Supersymmetric gauge theories, localization and holography

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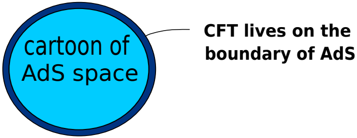
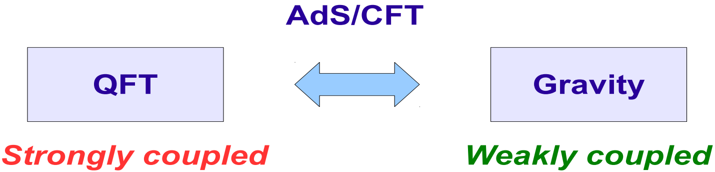
Outline

1. Overview
2. Rigid supersymmetry and localization
3. Holography

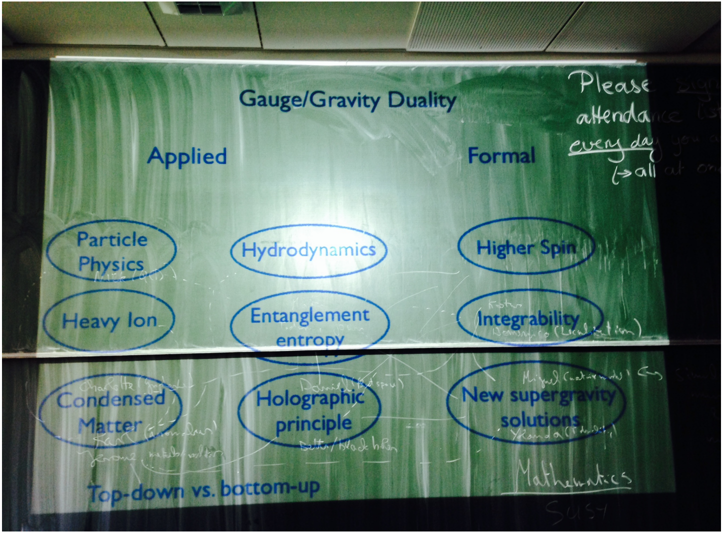
Holography or gauge/gravity duality

Equivalence between (quantum) **gravity** in bulk space-times and **quantum field theories** on their boundaries





Uses of the gauge/gravity duality



Localization

- For certain supersymmetric field theories defined on compact curved Riemannian manifolds the path integral may be computed exactly
- Localization: functional integral over all fields of a theory $\rightarrow$ integral/sum over a reduced set of field configurations
- Saddle point around a supersymmetric locus gives the exact answer
- A priori the path integral ("partition function" $\mathbf{Z}$) depends on the parameters of the theory and of the background geometry

Basic idea of localization

- The idea of using localization as a method to calculate obeservables in a quantum field theory goes back to [Witten 1988]
- Suppose we have an action $\mathbf{S}[\phi]$ invariant under a supersymmetry δ (more generally a Grassmann-odd symmetry) so that $\delta\mathbf{S}[\phi] = \mathbf{0}$, with $\delta^2 = \mathbf{0}$
- Then consider the path integral of a theory deformed by a δ-exact term $\delta\mathbf{V_F}$

$$\mathbf{Z(t)} = \int \mathcal{D}\phi \, \mathbf{e}^{-\mathbf{S}-\mathbf{t}\delta\mathbf{V_F}}$$

- The key point is that (assuming $\delta\mathcal{D}\phi = \mathbf{0}$) this is independent of $\mathbf{t}$

$$-\frac{\mathbf{dZ}}{\mathbf{dt}} = \int \mathcal{D}\phi \, \delta\mathbf{V_F} \, \mathbf{e}^{-\mathbf{S}-\mathbf{t}\delta\mathbf{V_F}} = \int \mathcal{D}\phi\delta \left(\mathbf{V_F}\mathbf{e}^{-\mathbf{S}-\mathbf{t}\delta\mathbf{V_F}}\right) = \mathbf{0}$$

$$\Rightarrow \qquad \mathbf{Z} \equiv \mathbf{Z(0)} = \mathbf{Z(t \to \infty)}$$

Basic idea of localization

- If we choose $\delta \mathbf{V_F}$ to have positive definite bosonic part, then as $\mathbf{t} \to \infty$ the integrand is suppressed exponentially, except when $\delta \mathbf{V_F}|_{\mathbf{bosonic}} = \mathbf{0}$
- In this case the saddle point approximation is not an approximation, it is an exact result

$$\mathbf{Z} = \int_{\delta-\mathbf{invariant\ fields}} \mathcal{D}\phi\, \mathbf{e}^{-\mathbf{S}}\, (\mathbf{1} - \mathbf{loop\ determinant})$$

- A typical choice of $\mathbf{V_F}$ is of the type $\mathbf{V_F} = \mathrm{Tr}[(\delta\psi)^\dagger \psi]$, where ψ is a fermion of the theory
- The path integral localizes on supersymmetric configurations

$$\delta\psi = \mathbf{0}$$

Uses of localization

Results have been obtained for supersymmetric theories defined on various manifolds $\mathbf{M_d}$, with different amounts of supersymmetry. A sample list of references calculating partition functions (or/and BPS observables) using localization is:

$\mathbf{d=1}$: supersymmetric quantum mechanics [Cordova-Shao], ...

