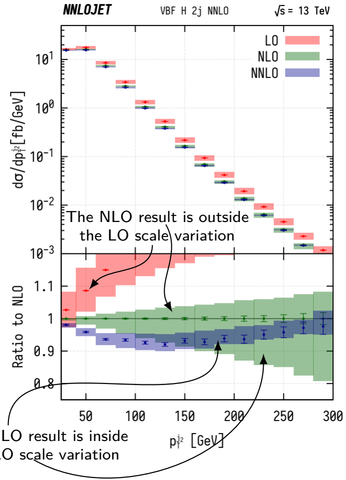
2) analytical expressions provide better understanding;

3) if suitably written, analytical expressions trivially generalise to all multiplicities;

4) ...

Why NLO?


NNLOJET
VBF H 2j NNLO
√s = 13 TeV
[arXiv:1802.02445]
LO
NLO
NNLO
dσ/dp$_T^{j_1}$[fb/GeV]
Ratio to NLO
p$_T^{j_1}$ [GeV]
The LO scale variation is
much bigger than the NLO one


NNLOJET
VBF H 2j NNLO
√s = 13 TeV
LO
NLO
NNLO
dσ/dp$_T^{j_2}$[fb/GeV]
Ratio to NLO
p$_T^{j_2}$ [GeV]
The NLO result is outside
the LO scale variation
The NNLO result is inside
the NLO scale variation

Loops vs Multiplicity - Higher Order Prediction Table

3	8	9	10	11
2	6	7	8	9
1	4	5	6	7
0	2	3	4	5
loops \ multi.	4	5	6	7

Table: g_S order for $\mathcal{A}$ as function of loops and multiplicity in QCD.

Colour Ordered Amplitudes: Gluons Only

The relation between the full amplitude and the colour ordered one is:

$$\mathcal{A}_n^{tree}(p_i, \lambda_i, a_i) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_\sigma(1)} ... T^{\sigma(n)}) A_n^{tree}(\sigma(1^{\lambda_1}), ..., \sigma(n^{\lambda_n}))$$

where sum is over all non cyclicly equivalent permuations, and A is the colour-ordered amplitude.

Crucially, A is gauge invariant, depends only on the momenta and helicities and the cyclic ordering reduces considerably the possible sigularities.

Similar expressions exist for more loops or different particle content.

Structure of One Loop Amplitudes

At one loop a basis of master integrals is known, meaning we can write the colour ordered amplitude as:

$$A^{1-loop} = \sum_i d_i I^i_{Box} + \sum_i c_i I^i_{Triangle} + \sum_i b_i I^i_{Bubble} + R$$

where d_i, c_i, b_i and R are rational coefficients of kinematic variables only.

I^i_{Box}, $I^i_{Triangle}$ and I^i_{Bubble} are the basis for 1 loop integrals. All tadpoles are zero in dimensional regularisation if the propagators are massless.

Basic Spinor Variables

The lowest lying representations of the Lorentz group are the $(1/2, 0)$ left- and $(0, 1/2)$ right-handed spinor representations, respectively:

$$\bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} \sqrt{p^0 + p^3} \\ \dfrac{p^1 - ip^2}{\sqrt{p^0 + p^3}} \end{pmatrix} \quad \& \quad \bar{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{\dot{\beta}} = \begin{pmatrix} \dfrac{p^1 - ip^2}{\sqrt{p^0 + p^3}} \\ -\sqrt{p^0 + p^3} \end{pmatrix}.$$

$$\lambda_{\alpha} = \begin{pmatrix} \sqrt{p^0 + p^3} \\ \dfrac{p^1 + ip^2}{\sqrt{p^0 + p^3}} \end{pmatrix} \quad \& \quad \lambda^{\alpha} = \epsilon^{\alpha\beta} \lambda_{\beta} = \begin{pmatrix} \dfrac{p^1 + ip^2}{\sqrt{p^0 + p^3}} \\ -\sqrt{p^0 + p^3} \end{pmatrix}$$

The spinor helicity notation is then:

$$\langle ij \rangle \equiv \lambda_i \lambda_j = (\lambda_i)^{\alpha} (\lambda_j)_{\alpha} \qquad \text{and} \qquad [ij] \equiv \bar{\lambda}_i \bar{\lambda}_j = (\bar{\lambda}_i)_{\dot{\alpha}} (\bar{\lambda}_j)^{\dot{\alpha}}$$

