Newton-Cartan geometry for holography

Jan Rosseel (TU Wien)

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+ work with Eric Bergshoeff, Jelle Hartong
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Outline

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- The Newton-Cartan formalism (or suitable generalizations thereof) has however appeared in recent applications.
  1. Construction of effective actions in condensed matter theory. (Hoyos, Son)
  2. Boundary geometry in non-relativistic holography : Lifshitz holography. (Ross; Christensen, Hartong, Obers, Rollier; Hartong, Kiritsis, Obers)

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  2. Boundary geometry in non-relativistic holography : Lifshitz holography. (Ross; Christensen, Hartong, Obers, Rollier; Hartong, Kiritsis, Obers)
- Here : develop formulation of Newton-Cartan geometry that can be used to study Lifshitz holography and understand certain aspects of it.

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  1. Temporal metric of rank 1 : $\tau_{\mu\nu} = \tau_\mu \tau_\nu$.
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- Newton's second law and the Poisson equation can now be covariantly written as:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,, \qquad R_{\mu\nu} = \rho\tau_\mu\tau_\nu\,.$$

Newton-Cartan in Lifshitz holography

- Lifshitz holography attempts to describe field theories invariant under space-time translations, spatial rotations and non-relativistic dilatations:

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- For holography, notion of 'Asymptotically Locally Lifshitz' space-time is needed. Via Fefferman-Graham-like expansion: (Ross;Korovin, Skenderis, Taylor;Chemissany, Geissbühler, Hartong, Rollier; Christensen, Hartong, Obers, Rollier)

$$ds^2 = e^{\Phi} \frac{dr^2}{r^2} + \left(-e_\mu{}^0 e_\nu{}^0 + e_\mu{}^a e_\nu{}^a\right) dx^\mu dx^\nu\,,$$

where $\quad e^0 = r^{-z} \tau_{(0)\mu}(x) dx^\mu + \cdots\,, \qquad e^a = r^{-1} e_{(0)\mu}{}^a(x) dx^\mu + \cdots\,.$

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- Moreover $m_{(0)\mu}$ plays a crucial role, e.g. appears in affine connection.

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- Bulk local Lorentz group contracts to spatial rotations and Galilean boosts:

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- In addition : boundary data only determined up to local Weyl rescalings.

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- This suggests that the internal symmetries are the ones of the Schrödinger algebra. So, we will 'gauge the Schrödinger algebra'. (with Bergshoeff, Hartong)

Torsional Newton-Cartan and the Schrödinger algebra

- Schrödinger algebra (for $z = 2$) consists of
  1. Bargmann algebra = central extension of Galilei algebra of non-relativistic space-time transformations : H, P_a, J_{ab}, G_a, Z
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- Gauging = associating gauge fields, transformation rules and covariant curvatures to all generators.

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$R_{\mu\nu}(H)$
space translations	P^a	${e_\mu}^a$	$\zeta^a(x^\nu)$	${R_{\mu\nu}}^a(P)$
boosts	G^a	${\omega_\mu}^a$	$\lambda^a(x^\nu)$	${R_{\mu\nu}}^a(G)$
spatial rotations	J^{ab}	${\omega_\mu}^{ab}$	$\lambda^{ab}(x^\nu)$	${R_{\mu\nu}}^{ab}(J)$
central charge transf.	Z	m_μ	$\sigma(x^\nu)$	$R_{\mu\nu}(Z)$
dilatations	D	b_μ	$\Lambda_D(x^\nu)$	$R_{\mu\nu}(D)$
spec. conf. transf.	K	f_μ	$\Lambda_K(x^\nu)$	$R_{\mu\nu}(K)$

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- Reproduces correct transformation laws for τ_μ, $e_\mu{}^a$ and m_μ.

Torsional Newton-Cartan and the Schrödinger algebra

- Other fields can be made dependent via curvature constraints. E.g. for the spin connections:

$$
\begin{aligned}
R_{\mu\nu}{}^{a}(P) &= 2\partial_{[\mu}e_{\nu]}{}^{a} - 2\omega_{[\mu}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^{a}\tau_{\nu]} - 2b_{[\mu}e_{\nu]}{}^{a} = 0\,, \\
R_{\mu\nu}(Z) &= 2\partial_{[\mu}m_{\nu]} - 2\omega_{[\mu}{}^{a}e_{\nu]a} = 0\,.
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Allow one to solve $\omega_{\mu}{}^{ab}$ and $\omega_{\mu}{}^{a}$ in terms of τ_{μ}, $e_{\mu}{}^{a}$, m_{μ}. A similar story holds for f_{μ}.

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- What about b_μ? The time-like component is pure gauge:

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\delta_K b_\mu = \Lambda_K \tau_\mu\,.
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  The spatial components are found as solutions of the constraint

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- Note that now

$$\partial_{[\mu}\tau_{\nu]} \neq 0 \qquad \text{but} \qquad \tau_{[\mu}\partial_\nu\tau_{\rho]} = 0 \qquad \Rightarrow \qquad \text{torsion}\,.$$

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- Given here for $z = 2$, but can be extended for arbitrary z. Can be extended to include more generic forms of torsion.