$\mathbf{d=2}$: $\mathbf{S^2}$ [Benini-Cremonesi], [Doroud et al], elliptic genera [Benini et al], ...

$\mathbf{d=3}$: $\mathbf{S^3}$ [Kapustin et al], [Hama et al], [Jafferis] and its deformations ..., $\mathbf{S^1\times S^2}$ [S. Kim], ...

$\mathbf{d=4}$: Riemannian [Witten '88], Kähler [Johansen '94], $\mathbf{S^4}$ [Pestun], Hopf surfaces [Assel et al], ...

$\mathbf{d=5}$: $\mathbf{S^5}$ and its deformations, Sasaki-Einstein [Källén,Qiu,Zabzine et al], ...
$\mathbf{S^1\times S^4}$, $\mathbf{S^1\times \mathbb{C}P^2}$ [Kim et al], ...

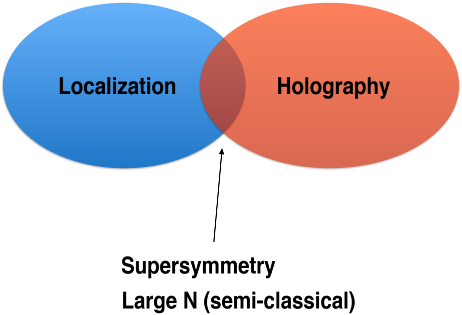
... many more ...

Uses of localization

- When the manifold $\mathbf{M_d}$ is of the form $\mathbf{S^1} \times \mathbf{M_{d-1}}$ the path integral may be interpreted as an index Tr $\mathbf{e^{-(operator)}}$, "counting" states in the field theory (Hamiltonian formalism). In this case the name "partition function" is more appropriate: think of $\mathbf{S^1}$ as compactified time, but there is no temperature
- Indices and other partition functions may be used to test conjectured non-perturbative Seiberg(-like) dualities
- Partition functions on $\mathbf{S^2}$ and $\mathbf{S^4}$ compute exact Kähler potential on the space of marginal deformations of supersymmetric SCFs

. . . relationships across dimensions, factorisation, "Higgs branch" vs "Coulomb branch", manifolds with boundaries, topological strings, Rényi/entanglement entropy, Hilbert series, . . .

Localization vs holography



- Beyond large **N** (semi-classical) $\rightarrow$ "quantum holography" Murthy's talk [Dabholkar, Drukker, Gomes, Murthy, . . .]

Comparing localization with holography

- When a field theory has a holographic dual, what can we attempt to compare on the two sides?
- Using localization we can compute exactly $\mathbf{n}$-point functions of BPS ($\delta\mathcal{O}_i = 0$) operators

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int_{\delta-\text{invariant fields}} \mathcal{D}\phi \, e^{-S} \, (1-\text{loop}) \, \mathcal{O}_1 \dots \mathcal{O}_n$$

- In the large $\mathbf{N}$ (semi-classical) limit we can compare with holographic $\mathbf{n}$-point functions of BPS operators computed using the gravity dual. The simplest case is one-point functions
- Even simpler is the path integral with no insertions, whose gravity dual is

$$e^{-S_{\text{supergravity}}[M_{d+1}]} = Z[M_d = \partial M_{d+1}]$$

where the supergravity action is evaluated on-shell on a $(d+1)$-dimensional solution M_{d+1} with d-dimensional conformal boundary M_d, on which the supersymmetric QFT is defined

Rigid supersymmetry and localization

Four dimensional $\mathcal{N} = 1$ supersymmetric field theories

- For concreteness, we now focus on $\mathbf{d = 4}$, $\mathcal{N} = \mathbf{1}$ supersymmetric gauge theories with matter
- Vector multiplet: gauge field $\mathcal{A}$; Weyl spinor $\boldsymbol{\lambda}$; auxiliary scalar $\mathbf{D}$, all transforming in the adjoint representation of a group $\mathbf{G}$
- Chiral multiplet: complex scalar ϕ; Weyl spinor ψ; auxiliary scalar $\mathbf{F}$, all transforming in a representation $\mathcal{R}$ of the group $\mathbf{G}$
- In flat space with Lorentzian signature, supersymmetric Lagrangians containing these fields are textbook material
- A first caveat in Euclidean space is that degrees of freedom in multiplets are *a priori* doubled: $\boldsymbol{\lambda}^\dagger \to \tilde{\boldsymbol{\lambda}}$, $\phi^\dagger \to \tilde{\phi}$, etcetera, where tilded fields are regarded as independent

Supersymmetry and Lagrangians (flat space)

- For example, the supersymmetry transformations of the fields in the vector multiplet are

$$\delta\mathcal{A}_\mu = i\zeta\sigma_\mu\widetilde{\lambda} \qquad \delta D = -\zeta\sigma^\mu D_\mu\widetilde{\lambda}$$

$$\delta\lambda = \mathcal{F}_{\mu\nu}\,\sigma^{\mu\nu}\zeta + iD\zeta \qquad \delta\widetilde{\lambda} = 0$$

  where ζ is a constant spinor parameter, $D_\mu = \partial_\mu - i\mathcal{A}_\mu\cdot$, and $\mathcal{F}_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$

