Stability of black holes and solitons in Anti-de Sitter space-time

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Work done in collaboration with:

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Outline

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AdS/CFT correspondence

- d-dim gravity theory in Anti-de Sitter (AdS) dual to (d-1)-dim SU(N) gauge theory on boundary of AdS
(Maldacena; Witten; Gubser, Klebanov & Polyakov (1998))
- Couplings

$$\lambda = (\ell/l_s)^4 = g^2 N \quad , \quad g_s \sim g^2$$

λ: 't Hooft coupling	N: rank of gauge group
ℓ: bulk scale/AdS radius	l_s: string scale
g: gauge coupling	g_s: string coupling

- classical gravity limit:

$$g_s \to 0 \qquad l_s/\ell \to 0$$

dual to **strongly coupled QFT with** $N \to \infty$

Breitenlohner-Freedman (BF) bound

(Breitenlohner & Freedman, 1982)

massive scalar field (mass parameter m^2) in AdS background

- Schrödinger-type equation for scalar field $\psi(t,r) \sim \exp(i\omega t)\phi(r)$

$$-\phi'' + \frac{r^2}{\ell^2}\left(m^2 - \left(\frac{d(d-2)}{4}\right)\right)\phi = \omega^2\phi$$

- for massive scalar fields in AdS with

$$m^2 \geq m^2_{\mathrm{BF,d}} \equiv -\frac{(d-1)^2}{4}$$

there are NO normalizable negative energy states

asymptotically AdS black holes

asymptotically AdS (aAdS) black holes with $m^2 > m^2_{\rm BF,d}$

- **uncharged** scalar field
  near-horizon geometry of **(near)-extremal** ($T_{\rm H} \approx 0$) black holes

$$AdS_2 \times M_{d-2} \ \ , \ \ M \ \mathrm{manifold}$$

  (Robinson (1959); Bertotti (1959); Bardeen & Horowitz (1999))
  if $m^2 < m^2_{\rm BF,2} \Rightarrow$ near-horizon AdS$_2$ unstable

- **scalar field charged** under $U(1)$, charge e

$$m^2_{\rm eff} = m^2 - e^2|g^{tt}|A_t^2$$

  $e^2|g^{tt}|$ large close to horizon if $T_{\rm H}$ small enough $\Rightarrow m^2_{eff} < m^2_{\rm BF,d}$
  $\Rightarrow$ near-horizon geometry unstable (Gubser (2008))

aAdS black holes form scalar “hair” below a given temperature

Outline

Action

Gauss-Bonnet gravity + scalar field ψ + U(1) gauge field A_μ

$$\begin{aligned}\mathcal{S} &= \frac{1}{16\pi G}\int d^5x\sqrt{-g}\,(16\pi G\mathcal{L}_{\rm matter} \\ &+ R - 2\Lambda + \frac{\alpha}{4}\left(R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right)\Big)\end{aligned}$$

with matter Lagrangian

$$\mathcal{L}_{\rm matter} = -\frac{1}{4}F_{MN}F^{MN} - (D_M\psi)^*\,D^M\psi - m^2\psi^*\psi \;\;,\;\; M,N = 0,1,2,3,4$$

$F_{MN} = \partial_M A_N - \partial_N A_M$ field strength tensor
$D_M\psi = \partial_M\psi - ieA_M\psi$ covariant derivative
$\Lambda = -6/L^2$: cosmological constant
G: Newton's constant , α: Gauss–Bonnet coupling
e: gauge coupling , m^2: mass of the scalar field ψ

Ansatz for static solutions

- Metric

$$ds^2 = -f(r)a^2(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}d\Sigma^2_{k,3}$$

with

$$d\Sigma^2_{k,3} = \begin{cases} d\Xi^2_3 & \text{for } \; k = -1 \; \text{ hyperbolic} \\ dx^2 + dy^2 + dz^2 & \text{for } \; k = 0 \; \text{ flat} \\ d\Omega^2_3 & \text{for } \; k = 1 \; \text{ spherical} \end{cases}$$

- Matter fields

$$A_M dx^M = \phi(r)dt \quad , \quad \psi = \psi(r)$$

Behaviour on the AdS boundary: matter fields

- **Gauge field**

$$\phi(r \gg 1) = \mu - \frac{Q}{r^2}$$

Q: charge ($k = 1$); charge density ($k = -1$, $k = 0$)

- **Scalar field**

$$\psi(r \gg 1) = \frac{\psi_-}{r^{\lambda_-}} + \frac{\psi_+}{r^{\lambda_+}}$$

with

$$\lambda_\pm = 2 \pm \sqrt{4 + m^2 L_{\rm eff}^2} \ , \ \ L_{\rm eff}^2 \equiv \frac{2\alpha}{1 - \sqrt{1 - 4\alpha/L^2}}$$

Behaviour on the AdS boundary: metric fields

- **Asymptotic behaviour**

$$f(r \gg 1) = k + \frac{r^2}{L_{\rm eff}^2} + \frac{f_2}{r^2} + O(r^{-4})$$
$$a(r \gg 1) = 1 + \frac{a_4}{r^4} + O(r^{-6})$$

f_2, a_4 constants (have to be computed numerically) and determine...

