Differential Entropy

Based on:

arXiv:1305.0856, arXiv:1310.4204, arXiv:1406.5859

with Vijay Balasubramanian, Borun Chowdhury, Bartek Czech and Michal Heller

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Related work in:

arXiv:1403.3416 - Myers, Rao, Sugishita
arXiv:1406.4889 - Czech, Dong, Sully
arXiv:1406.4611 - Hubeny
arXiv:1408.4770 - Headrick, Myers, Wien
arXiv:1409.4473 - Czech, Lamprou

There are many interesting connections between black hole entropy, entanglement entropy and space-time geometry.

Entanglement entropy is usually defined for QFT. It has been suggested that more generally in quantum gravity there is also a notion of entanglement entropy associated to a region and its complement which equals

Bianchi, Myers

$$S_{\mathrm{ent}} = \frac{A}{4G}$$

This result is *finite*, as opposed to entanglement entropy in QFT.

Qualitative idea: finiteness is due to built in UV regulator in quantum gravity. UV scale = Planck scale.

Indeed: $S_{\rm ent} = \frac{A}{\epsilon^{D-2}} + \ldots \sim \frac{A}{4G} \quad \text{for} \quad \epsilon = \ell_P$

Key question: can we make the link between $\frac{A}{4G}$ and some notion of entanglement entropy more precise?

Would provide an interesting new probe of space-time geometry and the dof of quantum gravity.

To assign an entanglement entropy to a region of space-time, one needs a factorization of the Hilbert space

$$\mathcal{H} = \mathcal{H}_{\text{outside}} \bigotimes \mathcal{H}_{\text{inside}}$$

Such a factorization is often used when computing Hawking radiation, when discussing the information loss paradox, and in many arguments pertaining to the (non)existence of firewalls.

Here we will argue this factorization fails in general.

Factorization of the Hilbert space is closely related to the existence of a closed subalgebra of the algebra of observables.

$$\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_{\bar{D}}$$

$$\mathcal{A} \supset \mathcal{A}_D$$

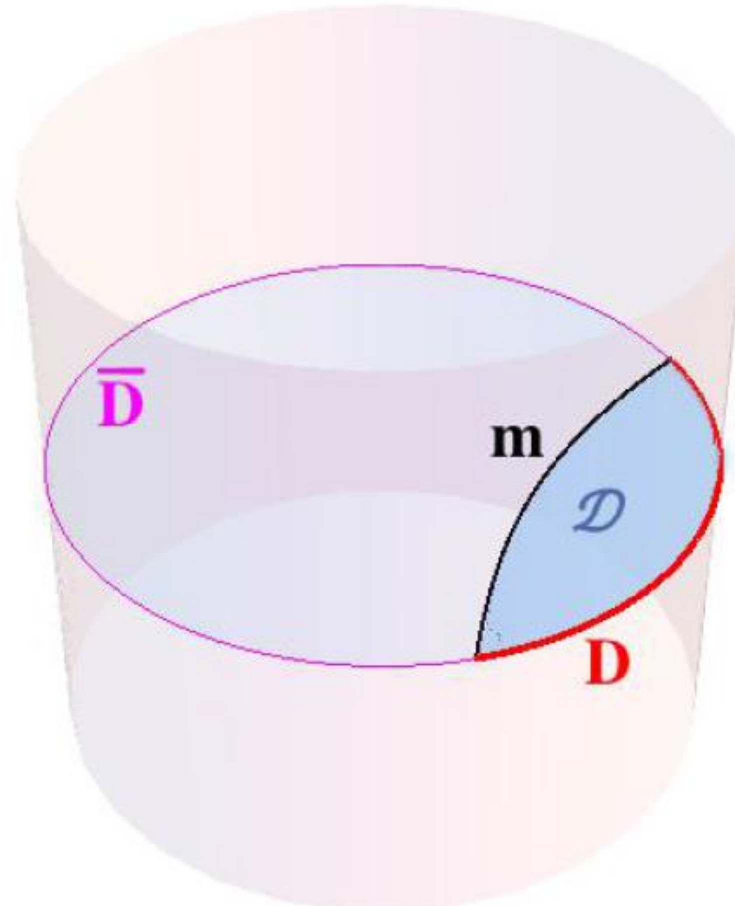
Operators in $\mathcal{A}_D$ act non-trivially on $\mathcal{H}_D$ and trivially on $\mathcal{H}_{\bar{D}}$.