- The supersymmetric Yang-Mills Lagrangian reads

$$\mathcal{L}_{\text{vector}} = \mathrm{tr}\left[\,\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}D^2 + i\widetilde{\lambda}\,\widetilde{\sigma}^\mu D_\mu\lambda\,\right]$$

- Similarly, there are supersymmetry transformations and supersymmetric Lagrangians for the fields in the chiral multiplet

Rigid supersymmetry on curved manifolds

- One can try to define supersymmetric field theories on Riemannian (or Lorentzian) curved manifolds: clearly $\partial_\mu \to \nabla_\mu$, but this is not enough
- The supersymmetry transformations and Lagrangians must be modified. [Witten]: "twist" $\mathcal{N} = 2$ SYM $\to$ supersymmetric on arbitrary Riemannian manifod
- Somewhat surprisingly, rigid supersymmetry in curved space (Euclidean or Lorentzian) addressed systematically only in the 2010's
- But local supersymmetry studied since long time ago $\to$ supergravity
- [Festuccia-Seiberg]: take supergravity with some gauge and matter fields and appropriately throw away gravity $\to$ "rigid limit"
- Important: in the process of throwing away gravity, some extra fields of the supergravity multiplet remain, but are non-dynamical $\to$ background fields

Rigid new minimal supersymmetry

- For $\mathbf{d=4}$ field theories with an $\mathbf{R}$ symmetry, one can use (Euclidean) new minimal supergravity [Sohnius-West]. Gravitini variations:

$$\delta\psi_\mu \sim (\nabla_\mu - \mathrm{i}A_\mu)\,\zeta + \mathrm{i}V_\mu\zeta + \mathrm{i}V^\nu\sigma_{\mu\nu}\zeta = 0$$

$$\delta\tilde{\psi}_\mu \sim (\nabla_\mu + \mathrm{i}A_\mu)\,\tilde{\zeta} - \mathrm{i}V_\mu\tilde{\zeta} - \mathrm{i}V^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} = 0$$

- A_μ, V_μ are background fields and ζ, $\tilde{\zeta}$ are supersymmetry parameters

- Existence of ζ or $\tilde{\zeta}$ is equivalent to Hermitian metric [Klare-Tomasiello-Zaffaroni], [Dumitrescu-Festuccia-Seiberg]

- The supersymmetry transformations of the vector multiplet are

$$\delta\mathcal{A}_\mu = \mathrm{i}\zeta\sigma_\mu\widetilde{\lambda} + \mathrm{i}\widetilde{\zeta}\,\widetilde{\sigma}_\mu\lambda$$

$$\delta\lambda = \mathcal{F}_{\mu\nu}\,\sigma^{\mu\nu}\zeta + \mathrm{i}D\zeta \qquad \delta\widetilde{\lambda} = \mathcal{F}_{\mu\nu}\,\widetilde{\sigma}^{\mu\nu}\widetilde{\zeta} - \mathrm{i}D\widetilde{\zeta}$$

$$\delta D = -\zeta\sigma^\mu\left(D_\mu\widetilde{\lambda} - \tfrac{3\mathrm{i}}{2}V_\mu\widetilde{\lambda}\right) + \widetilde{\zeta}\,\widetilde{\sigma}^\mu\left(D_\mu\lambda + \tfrac{3\mathrm{i}}{2}V_\mu\lambda\right)$$

where $D_\mu = \nabla_\mu - \mathrm{i}\mathcal{A}_\mu \cdot -\mathrm{i}q_R A_\mu$

Superconformal anomalies

- Putting an $\mathcal{N} = 1$ SCFT on a curved background yields trace and R-symmetry anomalies

$$\langle T^m_m \rangle = \frac{c}{16\pi^2}\mathscr{C}^2 - \frac{a}{16\pi^2}\mathscr{E} - \frac{c}{6\pi^2}F_{mn}F^{mn}$$

$$\langle \nabla_m J^m \rangle = \frac{c-a}{24\pi^2} R_{mnpq}\widetilde{R}^{mnpq} + \frac{5a-3c}{27\pi^2}F_{mn}\widetilde{F}^{mn}$$

where $F = d(A - \frac{3}{2}V)$, a and c are the central charges and

$$\mathscr{C}^2 \equiv C_{mnpq}C^{mnpq} = R_{mnpq}R^{mnpq} - 2R_{mn}R^{mn} + \frac{1}{3}R^2$$

$$\mathscr{E} \equiv \frac{1}{4}\epsilon^{mnpq}\epsilon^{rsuv}R_{mnrs}R_{pquv} = R_{mnpq}R^{mnpq} - 4R_{mn}R^{mn} + R^2$$

$$\mathscr{P} \equiv \frac{1}{2}\epsilon^{mnpq}R_{mnrs}R_{pq}{}^{rs} = \frac{1}{2}\epsilon^{mnpq}C_{mnrs}C_{pq}{}^{rs}$$