More complicated Spinor Variables

We can also define rank-two spinors as:

$$P^{\dot{\alpha}\alpha} = \bar{\lambda}^{\dot{\alpha}}\lambda^{\alpha} \text{ and } \bar{P}_{\alpha\dot{\alpha}} = \lambda_{\alpha}\bar{\lambda}_{\dot{\alpha}}.$$

Or directly in terms of four-momenta through the Infeld - van der Waerden symbols:

$$P^{\dot{\alpha}\alpha} = (\sigma^{\mu})^{\dot{\alpha}\alpha} P_{\mu} \text{ and } \bar{P}_{\alpha\dot{\alpha}} = (\bar{\sigma}^{\mu})_{\alpha\dot{\alpha}} P_{\mu}$$

where $(\sigma^{\mu}) = (1, \sigma^{i})$ and $\bar{\sigma}^{\mu} = (1, -\sigma^{i})$.

We can now understand the following notation:

$$\langle 1|2+3|4] = \lambda_1^{\alpha}\Big((\bar{P}_2)_{\alpha\dot{\alpha}} + (\bar{P}_3)_{\alpha\dot{\alpha}}\Big)\bar{\lambda}_4{}^{\dot{\alpha}}$$
$$\langle 1|2+3|4+5|6\rangle = \lambda_1^{\alpha}\Big((\bar{P}_2)_{\alpha\dot{\alpha}} + (\bar{P}_3)_{\alpha\dot{\alpha}}\Big)\Big((P_4)^{\dot{\alpha}\beta} + (P_5)^{\dot{\alpha}\beta}\Big)(\lambda_6)_{\beta}$$

Example on Jupyter Notebook

The Elegance of The Rational Coefficients

Working with colour-ordered amplitudes and the spinor helicity formalism helps expose the elegant structure of perturbative field theory results.

For example the 220 tree diagrams at 6pt can be summarised in 6 quantities, the most complicated of which looks like:

$$\frac{-1i\langle 26\rangle^4[35]^4}{\langle 12\rangle\langle 16\rangle[34][45]\langle 2|1+6|5]\langle 6|1+2|3]s_{345}}+$$

$$(345612,\ \mathrm{False})+$$

$$(561234,\ \mathrm{False})$$

Figure: *6g_pmpmpm/tree*

This Elegance Is Often Hidden

In the following the code might not have chosen the best spurious singularities. Still, it is a valid analytical expression for a 7 gluons box coefficient.

$$\frac{[23]\langle 34\rangle^4[34](-1/2i\langle 12\rangle^4[16]^4+2i\langle 23\rangle[36]\langle 12\rangle^3[16]^3-3i\langle 23\rangle^2[36]^2\langle 12\rangle^2[16]^2+2i\langle 23\rangle^3[36]^3\langle 12\rangle[16]-1/2i\langle 23\rangle^4[36]^4)}{s_{567}\langle 4|5+6|7]\langle 1|6+7|5][67][56]\langle 24\rangle^4\langle 12\rangle}+$$

$$\frac{[23]\langle 34\rangle^4[34][46](2i\langle 12\rangle^3[16]^3-6i\langle 23\rangle[36]\langle 12\rangle^2[16]^2+6i\langle 23\rangle^2[36]^2\langle 12\rangle[16]-2i\langle 23\rangle^3[36]^3)}{s_{567}\langle 4|5+6|7]\langle 1|6+7|5][67][56]\langle 24\rangle^3\langle 12\rangle}+$$

$$\frac{[23]\langle 34\rangle^4[34][46]^2(-3i\langle 12\rangle^2[16]^2+6i\langle 23\rangle[36]\langle 12\rangle[16]+2i\langle 24\rangle\langle 12\rangle[46][16]-3i\langle 23\rangle^2[36]^2-2i\langle 23\rangle\langle 24\rangle[36][46]-1/2i\langle 24\rangle^2[46]^2)}{s_{567}\langle 4|5+6|7]\langle 1|6+7|5][67][56]\langle 24\rangle^2\langle 12\rangle}+$$

