Integrable interpolations in σ-models and in supergravity

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- Construction: Nucl. Phys. **B880** (2014) 225, arXiv:1312.4560
- RG flows:
  - Phys. Lett. **B733** (2014) 265, arXiv:1404.3748 (with G. Itsios and K. Siampos)
  - Nucl. Phys. **B885** (2014) 583, arXiv:1405.7803 (with K. Siampos)
- Supergravity embedding: to appear (with D.Thompson)

General settings and context – Motivation

Non-Abelian T-duality

In the familiar Buscher's T-duality, $U(1)$ isometry

$$ds^2 = R^2 d\phi^2 + \cdots \quad \Longrightarrow \quad ds^2 = \frac{1}{R^2} d\phi^2 + \cdots \ .$$

Small/large symmetry. Variables, i.e. ϕ, remain compact.

Non-Abelian isometries: S^3 metric, has $SU(2)_L \times SU(2)_R$ isometry

- The T-dual metric w.r.t. $SU(2)_L$ is given by

$$ds^2 = dr^2 + \frac{r^2}{r^2+1} ds^2(S^2) \ .$$

- Less symmetry than the original, more like a warping.
- The variable r appears non-compact (unlike original).
  Global properties not well understood.

I will show how to understand these issues

Integrable σ-models

- Perturbations preserving integrability are non-trivial to find.
- When the perturbation parameter(s) grow large more difficult.

Possible in a class of WZW – like sigma – models

Exact β-functions in FT

- Quantum behaviour is encoded in the β-function eqs.
- 1st order non-linear dif. systems. Traditionally perturbatively.
- Is it possible to compute the β-function exactly?
  New fixed point theories towards the IR. A very difficult task.

Exact beta – function for the above class of sigma – models

Spacetimes

Embedding to type – II supergravity

Description and general properties of the models

The action

- We start with

$$S = S_{\mathrm{WZW},k}(g) + S_{\mathrm{PCM},\lambda}(\tilde{g}) \ ,$$

where $k \in \mathbb{Z}^{+}$, $g, \tilde{g} \in G$ nd λ is a real $\dim(G)$ square matrix.

- By gauging a subgroup acting as $g \to \Lambda^{-1} g \Lambda$, $\tilde{g} \to \Lambda^{-1}\tilde{g}$, with $\Lambda \in G$ and gauge fixing $\tilde{g} = \mathbb{I}$ we construct [KS 13]

$$\boxed{S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi}\int J_{+}^{a}(\lambda^{-1} - D^{T})^{-1}_{ab} J_{-}^{b}} \ ,$$

- The standard definitions are

$$J_{+}^{a} = -i\mathrm{Tr}(T_a \partial_{+} g g^{-1}) \ , \quad J_{-}^{a} = -i\mathrm{Tr}(T_a g^{-1}\partial_{-} g) \ ,$$
$$D_{ab} = \mathrm{Tr}(T_a g T_b g^{-1}) \ .$$

T_a, $a = 1, 2, \ldots, \dim(G)$ generators of the Lie algebra.

Basic properties

- For small λ_{ab} it becomes the non-Abelian anisotropic (bosonised) Thirring action

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi}\int \lambda_{ab} J_+^a J_-^b + \cdots \ .$$

- Symmetry broken for a generic matrice λ. For $\lambda_{ab} = \lambda\delta_{ab}$ its have $G_L \times G_R$.
- The theory is driven away from the conformal point; Relevant perturbation [Kadanoff-Brown 79, Chaudhuri-Schwartz 89].

$$\frac{d\lambda_{ab}}{dt} = \cdots \ , \qquad t = \ln\mu \ .$$

- A duality-type symmetry

$$\boxed{S_{-k,\lambda^{-1}}(g^{-1}) = S_{k,\lambda}(g)} \ .$$

Profoundly constrains all its properties.

Integrability

If we can cast the equations of motion in terms of current components as

$$\partial_\pm I^a_\mp = \mp\frac{1}{2} f_{abc} I^b_+ I^c_- \ .$$

- Construct a classical Lax pair as

$$\partial_+ L^a_- - \partial_- L^a_+ = f_{abc} L^b_+ L^c_- \quad \text{or} \quad \mathrm{d}L = L \wedge L \,,$$
$$L_\pm(\nu) = -\frac{\nu}{\nu \mp 1} I_\pm \ ,$$

  where $\nu \in \mathbb{C}$ is the spectral parameter.
- Then a classical monodromy matrix

$$M = P\exp\int_{-\infty}^{+\infty} \mathrm{d}\sigma\, L_1 \,, \qquad \partial_0 M = 0 \ .$$

- Expand in ν an infinite many conserved charges.