Supersymmetry tames the anomalies

- If the background is supersymmetric the anomalies are simplified substantially [Cassani-DM]
- For example, assuming the existence of both Killing spinors ζ abd $\tilde{\zeta}$ (in Lorentzian signature this is valid with ζ only)

$$\langle T^m_m \rangle = -\frac{a}{16\pi^2}\mathscr{E}$$

$$\langle \nabla_m J^m \rangle = \frac{a}{36\pi^2}\mathscr{P}$$

- In particular, this is true for complex manifolds with topology $S^1 \times S^3$ to be discussed next $\longrightarrow$ on these manifolds integrated anomalies vanish

Localization on four-manifolds: strategy outline

[Assel-Cassani-DM]

- Work in Euclidean signature and start with generic background fields $\mathbf{A}_\mu$, $\mathbf{V}_\mu$ associated to a Hermitian manifold
- Construct δ-exact Lagrangians for the vector and chiral multiplets $\rightarrow$ set-up localization on a general Hermitian manifold
- Restrict to backgrounds admitting a second spinor $\tilde{\zeta}$ with opposite R-charge
- Further restrict to manifolds with topology $\mathbf{M_4} \simeq \mathbf{S^1} \times \mathbf{S^3}$
- Prove that the localization locus is given by gauge field $\mathcal{A}_\tau =$ constant, with all other fields $(\lambda, \mathbf{D}; \phi, \psi, \mathbf{F})$ vanishing
- Partition function reduces to a matrix integral over $\mathcal{A}_\tau$ $\rightarrow$ integrand is infinite product of 3d super-determinants

Localizing Lagrangians and saddle point equations

- The bosonic parts of the localizing terms constructed with ζ only are

$$\mathcal{L}^{(+)}_{\text{vector}} = \text{tr}\left(\frac{1}{4}\mathcal{F}^{(+)}_{\mu\nu}\mathcal{F}^{(+)\,\mu\nu} + \frac{1}{4}\mathrm{D}^2\right)$$

$$\mathcal{L}_{\text{chiral}} = (\mathrm{g}^{\mu\nu} - \mathrm{i}\mathrm{J}^{\mu\nu})\,\mathrm{D}_\mu\widetilde{\phi}\mathrm{D}_\nu\phi + \widetilde{\mathrm{F}}\mathrm{F}$$

- With the obvious reality conditions on the fields, $\mathcal{A}, \mathrm{D}$ Hermitian, $\widetilde{\phi} = \phi^\dagger$, $\widetilde{\mathrm{F}} = \mathrm{F}^\dagger$, we obtain the saddle point equations

$$\textbf{vector}: \qquad \mathcal{F}^{(+)}_{\mu\nu} = 0\,, \qquad \mathrm{D} = 0$$

$$\textbf{chiral}: \qquad \mathrm{J}^\mu{}_\nu\mathrm{D}^\nu\widetilde{\phi} = \mathrm{i}\mathrm{D}^\mu\widetilde{\phi}\,, \qquad \mathrm{F} = 0$$

Geometries with two supercharges of opposite R-charge

- Suppose there exists a second spinor $\widetilde{\zeta}$, with opposite chirality, obeying the rigid new minimal equation

$$(\nabla_\mu + iA_\mu)\,\widetilde{\zeta} - iV_\mu\widetilde{\zeta} - iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} = 0$$

- Geometry is a special case of ambihermitian manifold, which may be neatly characterised by the complex holomorphic Killing vector field $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$
- The metric takes a canonical form in terms of complex coordinates z, w

$$ds^2 \;=\; \Omega^2[(dw + hdz)(d\bar{w} + \bar{h}d\bar{z}) + c^2 dz d\bar{z}]$$

with $\Omega(z,\bar{z})$, $c(z,\bar{z})$, $h(z,\bar{z})$ arbitrary functions

Hopf surfaces

- A Hopf surface is essentially a four-dimensional complex manifold with the topology of $\mathbf{S}^1 \times \mathbf{S}^3$. It can be described as a quotient of $\mathbb{C}^2 - (0,0)$, with coordinates z_1, z_2 identified as

$$(z_1, z_2) \sim (pz_1, qz_2)$$

  where p, q are in general two complex parameters

- We show that on a Hopf surface we can take a very general metric

$$ds^2 = \Omega^2 d\tau^2 + f^2 d\rho^2 + m_{IJ} d\varphi_I d\varphi_J \qquad I, J = 1, 2$$

  while preserving two spinors ζ and $\widetilde{\zeta}$

- τ is a coordinate on $\mathbf{S}^1$, while the 3d part has coordinates $\rho, \varphi_1, \varphi_2$, describing $\mathbf{S}^3$ as a $\mathbf{T}^2$ fibration over an interval

The matrix model

- The localizing locus simplifies drastically, e,g. $\rightarrow \mathcal{F}^{(+)} = \mathcal{F}^{(-)} = 0 \rightarrow$ full contribution comes from zero-instanton sector! Flat connections $\mathcal{A}_\tau =$ constant, and all other fields vanishing