Can one associate a closed algebra of observables to an arbitrary space-time domain?

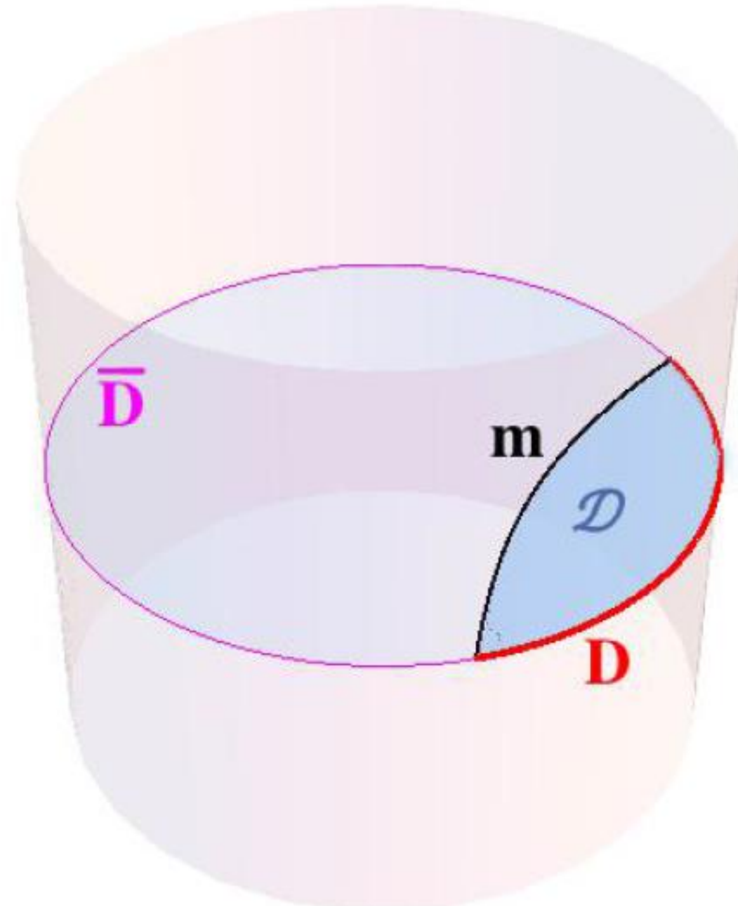
In AdS, consider an extremal domain $\mathcal{D}$ bounded by a minimal surface m.

This is the setup of Ryu-Takayanagi to compute entanglement entropy in AdS/CFT.

Clearly, the boundary Hilbert space factorizes and there is a subalgebra $\mathcal{A}_D$ of the algebra of field theory observables.



In this case, it is tempting to associate $\mathcal{H}_D$ and $\mathcal{H}_{\bar{D}}$ also to the bulk domains $\mathcal{D}$ and $\bar{\mathcal{D}}$.



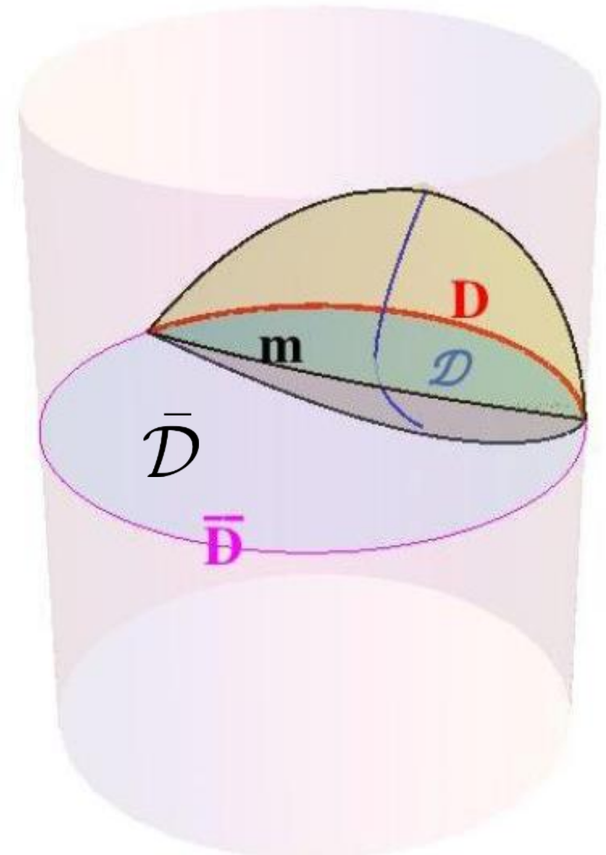
This would certainly agree with the idea that A/4G represents entanglement entropy between bulk degrees of freedom: Area(m)/4G is the right entanglement entropy.