$$\frac{-1/2i[13]^4[23]\langle 25\rangle^4\langle 34\rangle^8[34]}{[17]\langle 24\rangle^4\langle 45\rangle\langle 56\rangle\langle 4|2+3|1]\langle 4|5+6|7]\langle 2|1+7|5+6|4\rangle\langle 4|2+3|1+7|6\rangle}+$$

$$\frac{\langle 34\rangle^4\langle 25\rangle[13]^3\langle 23\rangle^2[23][34]\langle 45\rangle(2i\langle 35\rangle\langle 34\rangle[13]+2i\langle 45\rangle[14]\langle 34\rangle-2i\langle 35\rangle\langle 46\rangle[16]+2i\langle 45\rangle\langle 36\rangle[16])}{\langle 4|2+3|1+7|6\rangle\langle 2|1+7|5+6|4\rangle\langle 4|5+6|7]\langle 4|2+3|1]\langle 56\rangle\langle 24\rangle^3[17]}+$$

$$\frac{-2i[13]^3\langle 23\rangle^2[23]\langle 34\rangle^4[34]\langle 35\rangle\langle 45\rangle\langle 5|3+4|1]}{[17]\langle 24\rangle^2\langle 56\rangle\langle 4|2+3|1]\langle 4|5+6|7]\langle 2|1+7|5+6|4\rangle\langle 4|2+3|1+7|6\rangle}+$$

$$\frac{\cdots{}^2\langle 34\rangle^2[13]^2-3i\langle 45\rangle^2[14]^2\langle 34\rangle^2+2i[13]\langle 35\rangle^2\langle 34\rangle\langle 46\rangle[16]-2i[13]\langle 35\rangle\langle 34\rangle\langle 45\rangle\langle 36\rangle[16]+6i[14]\langle 35\rangle\langle 34\rangle\langle 45\rangle\langle 46\rangle[16]-6i[14]\langle 34\rangle\langle 45\rangle^2\langle 36\rangle[16]-3i\langle 35\rangle^2\langle 46\rangle^2[16]^2+6i\langle 35\rangle\langle 45\rangle\langle 36\rangle\langle 4\cdots}{\langle 4|2+3|1+7|6\rangle\langle 2|1+7|5+6|4\rangle\langle 4|5+6|7]\langle 4|2+3|1]\langle 56\rangle\langle 45\rangle\langle 24\rangle^2[17]}$$

$$\frac{\cdots](3i\langle 24\rangle[13]^2\langle 35\rangle^3-3i\langle 45\rangle^2\langle 24\rangle[14]^2\langle 35\rangle-6i\langle 35\rangle^2\langle 34\rangle\langle 25\rangle[13]^2+6i\langle 45\rangle^2[14]^2\langle 34\rangle\langle 25\rangle-2i\langle 35\rangle^2[16]\langle 46\rangle\langle 25\rangle[13]+2i\langle 25\rangle[13]\langle 35\rangle\langle 45\rangle\langle 36\rangle[16]-6i\langle 25\rangle[14]\langle 35\rangle\langle 45\rangle\langle 46\rangle[16]+6i\langle 45\rangle\cdots}{\langle 4|2+3|1+7|6\rangle\langle 2|1+7|5+6|4\rangle\langle 4|5+6|7]\langle 4|2+3|1]\langle 56\rangle\langle 45\rangle\langle 24\rangle[17]}$$

$$\frac{\cdots 14]\langle 45\rangle^2\langle 56\rangle\langle 36\rangle[16]^2+2i\langle 25\rangle[13]\langle 45\rangle\langle 56\rangle^2\langle 36\rangle[16]^3+2i\langle 45\rangle^3[14]^3\langle 46\rangle\langle 25\rangle[16]-3i\langle 25\rangle[14]^2\langle 45\rangle^2\langle 56\rangle\langle 46\rangle[16]^2+2i\langle 25\rangle[14]\langle 45\rangle\langle 56\rangle^2\langle 46\rangle[16]^3-1/2i\langle 56\rangle^3\langle 46\rangle\langle 25\rangle[16]^4-2i\langle 45\rangle\langle 3\cdots}{\langle 4|2+3|1+7|6\rangle\langle 2|1+7|5+6|4\rangle\langle 4|5+6|7]\langle 4|2+3|1]\langle 56\rangle\langle 45\rangle\langle 24\rangle[17]}$$

$$\frac{1/2i[23]\langle 34\rangle^4[34]\langle 2|3+4|1+6|7\rangle^4}{\langle 17\rangle\langle 24\rangle^4\langle 67\rangle\langle 1|6+7|5]\langle 2|3+4|5]\langle 4|2+3|1+7|6\rangle s_{167}s_{234}}$$

