Generalised Special Holonomy and $\mathcal{N} = 1$ Backgrounds

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Geometry of Supersymmetric Backgrounds

Consider a compactification of string theory to four-dimensional Minkowski spacetime. If the background is supersymmetric, then in the absence of fluxes the 6d internal manifold must satisfy

(Candelas, Horowitz, Strominger, Witten '85)

$$\nabla \epsilon = 0$$

$$\downarrow$$

Calabi-Yau 3-fold

This is an example of a compact manifold with special holonomy, i.e. a manifold in which there exist spinor fields parallel with respect to the Levi–Civita connection.

Geometry of Supersymmetric Backgrounds

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$$[\nabla + (\mathrm{Flux})\,] \cdot \epsilon = 0$$

If fluxes are turned on, the compatible connection is not torsion-free, so it is not a special holonomy manifold. So what is the geometry of the internal manifold?

- G-structures
- Generalised Complex Geometry
- Exceptional Generalised Geometry

Field Ansatz for Eleven-Dimensional Supergravity

Focus on eleven-dimensional supergravity reduced to four dimensions (but results also hold for IIA and IIB).

We keep only the components of the eleven-dimensional fields which are scalars in the external space.

Therefore we take the metric to be

$$\mathrm{d}s_{11}^2 = \mathrm{e}^{2\Delta}\eta_{\mu\nu}\mathrm{d}y^\mu\mathrm{d}y^\nu + g_{mn}\mathrm{d}x^m\mathrm{d}x^n,$$

and keep the components of the 4-flux $\mathcal{F}$

$$F_{m_1\dots m_4} = \mathcal{F}_{m_1\dots m_4}, \qquad \tilde{F}_{m_1\dots m_7} = (*_{11}\mathcal{F})_{m_1\dots m_7}\,.$$

These field strengths are globally defined closed forms, which means that we have "gerbe"-like gauge fields, the 3-form A_{mnp} and the 6-form $\tilde{A}_{mnpqrs}$.

The fermionic content is given by two components of the gravitino Ψ_M, the internal gravitino ψ_m and the trace of the external component ρ.

The Killing Spinor Equations

For supersymmetric vacua we set the fermions to zero and require the existence of at least one spinor ε globally defined on M such that the supersymmetric variations of all the fields with respect to ε vanish.

This implies that

$$\delta\rho = \left[\nabla\!\!\!/ - \tfrac{1}{4}F\!\!\!/ - \tfrac{1}{4}\tilde{F}\!\!\!/ + (\partial\!\!\!/\Delta)\right]\varepsilon = 0$$

$$\delta\psi_m = \big[\nabla_m + \tfrac{1}{288}F_{n_1\ldots n_4}\left({\Gamma_m}^{n_1\ldots n_4} - 8{\delta_m}^{n_1}\Gamma^{n_2n_3n_4}\right) - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{mn_1\ldots n_6}\Gamma^{n_1\ldots n_6}\big]\varepsilon = 0$$

These are the Killing Spinor Equations and we call ε the Killing spinor.

More independent Killing spinors imply that more supersymmetry is preserved.

$E_{7(7)} \times \mathbb{R}^+$ Generalised Geometry

Generalised Geometries, in analogy to the relation between Riemannian geometry and general relativity, are a new attempt at "geometrising" the bosonic symmetries of supergravity.

It introduces an extended notion of tangent space, where generalised vectors are patched together precisely according to the supergravity symmetries. By studying structures on these generalised tangent spaces, we can gain new insights into supergravity.

In previous work we showed that $E_{d(d)} \times \mathbb{R}^+$ generalised geometry can be used to fully reformulate eleven-dimensional supergravity restricted on a $d \leq 7$-dimensional compact manifold, making its larger local symmetries manifest.

Since we are looking at reductions down to four dimensions, we will use $E_{7(7)} \times \mathbb{R}^+$ generalised geometry

The Generalised Tangent Space

Let M be a 7-dimensional spin manifold.
The generalised tangent space E of M is given by

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M)$$

Globally, E is actually defined as a series of extensions, twisted by gerbes which encode the topology of the gauge fields.