Arguments in favor:

1. There are no observables which are localized in region D and which probe $\bar{\mathcal{D}}$. (Hubeny)

2. For spherical D, m is the horizon of a single Rindler observer, and one can think of $\mathcal{A}_D$ as the algebra of observables associated to the Rindler observer. (Casini, Huerta, Myers)

3. Matches with one-loop corrections (Faulkner, Lewkowycz, Maldacena)



Next, consider the union of two extremal domains

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$$

Clearly, the algebra of observables must contain

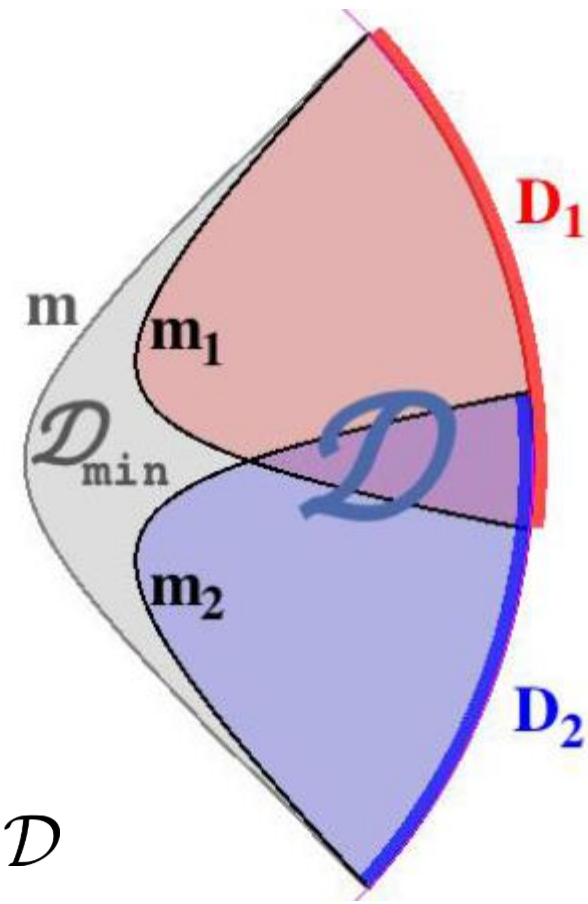
$$\mathcal{A}_{D_1} \cup \mathcal{A}_{D_2}$$