Figure: *7g_ppmpmpm/box*(77)

Generalising Results for All Multiplicities

Exposing the underlining structure can prove to be very powerful:

$$\frac{1/3i[12]^3\langle 17\rangle^2\langle 25\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 2|1+7|2]^3}+$$
$$\frac{1/2i[12]^2\langle 17\rangle\langle 25\rangle^4\langle 67\rangle}{\langle 12\rangle\langle 23\rangle\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 2|1+7|2]^2}+$$
$$\frac{-2i[12]^2\langle 17\rangle\langle 25\rangle^3\langle 57\rangle}{\langle 12\rangle\langle 23\rangle\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 2|1+7|2]^2}+$$
$$\frac{1i[12]\langle 25\rangle^4\langle 67\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 26\rangle^3\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 2|1+7|2]}+$$
$$\frac{-4i[12]\langle 25\rangle^3\langle 57\rangle\langle 67\rangle}{\langle 12\rangle\langle 23\rangle\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 2|1+7|2]}+$$
$$\frac{6i[12]\langle 25\rangle^2\langle 57\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 2|1+7|2]}+$$
$$\frac{-1/3i[16]^3\langle 17\rangle^2\langle 56\rangle^3}{\langle 16\rangle\langle 23\rangle\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 6|1+7|6]^3}+$$
$$\frac{1/2i\langle 12\rangle[16]^2\langle 17\rangle\langle 56\rangle^3\langle 67\rangle}{\langle 16\rangle^2\langle 23\rangle\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 6|1+7|6]^2}+$$
$$\frac{1i[16]^2\langle 17\rangle^2\langle 56\rangle^3}{\langle 16\rangle^2\langle 23\rangle\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 6|1+7|6]^2}+$$
$$\frac{-1i\langle 12\rangle^2[16]\langle 56\rangle^3\langle 67\rangle^2}{\langle 16\rangle^3\langle 23\rangle\langle 26\rangle^3\langle 34\rangle\langle 45\rangle\langle 6|1+7|6]}+$$
$$\frac{-3i[16]\langle 17\rangle\langle 27\rangle\langle 56\rangle^3}{\langle 16\rangle^2\langle 23\rangle\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 6|1+7|6]}$$

$$\frac{1/3i[12]^3\langle 18\rangle^2\langle 26\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 27\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 2|1+8|2]^3}+$$
$$\frac{1/2i[12]^2\langle 18\rangle\langle 26\rangle^4\langle 78\rangle}{\langle 12\rangle\langle 23\rangle\langle 27\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 2|1+8|2]^2}+$$
$$\frac{-2i[12]^2\langle 18\rangle\langle 26\rangle^3\langle 68\rangle}{\langle 12\rangle\langle 23\rangle\langle 27\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 2|1+8|2]^2}+$$
$$\frac{1i[12]\langle 26\rangle^4\langle 78\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 27\rangle^3\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 2|1+8|2]}+$$
$$\frac{-4i[12]\langle 26\rangle^3\langle 68\rangle\langle 78\rangle}{\langle 12\rangle\langle 23\rangle\langle 27\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 2|1+8|2]}+$$
$$\frac{6i[12]\langle 26\rangle^2\langle 68\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 27\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 2|1+8|2]}+$$
$$\frac{-1/3i[17]^3\langle 18\rangle^2\langle 67\rangle^3}{\langle 17\rangle\langle 23\rangle\langle 27\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 7|1+8|7]^3}+$$
$$\frac{1/2i\langle 12\rangle[17]^2\langle 18\rangle\langle 67\rangle^3\langle 78\rangle}{\langle 17\rangle^2\langle 23\rangle\langle 27\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 7|1+8|7]^2}+$$
$$\frac{1i[17]^2\langle 18\rangle^2\langle 67\rangle^3}{\langle 17\rangle^2\langle 23\rangle\langle 27\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 7|1+8|7]^2}+$$
$$\frac{-1i\langle 12\rangle^2[17]\langle 67\rangle^3\langle 78\rangle^2}{\langle 17\rangle^3\langle 23\rangle\langle 27\rangle^3\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 7|1+8|7]}+$$
$$\frac{-3i[17]\langle 18\rangle\langle 28\rangle\langle 67\rangle^3}{\langle 17\rangle^2\langle 23\rangle\langle 27\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 7|1+8|7]}$$