- ... **Energy** E of the solutions

$$\frac{16\pi G E}{V_3} = \sqrt{1 - \frac{\alpha}{L^2}} \left(-3 f_2 - 8 \frac{a_4}{L_{\rm eff}^2}\right)$$

V_3: volume of the 3-dimensional space

Black holes

- **Possess regular horizon** r_h with

$$f(r_h) = 0 \quad , \quad a(r_h) \ \ \text{finite}$$

- **entropy** S

$$\frac{S}{V_3} = \frac{r_h^3}{4G}\left(1 - \frac{6\alpha}{r_h^2}\right)$$

- **Temperature**

$$T_{\rm H} = \frac{f'(r_h)a(r_h)}{4\pi}$$

- **Free energy** $F = E - T_{\rm H}S - \mu Q$

Solitons

- Globally regular, no temperature naturally associated
- **Free energy** $F = E - \mu Q$
- $k = 1$ solutions exist for $r \in [0 : \infty[$ with

$$f(r=0) = 1 \quad , \quad a'(r)|_{r=0} = 0$$

- $k = 0$ solutions exist for $r \in [r_0 : \infty[$ with

$$f(r_0) = 0 \quad , \quad a(r_0) \text{ finite}$$

AND **either** x **or** y **or** z **needs to be compact with period** $\tau(r_0)$

Outline

Black holes without scalar hair

(Boulware & Deser, 1982; Cai, 2003)

$$\psi(r) \equiv 0 \quad , \quad \phi(r) = \frac{Q}{r_h^2} - \frac{Q}{r^2}$$

$$f(r) = k + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 - \frac{4\alpha}{L^2} + \frac{4\alpha M}{r^4} - \frac{4\alpha\gamma Q^2}{r^6}}\right) \quad , \quad a(r) \equiv 1$$

M: mass parameter
Q: charge (density)
r_h: event horizon

Special black hole cases for $k = -1$

- uncharged hyperbolic aAdS black holes $Q = 0$, $k = -1$
- **extremal solution** with $f(r_h^{(\mathrm{ex})}) = 0$, $f'(r)|_{r=r_h^{(\mathrm{ex})}} = 0$ exists
- $r_h^{(\mathrm{ex})} = L/\sqrt{2}$
- for $T_{\mathrm{H}} \approx 0$: horizon topology is $AdS_2 \times H^3$
  (Astefanesei, Banerjee & Dutta, 2008)
- Gauss-Bonnet black holes in $d = 5$ have AdS_2 radius

$$R = \sqrt{L^2/4 - \alpha}$$

(Y. Brihaye & B.H., PRD 84, 2011)

Special black hole cases for $k = 0$

- $\alpha = 0$, $Q = 0$, $k = 0$: **planar Schwarzschild-AdS**

$$ds^2 = -\left(\frac{r^2}{L^2} - \frac{{r_h}^4}{r^2 L^2}\right) dt^2 + \left(\frac{r^2}{L^2} - \frac{{r_h}^4}{r^2 L^2}\right)^{-1} dr^2 + \frac{r^2}{L^2}\left(dx^2 + dy^2 + dz^2\right)$$

- $\alpha = 0$, $Q \neq 0$, $k = 0$: **planar Reissner-Nordström-AdS**

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}\left(dx^2 + dy^2 + dz^2\right)$$

with

$$f(r) = \frac{r^2}{L^2} - \frac{2M}{r^2} + \frac{GQ^2}{r^4} \quad , \quad A_M dx^M = \frac{Q}{r_h^2}\left(1 - \frac{r_h^2}{r^2}\right) dt$$