On an open subset $U_{(i)} \subset M$ we can write

$$V_{(i)} \in \Gamma(TU_i \oplus \Lambda^2 T^*U_i \oplus \Lambda^5 T^*U_i \oplus (T^*U_i \otimes \Lambda^7 T^*U_i))$$

Then the patching on the overlap $U_{(i)} \cap U_{(j)}$ is given by

$$\begin{aligned} V_{(i)} &= v_{(i)} + \omega_{(i)} + \sigma_{(i)} + \tau_{(i)} \\ &= v_{(j)} + \omega_{(j)} + i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)} + \sigma_{(j)} + i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)} + \omega_{(j)} \wedge \mathrm{d}\Lambda_{(ij)} + \dots \end{aligned}$$

where $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally 2- and 5-forms which satisfy certain consistency conditions on higher order overlaps. This matches precisely the gauge transformations of supergravity.

Crucially, the symmetry transformations $GL(7,\mathbb{R}) \ltimes$ "Gauge" $\subset E_{7(7)} \times \mathbb{R}^+$.

P. Pacheco, D. Waldram '08
A.C., C. Strickland-Constable, D. Waldram '11, '12

The Generalised Tangent Space

In fact, the fiber E_x at $x \in M$ forms the $\mathbf{56_1}$ representation space of $E_{7(7)} \times \mathbb{R}^+$.

Frames for E form an $E_{7(7)} \times \mathbb{R}^+$ principal bundle, the generalised frame bundle $\tilde{F}$. Generalised tensors will then be associated to different representations of $E_{7(7)} \times \mathbb{R}^+$.

Several familiar notions from Riemannian geometry can be defined for the $E_{7(7)} \times \mathbb{R}^+$ generalised tangent bundle.

Dorfman Derivative

The differential structure of E is given by the Dorfman bracket, a generalisation of the Lie derivative which combines the action of infinitesimal diffeomorphisms and gauge transformations

$$L_V W^M = V^N \partial_N W^M - (\partial \times_{\mathrm{ad}} V)^M{}_N W^N$$

where

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m \\ 0 & \text{else} \end{cases} \in E^*$$

The Dorfman bracket is not antisymmetric, but it does satisfy the Leibniz property, i.e. E is a Leibniz algebroid.

Generalised Connections

A generalized connection is a first-order linear differential operator which acts on generalised vectors as

$$D_M W^N = \partial_M W^N + \Omega_M{}^N{}_P W^P$$

where $\Omega_V = V^M \Omega_M{}^N{}_P \in \mathrm{ad}\,\tilde{F}$.

The generalised torsion of a generalised connection is defined as usual by

$$T(V,W) = L_V^D W - L_V W$$

now with the Dorfman derivative instead of the Lie derivative.

We find that the generalised torsion constraints some components of the connection

$$T \in W \subset E^* \otimes \mathrm{ad}\,\tilde{F}$$

with W in the $\mathbf{912_{-1}} + \mathbf{56_{-1}}$ representation of $E_{7(7)} \times \mathbb{R}^+$.

Generalised Metric and Spinors

We now introduce extra structure, in analogy with Riemannian geometry. Consider the maximal compact subgroups $SU(8)/\mathbb{Z}_2 \subset E_{7(7)}$.

An $SU(8)/\mathbb{Z}_2$ structure on E is defined by a generalised metric H which at each point parametrises the coset

$$H \in \frac{E_{7(7)} \times \mathbb{R}^+}{SU(8)/\mathbb{Z}_2}$$

This precisely corresponds to the degrees of freedom of the bosonic supergravity fields, which are thus unified in a single object

$$\{g, A, \tilde{A}, \Delta\} \in H$$

$Spin(7)$ spinors can be identified as transforming under the double cover $SU(8)$. The fermion fields ψ_m and ρ are thus thought of as $SU(8)$ objects.