In the field theory, the smallest algebra that contains these is

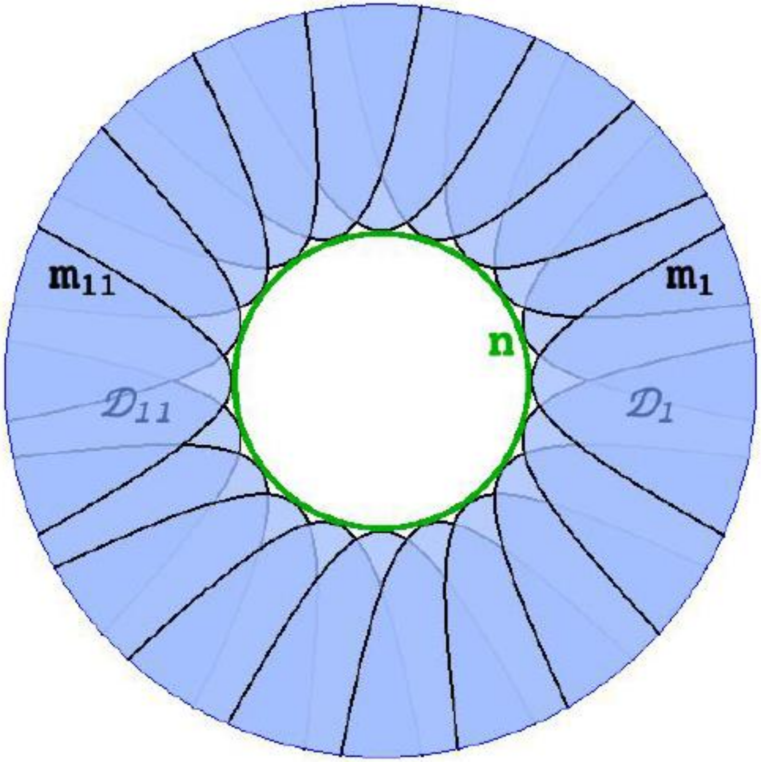
$$\mathcal{A}_{D_1 \cup D_2}$$

but this corresponds to a larger bulk domain $\mathcal{D}_{\min}$!

So we cannot associate a closed algebra nor entanglement entropy to $\mathcal{D}$



What is the right generalization?



Proposal: for $\mathcal{C} = \cup_i \mathcal{D}_i$

Observables on $\mathcal{C} = \oplus_i \mathcal{A}_{D_i}$

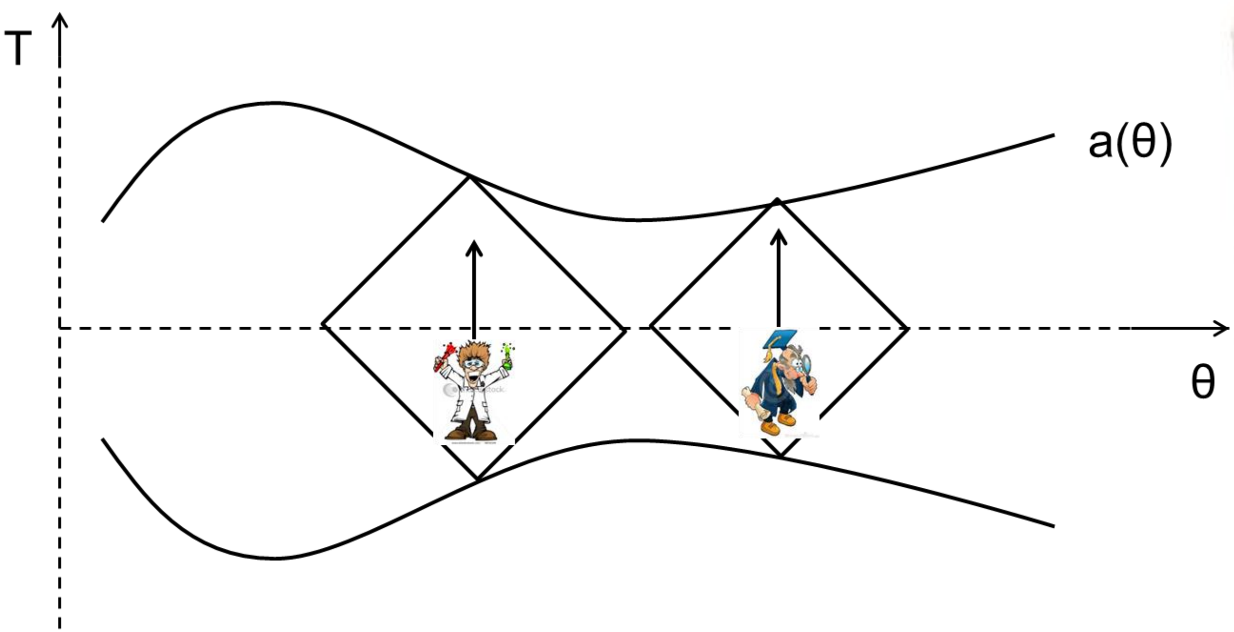
Relevant notion of entropy is differential entropy