Planar AdS soliton without scalar hair for $k = 0$

- **double Wick rotation** of planar Schwarzschild-AdS:

$$\begin{aligned} ds^2 &= -\frac{r^2}{L^2}dt^2 + \left(\frac{r^2}{L^2} - \frac{{r_0}^4}{r^2L^2}\right)^{-1} dr^2 + \left(\frac{r^2}{L^2} - \frac{{r_0}^4}{r^2L^2}\right) d\eta^2 \\ &+ \frac{r^2}{L^2}\left(dx^2 + dy^2\right) \end{aligned}$$

where η **periodic** with period

$$\tau_\eta = \frac{\pi L^2}{r_0} \qquad \text{where } r_0 > 0$$

Outline

Uncharged black holes with $k = -1$

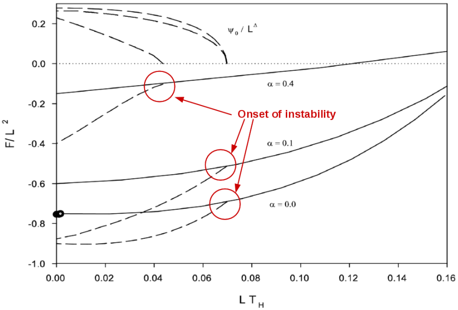
- Uncharged black holes $Q = 0$
- **uncharged scalar field** $e = 0$
- asymptotic AdS_5 stable, near-horizon AdS_2 unstable for

$$m^2_{\mathrm{BF},5} = -\frac{4}{L^2_{\mathrm{eff}}} \le m^2 \le -\frac{1}{4R^2} = m^2_{\mathrm{BF},2}$$

Uncharged black holes with $k = -1$

(Y. Brihaye & B.H., PRD 84, 2011)

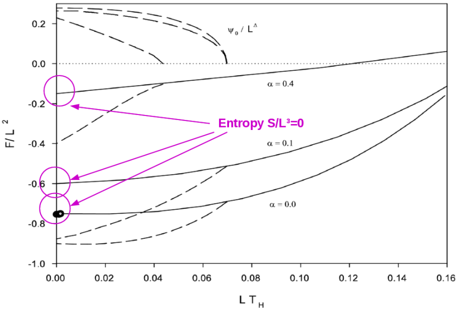
- black holes with scalar hair **thermodynamically preferred**



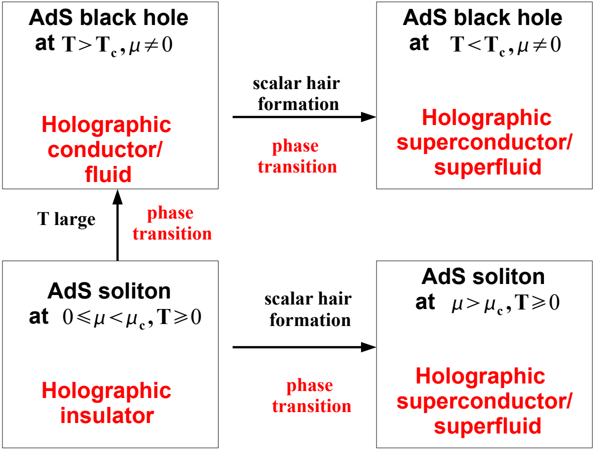
Uncharged black holes with $k = -1$

(Y. Brihaye & B.H., PRD 84, 2011)

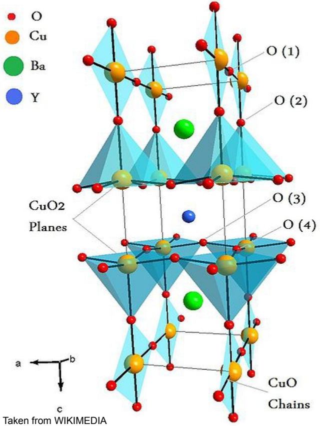
- the larger α the lower $T_{\rm H}$ at which instability appears



Charged black holes and solitons with $k = 0$



Applications to holographic superconductors



Taken from WIKIMEDIA

- Example of high temperature superconductor: Yttrium(Y)-barium(BA)-copper(Cu)-oxide(O)
- highest possible $T_c = 92$K
- **superconductivity associated to CuO_2-planes**
- BCS theory does not explain experimental results well
- **strongly interacting Quantum field theories**

Charged black holes with $k = 0$

- **Mermin-Wagner theorem**: spontaneous symmetry breaking of a continuous symmetry forbidden in (2+1) dimensions at finite temperature, but holographic superconductors (in Einstein gravity) have been constructed (see e.g. (Hartnoll, Herzog & Horowitz, 2008))

Q: Can Gauss-Bonnet corrections suppress condensation?