Integrability of the σ-model proven in special cases:

- For any semi-simple group G and $\lambda_{ab} = \lambda\delta_{ab}$ [KS 13]
- For $G = SU(2)$ and anisotropic $\lambda = \mathrm{diagonal}(\lambda_1, \lambda_2, \lambda_3)$ [KS-Siampos 14, to appear]
- For symmetric coset G/H spaces [Hollowood-Miramontes-Schmidtt 14] (then necessarily λ is the Cartan–Killing metric)

Algebraic structure:
The $I_\pm^a$'s obey a Poisson algebra; deformation of that for PCM.

Computing the RG flow eqs

Under an RG flow the $G_{\mu\nu}$ and $B_{\mu\nu}$ change. The one-loop β-functions are

$$\frac{\mathrm{d}G_{\mu\nu}}{\mathrm{d}t} + \frac{\mathrm{d}B_{\mu\nu}}{\mathrm{d}t} = R^-_{\mu\nu} + \underbrace{\nabla^-_\nu \xi_\mu}_{\text{diffs}} \ ,$$

- The covariant derivatives and tensors include the torsion.
- In several instances $G_{\mu\nu}$ and $B_{\mu\nu}$ retain their form. Then what flows with energy are the couplings, i.e. the λ_{ab}'s and the fields, i.e. coordinates in $g \in G$.
- A tour de force computation.
  - Performing a diffeomorphism along the flow is instrumental.
  - I skip the details.

The RG-flow equations

A general expression can be given for $\frac{d\lambda_{ab}}{dt}$ [KS-Siampos 14].

For the diagonal case $\lambda_{ab} = \lambda\delta_{ab}$ it is simply [Itsios-KS-Siampos 14]

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\frac{c_G\,\lambda^2}{2k(1+\lambda)^2}}\,,$$

where c_G the quadratic Casimir in the adjoint rep.

- Sums up all perturbative contributions.
- For large rank groups exact coincides with perturbative.

The anisotropic $SU(2)$ case

Consider the case with $G = SU(2)$ and

$$\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \ .$$

Then the RG flow eqs become

$$\boxed{\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = -\frac{2}{k}\frac{(\lambda_2 - \lambda_1\lambda_3)(\lambda_3 - \lambda_1\lambda_2)}{(1-\lambda_2^2)(1-\lambda_3^2)}} \ , \quad \text{and cyclic in } 1,2,3.$$

Choosing λ diagonal is RG flow compatible.

- For $\lambda_a \ll 1$ becomes the the Lagrange system ("Euclidean version" of Euler's top eqs)
- For $\lambda_a \to 1$ the Darboux–Halphen system (monopole moduli space – Atiyah–Hitchin metric)

Prototype for "regularizing" non-Abelian T-duality

Consider the case with

$$\lambda_{ab} = \lambda \delta_{ab} \ .$$

Let $G = SU(2)$ and parametrize

$$g = e^{i\alpha \hat{n}_i \sigma_i} \ , \qquad \hat{\mathbf{n}} = (-\sin\beta \sin\gamma, \sin\beta\cos\gamma, \cos\beta) \ ,$$

The corresponding σ-model has metric

$$ds^2 = k\left(\frac{1+\lambda}{1-\lambda} d\alpha^2 + \frac{1-\lambda^2}{\Delta(\alpha)} \sin^2\alpha \ ds^2(S^2)\right) \ ,$$

and antisymmetric tensor

$$B = k\left(-\alpha + \frac{(1-\lambda)^2}{\Delta(\alpha)} \cos\alpha \sin\alpha\right) \mathrm{Vol}(S^2) \ ,$$

where

$$\Delta(\alpha) = (1-\lambda)^2 \cos^2\alpha + (1+\lambda)^2 \sin^2\alpha \ .$$

- The $SU(2)$ WZW: In the UV $\lambda \to 0$ we obtain

$$S^3: \qquad ds^2 = k\left(d\alpha^2 + \sin^2\alpha\ ds^2(S^2)\right) ,$$
$$B = -k\alpha \mathrm{Vol}(S^2) .$$

- The non-Abelian T-dual: Towards the IR at $\lambda \to 1$ the limit is more delicate. Letting

$$\alpha = \frac{r}{2k} , \qquad \lambda = 1 - \frac{1}{k} + \cdots , \qquad k \to \infty ,$$

gives

$$ds^2 = dr^2 + \frac{r^2}{r^2+1} ds^2(S^2) , \qquad B = -\frac{r^3}{r^2+1} \mathrm{Vol}(S^2) .$$

This is a zooming into the geometry. It explains the non-compactness of the variables.