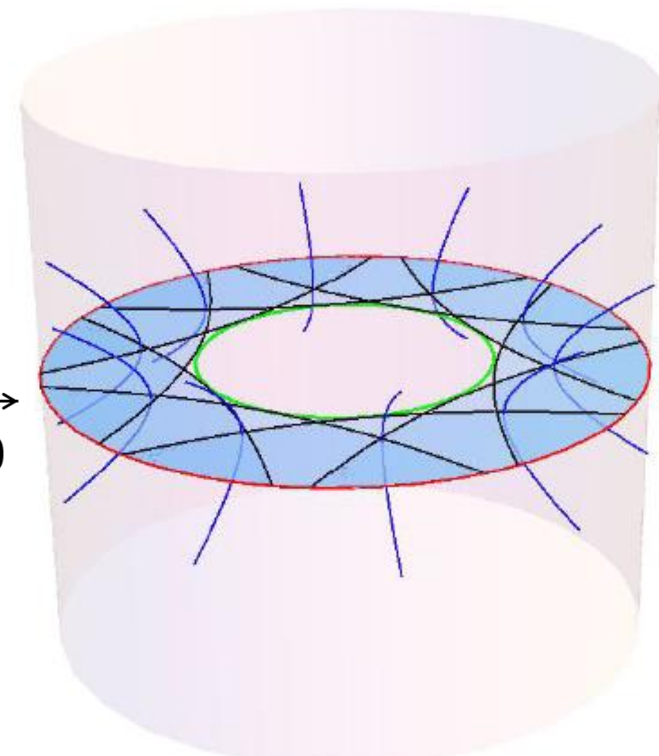
$$S = (\lim_{k\to\infty}) \sum_{i=1}^{k} S(D_i) - S(D_i \cap D_{i+1})$$

Agrees with the length of the curve in AdS_3.

Differential entropy is a measure of the amount of information about the state of the system inaccessible to local Rindler observers.

Alternatively, one can think of it as associating entropy to a field theory on a finite time strip.





Differential entropy is a quantity associated to a family of intervals on the boundary.

$$S = (\lim_{k\to\infty}) \sum_{i=1}^{k} S(D_i) - S(D_i \cap D_{i+1})$$

It computes the length of a curve which is tangent to all bulk extremal curves whose endpoints coincide with the endpoints of the intervals.

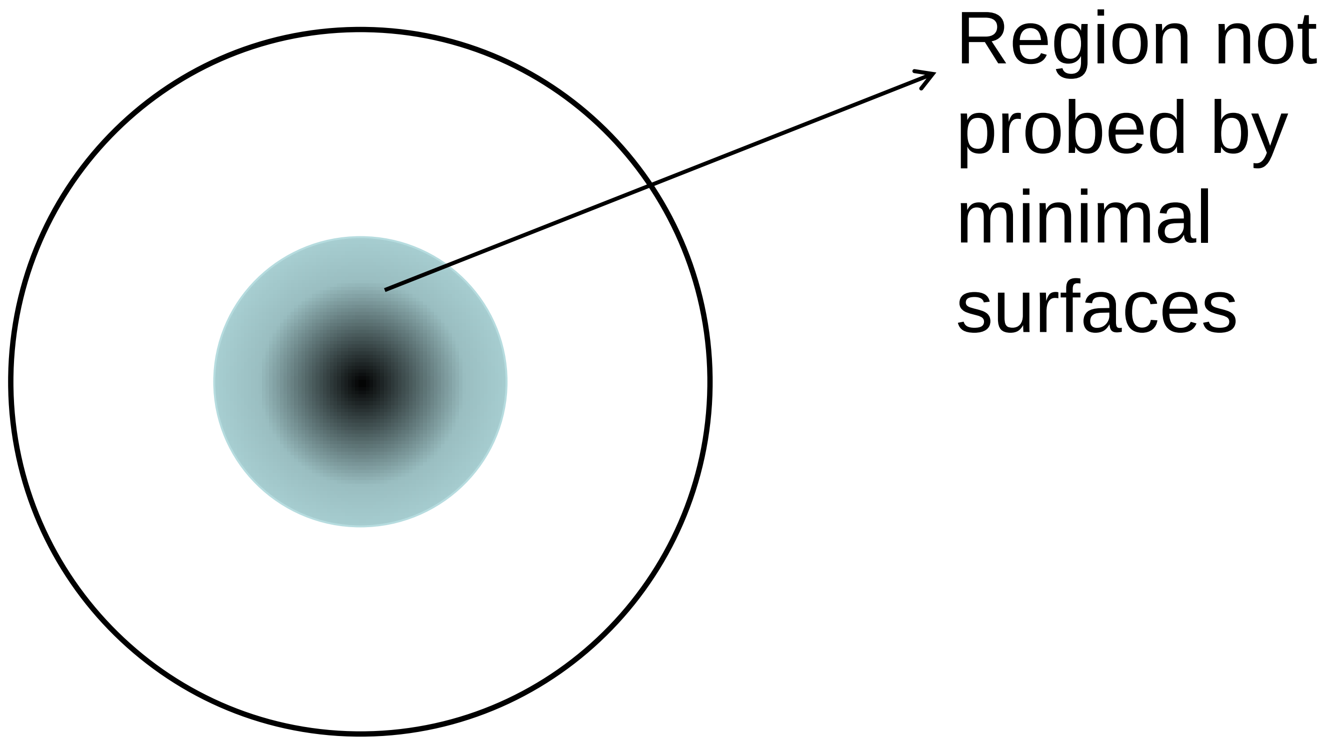
It is a new probe of a quantity localized in the interior of AdS.