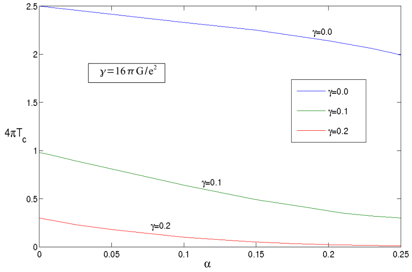
- for $G = 0$ (no backreaction): condensation can **not** be suppressed for (3+1)-dimensional Holographic Gauss-Bonnet superconductors (Gregory, Kanno & Soda, 2009)

Q: Can backreaction suppress condensation?

Charged black holes with $k = 0$

(Brihaye & B. Hartmann, Phys. Rev. D 81, 2010)

- Gauss-Bonnet coupling $0 \leq \alpha \leq L^2/4$



$\Longrightarrow$ **condensation gets harder for $\alpha > 0$, but not suppressed**

Charged solitons with $k = 1$ and $\alpha = 0$

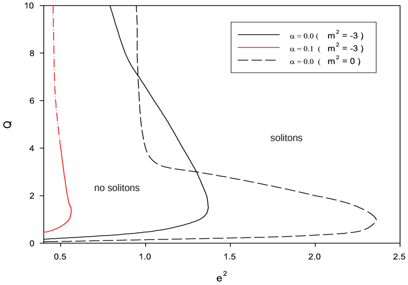
- For $\alpha = 0$ solitons regular on $r \in [0 : \infty[$ exist
(Basu, Mukherjee & Shieh, 2009;
Dias, Figueras, Minwalla, Mitra, Monteiro & Santos, 2011)
- Need: appropriate boundary conditions for scalar and gauge field at $r = 0$

$$\phi'(0) = 0 \ \ , \ \ \psi'(0) = 0$$

Charged solitons with $k = 1$ and $\alpha \neq 0$

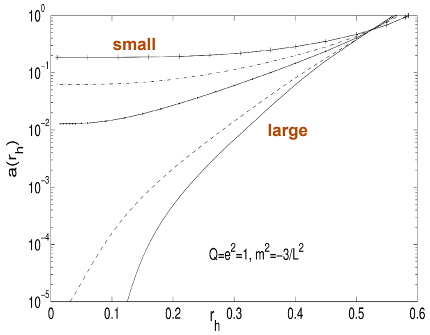
(Brihaye & B. Hartmann, PRD 85 (2012) 124024)

- exist only in limited domain of Q-e^2-plane
- at $e = e_c(Q, \alpha)$: numerical results suggest $a(0) \to 0$
- **for $\alpha \neq 0$ range of allowed Q and e values enlarged**



Charged black hole with $k = 1$

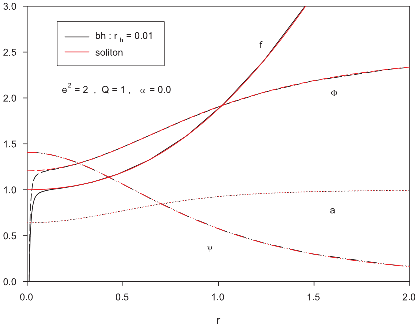
(Brihaye & B. Hartmann, PRD 85 (2012) 124024)



- For small α: solution exists down to $r_h = 0$
  $\rightarrow$ **soliton?**
- For large α: solution has $a(r_h) \rightarrow 0$ for $r_h \rightarrow r_h^{(cr)} > 0$
  $\rightarrow$ **extremal black hole?**

Charged black hole with $k = 1$

(Brihaye & B. Hartmann, PRD 85 (2012) 124024)

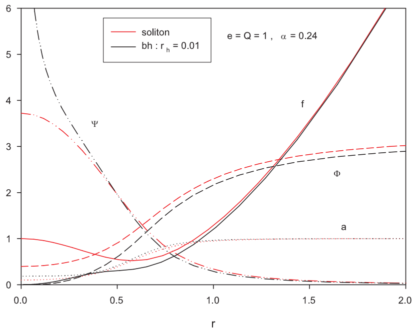


For $\alpha = 0$:

- Black hole tends to soliton solutions in the limit $r_h \to 0$

Charged black hole with $k = 1$

(Brihaye & B. Hartmann, PRD 85 (2012) 124024)



$\alpha \neq 0$:

- Gauss-Bonnet solitons with scalar hair exist
- black holes with scalar hair do **not** tend to corresponding solitons for $r_h \to 0$

Charged black hole with $k = 1$

(Brihaye & B. Hartmann, PRD 85 (2012) 124024

There exist no extremal Gauss-Bonnet black holes with scalar hair.