- The localized path integral is reduced an infinite products of $\mathbf{d = 3}$ super-determinants, that may be computed explicitly using the method of pairing of eigenvalues [Hama et al], [Alday et al]

- Infinite products regularised using formulas for elliptic gamma functions

$$Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho \in \Delta_{\mathcal{R}}} \prod_{n \in \mathbb{Z}} Z^{\text{chiral}}_{\text{1-loop (3d)}} \left[\sigma_0^{(n,\rho)}\right]$$

$$\rightarrow \; e^{i\pi \Psi^{(0)}_{\text{chi}}} e^{i\pi \Psi^{(1)}_{\text{chi}}} \prod_{\rho \in \Delta_{\mathcal{R}}} \Gamma_e \left(e^{2\pi i \rho_{\mathcal{A}_0}} (pq)^{\frac{r}{2}}, p, q \right)$$

Supersymmetric index

- The prefactor $\Psi^{(1)}_{\rm chi}$ is anomalous and must cancel after combining with the vector multiplet contribution $\to$ anomaly cancellation conditions "for free"

- The rest combines into the following formula

$$Z[\mathcal{H}_{p,q}] \;=\; e^{-\mathcal{F}(p,q)}\, \mathcal{I}(p,q)$$

where $\mathcal{I}(p,q)$ is the supersymmetric index with p, q fugacities

$$\mathcal{I}(p,q) \;=\; \frac{(p;p)^{r_G}(q;q)^{r_G}}{|\mathcal{W}|}\int_{T^{r_G}} \frac{dz}{2\pi i z} \prod_{\alpha\in\Delta_+} \theta\left(z^{\alpha},p\right)\theta\left(z^{-\alpha},q\right) \prod_J \prod_{\rho\in\Delta_J} \Gamma_e\left(z^{\rho}(pq)^{\frac{r_J}{2}},p,q\right)$$

which may be defined as a sum over states as

$$\mathcal{I}(p,q) \;=\; \mathrm{Tr}[(-1)^F p^{J+J'-\frac{R}{2}} q^{J-J'-\frac{R}{2}}]$$

- The fact that the index is computed by the localized path integral on a Hopf surface was anticipated by [Closset-Dumitrescu-Festuccia-Komargodski]

Supersymmetric Casimir energy

- The path integral + regularisation produces a pre-factor $\mathcal{F}(\mathbf{p},\mathbf{q})$ explicitly given by ($\mathbf{p} \equiv \mathbf{e}^{-2\pi|\mathbf{b}_1|}$, $\mathbf{q} \equiv \mathbf{e}^{-2\pi|\mathbf{b}_2|}$)

$$\mathcal{F}(\mathbf{p},\mathbf{q}) = \frac{4\pi}{3}\left(|\mathbf{b}_1| + |\mathbf{b}_2| - \frac{|\mathbf{b}_1| + |\mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}\right)(\mathbf{a} - \mathbf{c}) + \frac{4\pi}{27}\frac{(|\mathbf{b}_1| + |\mathbf{b}_2|)^3}{|\mathbf{b}_1||\mathbf{b}_2|}(3\,\mathbf{c} - 2\,\mathbf{a})$$

$$\mathbf{a} = \frac{3}{32}\left(3\,\mathrm{tr}\mathbf{R}^3 - \mathrm{tr}\mathbf{R}\right), \qquad \mathbf{c} = \frac{1}{32}\left(9\,\mathrm{tr}\mathbf{R}^3 - 5\,\mathrm{tr}\mathbf{R}\right)$$

- Invariant depending only on complex structure and the trace anomaly coefficients $\mathbf{a}, \mathbf{c}$ $\rightarrow$ expect to encode physical properties

- It captures the "vacuum energy" $\rightarrow$ supersymmetric Casimir energy $\mathbf{E}_{\mathbf{susy}}$

More comments on the supersymmetric Casimir energy

- How does one know the result does not depend on the regularisation procedure, e.g. zeta-function?
- One must show that there are no finite, supersymmetric, counterterms, i.e. integrals of local densities [Assel-Cassani-DM] (to appear)
- Supersymmetric Casimir energy can be recovered from the Hamiltonian formalism [Lorenzen-DM] (to appear)

$$\langle \mathbf{0} | \mathbf{H}_{\mathbf{BPS}} | \mathbf{0} \rangle = \mathbf{E}_{\mathbf{susy}}$$

where $\mathbf{H}_{\mathbf{BPS}}$ is an appropriate supersymmetric Hamiltonian

Partition function on $\mathbf{M_3}$

- From the 4d partition function is possible to extract the partition function of an $\mathcal{N} = 2$, $\mathbf{d} = 3$ theory on a manifold $\mathbf{M_3} \simeq \mathbf{S^3}$ arising as the dimensional reduction on $\mathbf{S^1}$, previously computed in [Alday et al]

$$\mathbf{Z}_\beta = \int \mathrm{d}\sigma_0\, \mathrm{e}^{-\frac{\mathrm{i}\pi\mathrm{k}}{|\mathrm{b_1 b_2}|}\mathrm{Tr}\,\sigma_0^2} \prod_{\alpha\in\Delta_+} 4\sinh\frac{\pi\sigma_0\alpha}{|\mathbf{b_1}|}\sinh\frac{\pi\sigma_0\alpha}{|\mathbf{b_2}|}$$

$$\cdot \prod_\rho \mathbf{s}_\beta\left[\frac{\mathrm{i}(\beta+\beta^{-1})}{2}(1-\mathbf{r}) - \frac{\rho(\sigma_0)}{\sqrt{|\mathbf{b_1 b_2}|}}\right]$$