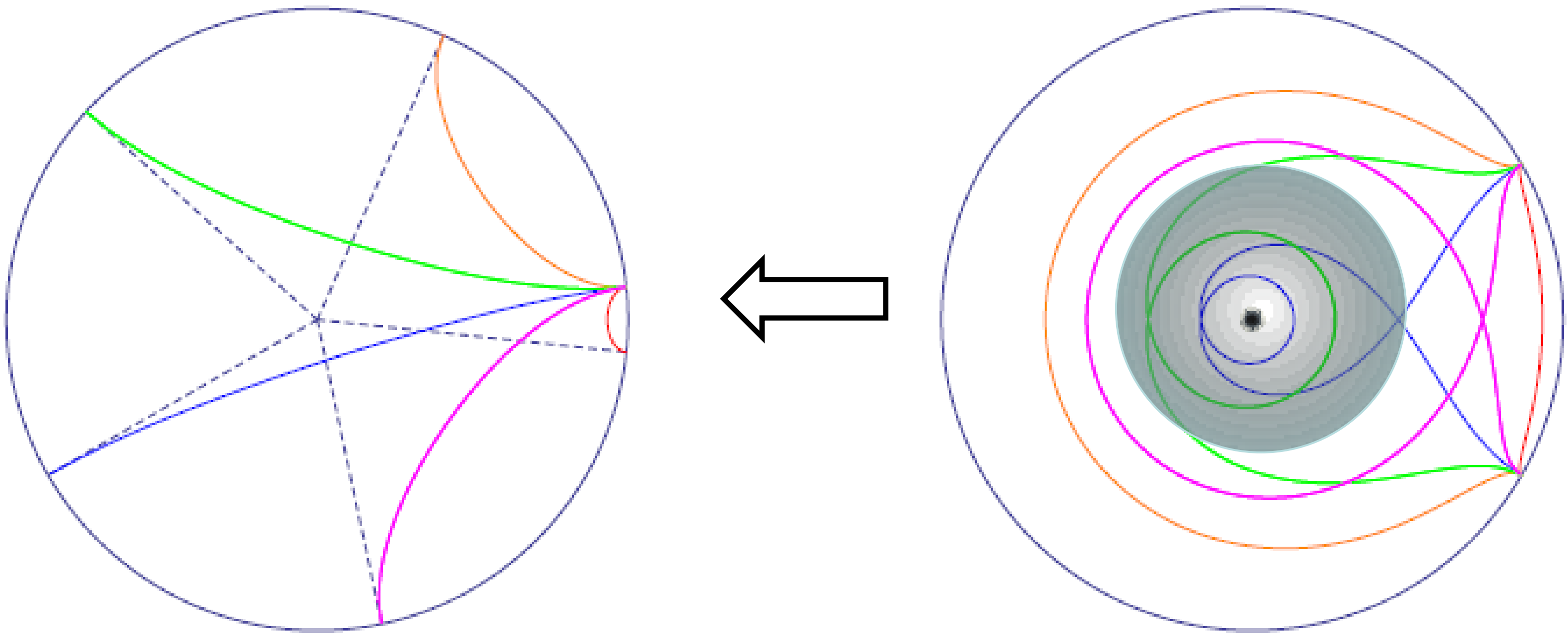
It does not compute any form of entanglement entropy.