Charged black hole with scalar hair, $k = 1$

Proof:

- assume near-horizon geometry to be $AdS_2 \times S^3$:

$$ds^2 = v_1 \left(-\rho^2 d\tau^2 + \frac{1}{\rho^2} d\rho^2\right) + v_2 \left(d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)$$

  v_1, v_2: positive constants

- Combination of equations of motion yields

$$0 = 16\pi G \left(\frac{\rho^2}{v_1}\psi'^2 + \frac{e^2\phi^2\psi^2}{\rho^2 v_1}\right)$$

  This leads to: $\psi' = 0$ and $\phi^2\psi^2 = 0$ in near horizon geometry

- $\phi^2 = 0$ ruled out $\rightarrow \psi \equiv 0$ **in near horizon geometry** q.e.d.

Conclusions & Outlook

Conclusions

- Two mechanisms can make AdS black holes unstable to scalar condensation
  - **uncharged scalar field**: black holes with AdS_2 factor in near-horizon geometry (near-extremal black holes)
    Example in this talk: uncharged, static black holes with hyperbolic horizon ($k = -1$)
  - **charged scalar field**: coupling to gauge field lowers effective mass of scalar field
    Examples in this talk: charged, static black holes with flat or spherical horizon ($k = 0$, $k = 1$)
- Black holes with scalar hair thermodynamically preferred

Outlook

- Instabilities of other black holes and solitons in AdS_d
  - charged and rotating Einstein black holes:
    Y. Brihaye & B.H., JHEP 1203 (2012) 050
  - charged and rotating Gauss-Bonnet black holes:
    Y. Brihaye, B.H. & S. Tojiev, Phys. Rev. D (2013) 024040
  - solitons in conformal gravity and/or scalar field models
    Y. Brihaye, B.H. & S. Tojiev, Phys. Rev. D 88 (2013) 104006

THANK YOU FOR YOUR ATTENTION!

Equations of motion

$$
\begin{aligned}
f' &= 2r\frac{k-f+2r^2/L^2}{r^2+2\alpha(k-f)} \\
&- \gamma\frac{r^3}{2fa^2}\left(\frac{2e^2\phi^2\psi^2+f(2m^2a^2\psi^2+\phi'^2)+2f^2a^2\psi'^2}{r^2+2\alpha(k-f))}\right) \\
a' &= \gamma\frac{r^3(e^2\phi^2\psi^2+a^2f^2\psi'^2)}{af^2(r^2+2\alpha(k-f))} \\
\phi'' &= -\left(\frac{3}{r}-\frac{a'}{a}\right)\phi'+2\frac{e^2\psi^2}{f}\phi \\
\psi'' &= -\left(\frac{3}{r}+\frac{f'}{f}+\frac{a'}{a}\right)\psi'-\left(\frac{e^2\phi^2}{f^2a^2}-\frac{m^2}{f}\right)\psi
\end{aligned}
$$

where $\gamma = 16\pi G$

Conditions on the horizon (Black holes)

- Regular horizon at $r = r_h > 0$

$$f(r_h) = 0 \quad , \quad a(r_h) \ \text{finite}$$

$$\phi(r_h) = 0 \ , \ \psi'(r_h) = \left. \frac{m^2\psi\left(r^2 + 2\alpha k\right)}{2rk + 4r/L^2 - \gamma r^3\left(m^2\psi^2 + \phi'^2/(2a^2)\right)} \right|_{r=r_h}$$

The planar limit

($\alpha = 0$: Gentle, Pangamani & Withers, (2011))
($\alpha \neq 0$: Brihaye & B. Hartmann, PRD 85 (2012) 124024)

- apply rescaling $r \to \lambda r$, $t \to \lambda^{-1} t$, $M \to \lambda^4 M$, $Q \to \lambda^3 Q$ to Boulware-Deser-Cai solution with

$$f(r) = r^2 \left(\frac{1}{r^2} + \frac{1}{2\alpha} \left(1 - \sqrt{1 - \frac{4\alpha}{L^2} + \frac{4\alpha M}{r^4} - \frac{4\alpha\gamma Q^2}{r^6}} \right) \right)$$

- For $\lambda \to \infty$: $\lambda^2 d\Omega_3^2 \to dx^2 + dy^2 + dz^2$
- For $Q \to \infty$ the $k = 1$ **solution becomes** $k = 0$ **solution**, i.e. becomes comparable in size to AdS radius L
- electromagnetic repulsion balanced by gravitational attraction present in AdS