Spacetimes - type-II Supergravity

We aim at using these σ-models as building blocks for constructing solutions of type-II Supergravity.

Need to:

- Decide which part of a 10-dim space to deform
- Use for the NS-sector the σ-models field and for the dilaton

$$\Phi = -\frac{1}{2}\,\mathrm{det}(\lambda^{-1} - D^T)\ .$$

- Support these NS-sector with RR-fluxes.
- Find the rules determining these RR fields.
  It turns out that these are essentially the same as those for non-Abelian T-duality [KS-Thmopson 10].

We have constructed several examples of λ-deformations

[KS-Thompson 14, to appear]:

- $AdS_3 \times S^3 \times T^4$ using the $SU(2) \times SL(2, \mathbb{R})$ isometry of the group.
- $AdS_2 \times S^2 \times T^6$ using the $SU(2) \times SL(2, \mathbb{R})$ isometry of the coset
- $AdS_3 \times S^3 \times T^4$ using the $SU(2) \times SU(2) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ isometry of the group

Explicit example: A new twist to the old black hole

The NS sector: The metric is

$$ds^2 = k\left(\frac{1-\lambda}{1+\lambda}(-\coth^2\rho dt^2 + d\rho^2) + \frac{4\lambda}{1-\lambda^2}(\cosh t d\rho + \sinh t \coth\rho dt)^2\right)$$
$$+ k\left(\frac{1-\lambda}{1+\lambda}(d\omega^2 + \cot^2\omega d\phi^2) + \frac{4\lambda}{1-\lambda^2}(\cos\phi d\omega + \sin\phi\cot\omega d\phi)^2\right)$$
$$+ \sum_{i=4}^{9} dx_i^2 \; .$$

- 1st line: A deformation of the $SL(2,\mathbb{R})/U(1)$ exact CFT.
- 2nd line: A deformation of the $SU(2)/U(1)$ exact CFT.

In addition there is a dilaton

$$e^{-2\Phi} = \sin^2\omega \sinh^2\rho \; .$$

The antisymmetric vanishes.

The RR-sector:

- First define the frames

$$e^0 = \sqrt{k\frac{1-\lambda}{1+\lambda}}(\sinh t d\rho + \cosh t \coth \rho dt) \ , \quad e^1 = \cdots \ .$$

so that the metric is

$$ds^2 = \eta_{ab} e^a e^b \ , \qquad \eta^{ab} = \mathrm{diag}(-1, 1, \ldots, 1) \ .$$

- In $\mathbb{R}^6$ denote by

$$J_2 : \quad \text{Kahler form} \ ,$$
$$J_3 : \quad \text{Real part of complex differential form of type } (3,0) \ .$$

Then, there are two possibilities:

- Type-IIB

$$F_5 = (1 + \star) f_5 \ ,$$
$$f_5 = \frac{1}{\sqrt{k}} \sqrt{\frac{4\lambda}{1-\lambda^2}} \sin\omega \sinh\rho \ e^0 \wedge e^3 \wedge J_3 \ .$$

- Type–IIA

$$F_2 = \frac{1}{\sqrt{k}} \sqrt{\frac{4\lambda}{1-\lambda^2}} \sin\omega \sinh\rho \ e^0 \wedge e^3 \ ,$$
$$F_4 = \frac{1}{\sqrt{k}} \sqrt{\frac{4\lambda}{1-\lambda^2}} \sin\omega \sinh\rho \ e^1 \wedge e^2 \wedge J_2 \ .$$

Deformation of the black hole found in [Witten 91].

Concluding remarks

- New integrable theories, interpolating between exact CFT WZW or gauged WZW models and the non-Abelian T-duals of PCM for group or coset spaces.
- The action is the effective all-loop action for the non-Abelian (bosonized) anisotropic Thirring model.
  Produced same RG flow eqs as the standard Thirring action, computed with CFT methods by [Gerganov-LeClair-Moriconi 01].
- The isotropic models have a Yangian symmetry and provide non-trivial solutions to the modified Yang-Baxter equation via the computation of the Maillet brackets [Itsios-KS-Siampos-Torrielli, 14].
- A type-II supergravity embedding with non-trivial RR fields is possible [KS-Thompson, to appear].
- Related work on these models by [Hollowood-Miramontes-Schmidtt 14] who show that they correspond to quantum deformations in the string theory with q a root of unity.