- Here $\beta = \sqrt{|\mathbf{b_1}/\mathbf{b_2}|}$ and $\mathbf{s}_\beta(\mathbf{z})$ denotes the double sine function

- It depends on $\mathbf{M_3} \simeq \mathbf{S^3}$ only through the almost contact structure, via the Killing vector $\mathbf{K} = \mathbf{b_1}\partial_{\varphi_1} + \mathbf{b_2}\partial_{\varphi_2}$, where $\partial_{\varphi_1}, \partial_{\varphi_2}$ is a basis of $\mathbf{U(1)^2}$

Exact free energy and large $\mathbf{N}$ limit

- The exact free energy $\mathcal{F} = -\log Z$ of a general $\mathcal{N} = 2$ supersymmetric Chern-Simons theory defined on an M_3 depends on the gauge group G, matter representation $\mathcal{R}$ Chern-Simons levels k, as well as b_1, b_2

- For a quiver gauge group $G = SU(N)^p$ we can consider the large $\mathbf{N}$ limit: $N \to \infty$, at fixed k

- For rather general $\mathcal{N} = 2$ theories one finds that in this limit the dependence on β factorises as [DM-Passias-Sparks]

$$\mathcal{F}_\beta = \frac{(\beta + \beta^{-1})^2}{4} \mathcal{F}_{\beta=1} + O(N^{1/2})$$

where $\mathcal{F}_{\beta=1} \sim O(N^{3/2})$ is the large N free energy on the round three-sphere (with standard almost contact structure)

- In the final part of the talk I will discuss how to reproduce this result from a holographic computation, independently of many details of the dual supergravity solutions

Holography

Gravity duals of $\mathcal{N} = 2$ Chern-Simons theories on $\mathbf{S}^3$

- The holographic dual to the ABJM model is the $\mathcal{N} = 6$ solution of $\mathbf{d} = 11$ supergravity solution with metric $\mathrm{AdS}_4 \times \mathbf{S}^7/\mathbb{Z}_k$
- A large class of $\mathcal{N} = 2$ quiver gauge theories with Chern-Simons terms are dual to $\mathrm{AdS}_4 \times \mathbf{Y}_7$, with $\mathbf{Y}_7$ a Sasaki-Einstein manifold
- In order to compare with field theories on the round $\mathbf{S}^3$ it suffices to consider Euclidean AdS_4, with round $\mathbf{S}^3$ at its conformal boundary
- Key ingredient: the bulk Killing spinor ϵ in AdS_4 induces a rigid Killing spinor χ on the boundary round $\mathbf{S}^3$

$$\nabla_\mu \epsilon \;=\; -\frac{1}{2}\Gamma_\mu \epsilon \quad \longrightarrow \quad \nabla_i \chi \;=\; \tfrac{i}{2}\gamma_i \chi$$

- The holographically renormalised on-shell sugra action reproduces the localized free energy $\mathcal{F}_{\beta=1}$ at large $\mathbf{N}$ [Drukker-Marino-Putrov]

Gravity duals for more general $\mathbf{M_3}$

Idea: find a supersymmetric filling $\mathbf{M_4}$ of the given $\mathbf{M_3}$ in $\mathbf{d = 4}$, $\mathcal{N} = \mathbf{2}$ gauged supergravity (Einstein-Maxwell theory), and use the fact that any such solution uplifts to a supersymmetric solution $\mathbf{M_4 \times Y_7}$ of $\mathbf{d = 11}$ supergravity

Action: $\mathbf{S} = -\frac{\mathbf{1}}{\mathbf{16\pi G_4}} \int \mathrm{d}^4\mathbf{x}\sqrt{\mathbf{g}}\left(\mathbf{R} + \mathbf{6} - \mathbf{F}^2\right)$

Killing Spinor Equation: $\left(\nabla_\mu - \mathbf{i}\mathbf{A}_\mu + \frac{\mathbf{1}}{\mathbf{2}}\Gamma_\mu + \frac{\mathbf{i}}{\mathbf{4}}\mathbf{F}_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\epsilon = \mathbf{0}$

Dirichlet problem: find $\mathbf{(M_4, g_{\mu\nu}, A)}$ such that

- The conformal boundary of $\mathbf{M_4}$ is $\mathbf{M_3}$
- The bulk gauge field $\mathbf{A}$ restricts to a background $\mathbf{A^{(3)}}$ on the boundary
- The bulk Killing spinor ϵ restricts to the boundary rigid Killing spinor χ

Check: The on-shell sugra action reproduces the localized free energy at large $\mathbf{N}$

General class of four dimensional gravity duals

1 Given an $\mathcal{N} = \mathbf{2}$ supersymmetric field theory defined on $\mathbf{M_3} \simeq \mathbf{S^3}$ with arbitrary metric, together with a choice of almost contact structure, can we construct its 4d gravity dual?