Torsion Free, $SU(8)$ Connections

Given an $SU(8)$ structure $P_{SU(8)}$, we have that generalised spin connections are of the form

$$D_M W^N = \partial_M W^N + \Omega_M{}^N{}_P W^P$$

where now $\Omega_V \in \mathrm{ad}\, P_{SU(8)}$.
Take two such $SU(8)$ connections and define

$$\Sigma = D - D'$$

Then we have that Σ spans a space

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$$\tau(\Sigma) = T(D) - T(D') \in W$$

It is easy to check that

$$\tau(K) \subset W \qquad \ker(\tau) \neq 0$$

Thus there always exists a generalised connection which is both metric compatible and torsion free, but it is not unique.

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Killing Spinor Equations Revisited

However, it turns out that all the supergravity equations can be written in $SU(8)$ invariant language which is independent of the choice of connection, as all undetermined elements project out. In particular, given any D such that $T(D) = 0$, the supersymmetry transformations of the fermions are just

$$\delta\psi^{\alpha\beta\gamma} = D^{[\alpha\beta}\varepsilon^{\gamma]}$$
$$\delta\bar{\rho}_{\alpha} = -\bar{D}_{\alpha\beta}\varepsilon^{\beta}$$

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Already looks very close to the special holonomy equations.
Clearly if we have a torsion-free connection with

$$D\varepsilon = 0 \Rightarrow \bar{D}_{\alpha\beta}\varepsilon^{\beta} = 0, \qquad D^{[\alpha\beta}\varepsilon^{\gamma]} = 0$$

so the background is supersymmetric.
Does the converse hold? Given a supersymmetric background can we find a torsion-free D' such that $D'\varepsilon = 0$?

Generalised $SU(7)$-structures

A nowhere vanishing spinor ε defines an $SU(7) \subset SU(8)$ structure.

So the question is, given a supersymmetric background, can we find an $SU(7)$-connection which is torsion-free?

Start with a generic torsion-free $SU(8)$ connection D and let $\hat{D} = D + \hat{\Sigma}$ with $\tau(\hat{\Sigma}) = 0$ so that $\hat{D}$ is also torsion-free and $SU(8)$ compatible (remember there are several such connections!).

Can we choose $\hat{\Sigma}$ such that

$$\hat{D}\varepsilon = D\varepsilon + \hat{\Sigma} \cdot \varepsilon = 0 \ ?$$

If so $\hat{D}$ is compatible with the reduced structure group $SU(7)$.

So we must prove that we can solve

$$D\varepsilon = -\hat{\Sigma} \cdot \varepsilon$$

for $\hat{\Sigma}$, given that ε satisfies the Killing Spinor Equations.

Torsion-free $SU(7)$-structures

The strategy is then

$$D\varepsilon \qquad\qquad \hat{\Sigma} \cdot \varepsilon$$

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We conclude there is enough freedom to set

$$\hat{\Sigma}\cdot\varepsilon = -D\varepsilon \quad \Rightarrow \quad \hat{D}\varepsilon = 0$$

as we wanted.

$SU(7)$ Generalised Special Holonomy Manifolds

Thus setting the Killing Spinor Equations to zero is equivalent to demanding the vanishing of the generalised intrinsic torsion.

$$\delta\rho = 0, \quad \delta\psi = 0$$
$$\Updownarrow$$
$$\exists\, D' :\, D'\varepsilon = 0, \quad T(D') = 0$$

in which case we have the generalised analogue of special holonomy.

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Manifolds with a generalised torsion-free $SU(7)$*-structure are* $\mathcal{N} = 1$ *supersymmetric backgrounds of M theory and vice-versa.*

(can think of these manifolds as "exceptional generalised Calabi-Yau")

Conclusion

$E_{7(7)} \times \mathbb{R}^+$ generalised geometry allows us to "geometrise" the full bosonic sector of four-dimensional backgrounds of eleven-dimensional supergravity.

This enabled us to re-interpret $\mathcal{N} = 1$ flux backgrounds as manifolds with $SU(7)$ generalised special holonomy $\rightarrow$ "integrability" condition that works for all possible fluxes.

What about higher $\mathcal{N}$? Work in progress: result almost certainly holds, proof a bit more subtle.

Would be interesting to know if similar results also hold for other types of generalised geometry, which are used to describe other supergravities. Easy to show it holds for $O(d, d)$ generalised geometry. Is it a general feature?

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Thank you very much.