There are many interesting subtleties and generalizations.

- Works also for certain higher derivative theories. Nice general argument given by Headrick, Myers, Wien
- Works in higher dimensions, need foliation by codimension one surfaces (Czech, Dong, Sully; Headrick, Myers, Wien)
- There are issues for non-convex curves.
- Need extremal surfaces, not necessarily minimal. Field theory interpretation of such extremal surfaces is unknown.

E.g. conical defect geometry





Regular geodesics in covering space.

Covering space = a “long string” sector of dual CFT.

Long geodesics can penetrate this region

Does the length of these long geodesics have a field theory dual?

This requires us to go to the long string picture and ungauge the Z_n symmetry, compute the entanglement entropy there, and then sum over gauge copies. (Balasubramanian, Chowdhury, Czech, JdB)

Ungauging is often necessary as an intermediate step in defining entanglement entropy in gauge theories. (see e.g. Donnelly; Agon, Headrick, Jafferis, Kasko; Casini, Huerta, Rosabal)

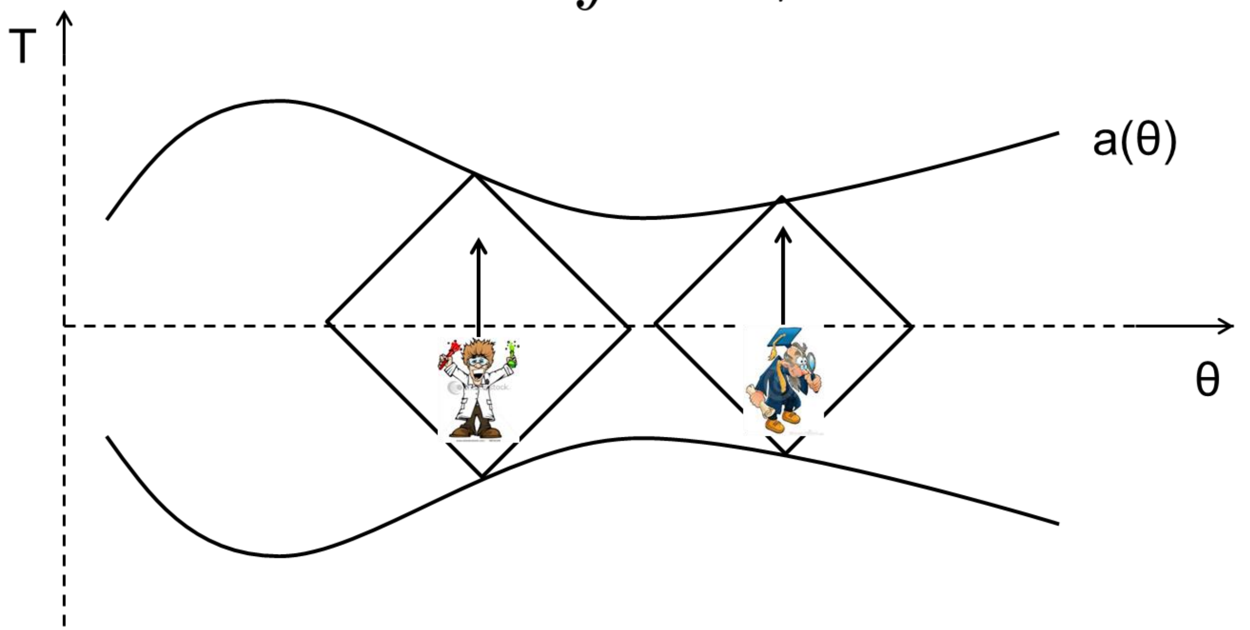
The gauge theory description is valid at the weakly coupled orbifold point, but may survive to strong coupling.

Since the long string contains fractionated (matrix) degrees of freedom, we apparently need entanglement between fractionated degrees of freedom to resolve the deep interior and near horizon regions in AdS.

Direct reconstruction of bulk from boundary (Czech, Lamprou)

Idea: points in space-time correspond to collections of intervals for which the differential entropy vanishes. Introduce

$$I = \int d\theta \sqrt{-\frac{d^2 S}{da^2}(1 - a'(\theta)^2)}$$



Extrema of

$$I = \int d\theta \sqrt{-\frac{d^2 S}{da^2}(1 - a'(\theta)^2)}$$

indeed correspond to points. Construction is state-dependent! Given two solutions one can define a distance

$$d(a_1, a_2) = \frac{1}{2} S_{\text{diff}}(\min(a_1(\theta), a_2(\theta))$$

and this turns out to reproduce the geodesic distance between points.