2 Can we compute the holographic free energy for any such solution, and check that it agrees with the large $\mathbf{N}$ limit of the localized partition function on $\mathbf{M_3}$?

- Affirmative answer to both questions, working in minimal gauged supergravity in four dimensions, and focusing on (anti-)self-dual metrics on the four-ball [Farquet-Lorenzen-DM-Sparks]

Self-dual supersymmetric solutions

- The local form of Euclidean supersymmetric solutions of Euclidean $\mathbf{d = 4}$ minimal gauged supergravity given in [Dunajski-Gutowski-Sabra-Tod]

- When the graviphoton is real there exist a canonical Killing vector

$$\mathbf{K} \;=\; \mathrm{i}\epsilon^\dagger \Gamma^\mu \Gamma_5 \epsilon \partial_\mu \;=\; \partial_\psi$$

- There is a special class of "self-dual solutions" in which $*_4\mathbf{F} = -\mathbf{F}$ is anti-self-dual and the four-metric is Einstein with anti-self-dual Weyl tensor

- These solutions are (locally) conformal to a scalar-flat Kähler metric

$$\mathrm{ds}^2_4 \;=\; \frac{1}{\mathbf{y}^2}\mathrm{ds}^2_{\text{Kahler}} \;=\; \frac{1}{\mathbf{y}^2}\left[\mathbf{V}^{-1}(\mathrm{d}\psi + \phi)^2 + \mathbf{V}(\mathrm{d}\mathbf{y}^2 + 4\mathrm{e}^{\mathbf{w}}\mathrm{d}\mathbf{z}\mathrm{d}\bar{\mathbf{z}})\right]$$

where $\mathbf{V} = 1 - \frac{1}{2}\mathbf{y}\partial_\mathbf{y}\mathbf{w}$, with the metric determined entirely by a solution to the Toda equation

$$\partial_\mathbf{z}\partial_{\bar{\mathbf{z}}}\mathbf{w} + \partial^2_\mathbf{y}\mathrm{e}^\mathbf{w} \;=\; 0$$

Self-dual supersymmetric solutions

A self-dual supersymmetric solution can be constructed starting from a metric $\mathrm{ds}^2_{\mathsf{Kahler}}$ and a Killing vector **K**

- The function/coordinate **y** can be computed from the Killing vector **K** as

$$\Psi \equiv \frac{1}{2}(\mathrm{d}K^\flat)^+ \qquad \rightarrow \qquad \frac{2}{y^2} = |\Psi|^2$$

- The (bulk) gauge field depends on the choice of **K**, e.g. through the formula

$$F = \frac{1}{2}\mathcal{R}$$

where $\mathcal{R}$ is the Ricci curvature of the Kähler metric $\mathrm{ds}^2_{\mathrm{Kahler}}$, computed with the complex structure defined by **K**:

$$J^\mu{}_\nu = -y g^{\mu\rho}\left(\mathrm{d}K^\flat\right)^+_{\rho\nu}$$

Constructing gravity duals to $\mathbf{M_3 \simeq S^3}$

- Our strategy for constructing gravity duals to the boundary geometries on $\mathbf{M_3 \simeq S^3}$ is to begin with an arbitrary $\mathbf{U(1) \times U(1)}$-invariant self-dual Einstein metric on a four-ball $\mathbf{B_4}$, which is asymptotically locally AdS with conformal boundary $\partial \mathbf{B_4} = [\mathbf{M_3}]$

- These metrics can be written down (locally) in explicit form, and are labeled by an arbitrary number of parameters [Calderbank-Pedersen] $\rightarrow$ solutions a la multi-center (**m**-pole solutions)

- Then we pick an arbitrary Killing vector $\mathbf{K} = \mathbf{b_1} \partial_{\varphi_1} + \mathbf{b_2} \partial_{\varphi_2}$, where $\partial_{\varphi_1}, \partial_{\varphi_2}$ are a basis of $\mathbf{U(1) \times U(1)}$

- By construction, for each metric and each choice of Killing vector **K** we locally get a supersymmetric supergravity solution

- Finally, we prove that for any fixed choice of self-dual Einstein metric, this leads to a one-parameter family of solutions labelled by $\mathbf{b_1/b_2}$, which are globally regular

Conformal boundary

- Asymptotically (near to $\mathbf{y=0}$) the bulk KIlling spinor has the expansion

$$\epsilon \;=\; \mathbf{y}^{-1/2}\left[\left(\mathbf{1}+\Gamma_0+\tfrac{1}{4}\mathbf{y}\mathbf{w}_{(1)}\Gamma_0\right)\left(\begin{array}{c}\chi\\ \mathbf{0}\end{array}\right)+\mathcal{O}(\mathbf{y}^2)\right]$$

where χ is a three-dimensional spinor satisfying the rigid (new minimal) Killing spinor equation and

$$\mathbf{w}(\mathbf{y},\mathbf{z},\bar{\mathbf{z}})=\mathbf{w}_{(0)}(\mathbf{z},\bar{\mathbf{z}})+\mathbf{y}\mathbf{w}_{(1)}(\mathbf{z},\bar{\mathbf{z}})+\mathcal{O}(\mathbf{y}^2)$$