Figure: $7g\_ppppmpm/bubble(3)$ & $8g\_pppppmpm/bubble(3)$

In the above, the 8 gluon result is trivially obtained by generalising the 7 gluon one to general "n".

Thank you!

3pt Amplitudes from Little Group Scaling

We can obtain the 3pt amplitudes essentially for free!

Assuming $\mathcal{A}(++-)$ depends only on square brackets, we have:

$$\mathcal{A}(++-) \propto [12]^{x_{12}}[13]^{x_{13}}[23]^{x_{23}}$$

Then the Little Group Scalings tell us:

$$\begin{aligned} -2 &= -x_{12} - x_{13} \\ -2 &= -x_{12} - x_{23} \\ 2 &= -x_{13} - x_{23} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_{12} &= 3 \\ x_{13} &= -1 \\ x_{23} &= -1 \end{aligned} \quad \Rightarrow \quad \mathcal{A} \propto \frac{[12]^3}{[13][23]}$$

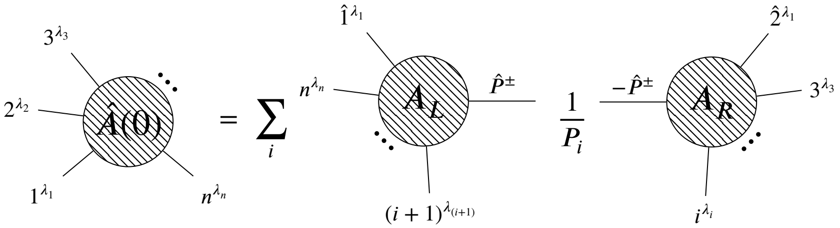
If you assumed it depended only on angle brackets the result would be dimensionally wrong, therefore you know this is the correct one.

Similarly 3pt amplitudes involving not only gluons can be obtained.

Higher Multiplicity Tree Amplitudes from BCFW

We can picture this last formula graphically:

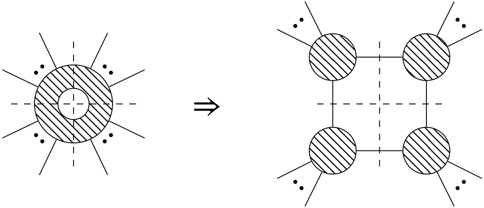
$$\hat{A}(0) = -\sum_i \frac{Res\ \hat{A}(z)|_{z=z_i}}{z_i} = \sum_i \hat{A}_L \frac{1}{P_i^2} \hat{A}_R.$$



One Loop from Tree Level via Generalised Unitarity (Maximal Cut)

So far we discussed (standard) unitairty, where two propagators are set to be on-mass-shell. Although this yields a lot of information, even more powerful result can be obtained by cutting more propagators.

In four dimension the loop momentum has 4 components, therefore we can at most impose 4 on-mass-shell conditions. This is a maximal cut:



Clearly this becomes the product of 4 tree level amplitudes.

Box Coefficients from Maximal Cuts

In terms of our master integral expansion, maximal cuts pick out specific box coefficients.

Summing over the helicities of the cut loop momentum and taking into account a Jacobian factor from the integral over 4 delta functions, we get:

$$d_i = \frac{1}{2}\left(d_i^+ + d_i^-\right)$$
$$d_i^\pm = A_1^{tree}(l^\pm)A_2^{tree}(l^\pm)A_2^{tree}(l^\pm)A_2^{tree}(l^\pm)$$

For triangles and boxes things become a bit more complicated, but essentially triple cuts yield triangle coefficients polluted with box coefficients - since they share the triple poles. Similarly double (standard) cuts yield box coefficients polluted with boxes and triangles. Also, note how triple and double cuts don't fully fix the loop momentum, therefore some integration is still necessary in those cases.

Coefficient Pollution