- The structure induced on the conformal boundary (at $\mathbf{y=0}$) is precisely the 3d background geometry [Closset-Dumitrescu-Festuccia-Komargodski], so χ obeys that rigid new minimal supersymmetry equation (similar to 4d version we saw earlier)

Holographic free energy

- The holographic free energy is the on shell supergravity Euclidean action evaluated on a solution, regularised and renormalised, using the prescription of holographic renormalisation

$$-\log Z_{\text{gravity}} = S_{\text{Einstein}-\text{Maxwell}} + S_{\text{Gibbons}-\text{Hawking}} + S_{\text{counterterms}}$$

- The individual terms do depend on the detailed solution, for example

$$\frac{1}{16\pi G_4}\int_{B_4} F^2\sqrt{\det g}\,\mathrm{d}^4x \;=\; -\frac{\pi(|b_1|+|b_2|)^2}{8G_4 b_1 b_2} + \frac{1}{256\pi G_4}\int_{M_3}\left(3w_{(1)}^3 + 4w_{(1)}w_{(2)}\right)\sqrt{\det g_3}\,\mathrm{d}^3x$$

Holographic free energy

- However, remarkably the final result is

$$-\log \mathbf{Z}_{\text{gravity}} = \mathbf{S}_{\text{on shell}} = \frac{(|\mathbf{b}_1| + |\mathbf{b}_2|)^2}{4|\mathbf{b}_1||\mathbf{b}_2|} \cdot \frac{\pi}{2\mathbf{G}_4}$$

- This formula is derived without knowledge of any specific metric! We have assumed only that a solution with the correct global properies exists (the **m**-pole provide infinitely many explicit examples)

- It is analogous to the formula for the volume of a Sasakian manifold in terms of an arbitrary Reeb Killing vector, that was shown to be essentially independent of the explicit metric [DM-Sparks-Yau]

- It agrees perfectly with large **N** limit of $-\log \mathbf{Z}$ computed using localization! Result analogous to general check of **a**-maximisation = **Z**-minimisation in AdS_5/CFT_4 ✓

Other examples

There are a few other examples in other dimensions, but a more systematic understanding is still lacking

4d/5d [Cassani-DM] supersymmetric Casimir energy of $\mathcal{N}=\mathbf{1}$ field theories on $\mathbf{S^1} \times \mathbf{S^3_{sqaushed}}$ compared to newly constructed supersymmetric asymptotically locally AdS_5 solution of type IIB supergravity

5d/6d [Jafferis-Pufu] large **N** free energy of susy gauge theories on $\mathbf{S^5}$ matched to holographic free energy and entanglement entropy of supersymmetric AdS_6 in massive type IIA supegravity

5d/6d [Alday et al] match free energy of field theories on examples of deformed $\mathbf{S^5}$ to holographic computations in newly constructed asymptotically locally AdS_6 solutions of massive type IIA supergravity

Other examples

4d/5d [Bobev et al] large **N** free energy of $\mathcal{N} = \mathbf{2}^*$ SYM (mass-deformed $\mathcal{N} = \mathbf{4}$ SYM) on round $\mathbf{S}^4$ compared to new aymptotically AdS_5 solution of type IIB supergravity

3d/4d [Huang-Rey-Zhou] and [Nishioka] large **N** free energy of susy gauge theories on branched $\mathbf{S}^3$ = "supersymmetric Rényi entropy", matched to on-shell action of supersymmetric topological (hyperbolically sliced) black hole

4d/5d [Huang-Zhou] and [Crossley et al] large **N** supersymmetric Rényi entropy in $\mathcal{N} = \mathbf{4}$ SYM matched to on-shell action of hyperbolically sliced supersymmetric black hole solution of type IIB supergravity

Outlook

- Push the localization technique: how many more path integrals can we compute exactly and explicitly, and what can we learn from them? Especially in higher dimensions ($\mathbf{d = 4, 5}$) rigid supersymmetry allows for large classes of geometries (in $\mathbf{d = 5}$ a systematic classification is still missing)

- Supersymmetric localization yields very precise predictions for the gauge/gravity duality, allowing to perform detailed tests in situations without superconformal invariance. Supergravity solutions should reproduce exactly numbers and functions, rather than qualitative features of the putative field theory dual!

- This is forcing us to refine the holographic dictionary and think about "why" computations on the two sides match $\longrightarrow$ progress towards "proving" the gauge/gravity duality in islands of growing size (as opposed to checking it in a large number of isolated examples)

- Localization may be used to perform exact quantum computations in gravitational theories. Gauge/gravity duality tested beyond the semi-classical/large $\mathbf{N}$ limit