Outlook:

- Differential entropy is a new probe of bulk physics. It can perhaps be generalized to other quantities (differential Wilson loops…)
- Apparently degrees of freedom cannot be localized in quantum gravity. Breakdown of locality. To describe the outside of a hole, we already need all degrees of freedom. To describe interior, we use the same degrees of freedom. Complementarity?
- Extremal rather than causal surfaces play a crucial role. Why?
- Role of finite time limitations on measurements for differential entropy?

Open questions:

- Interpretation of differential entropy? Ignorance of collections of observers? State swapping protocol? Relation to MERA? (Czech, Hayden, Lashkari, Swingle)
- New ways to reconstruct bulk physics.
- Implications for black hole physics?
- Implications for cosmology? Can only localize degrees of freedom in regions bounded by extremal surfaces?
- Can write down a first law for differential entropy. New way to derive Einstein Equations in the bulk? (cf Faulkner, Guica, Hartman, Myers, van Raamsdonk)

$$
\begin{tikzcd}
& \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\right) \arrow[dl, "\Pi_{12}"'] \arrow[d, "\Pi_{23}"] \arrow[dr, "\Pi_{31}"] & \\
\frac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right) & \frac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right) & \frac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right) \\
& \frac{1}{2}\left(|\uparrow\uparrow\uparrow\rangle\langle\uparrow\uparrow\uparrow| + |\downarrow\downarrow\downarrow\rangle\langle\downarrow\downarrow\downarrow|\right) \arrow[ul, "\Pi_{12}"] \arrow[u, "\Pi_{23}"] \arrow[ur, "\Pi_{31}"'] &
\end{tikzcd}
$$

Local observers who can only measure two spin subsystems cannot distinguish the pure state from the mixed state. We would associate Residual Entropy S= log 2 to this system.

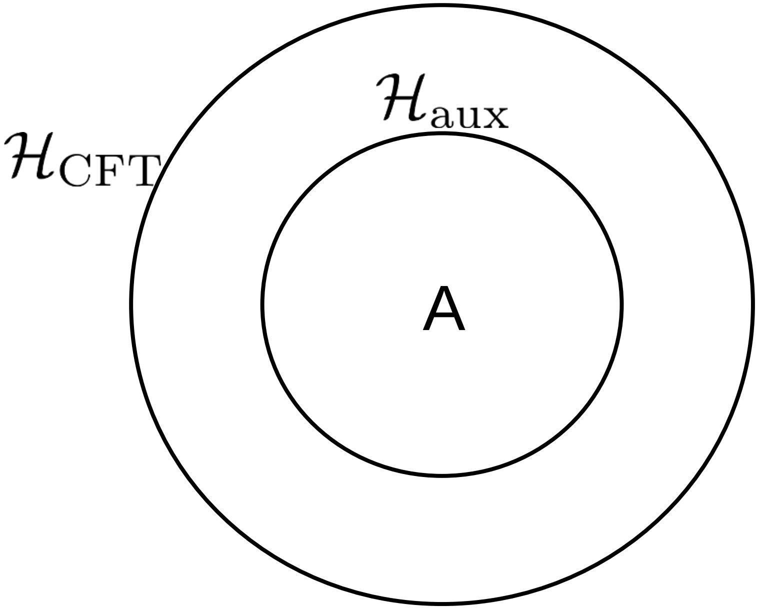
Interpretation of differential entropy/residual entropy?

Suppose it corresponds indeed to the entropy of some density matrix ρ, but there is no evidence that this is the reduced density matrix of some tensor factor in the Hilbert space.

If not, what does it have to do with the entanglement of quantum gravitational degrees of freedom? How do we reconstruct the original vacuum state from ρ if we cannot purify it?

Idea:

In quantum gravity we usually need to associate Hilbert spaces to boundaries of space-time. Think Wheeler-de Witt wavefunctions, Chern-Simons theory, etc.



Now suppose that to the outside we should really associate a state in

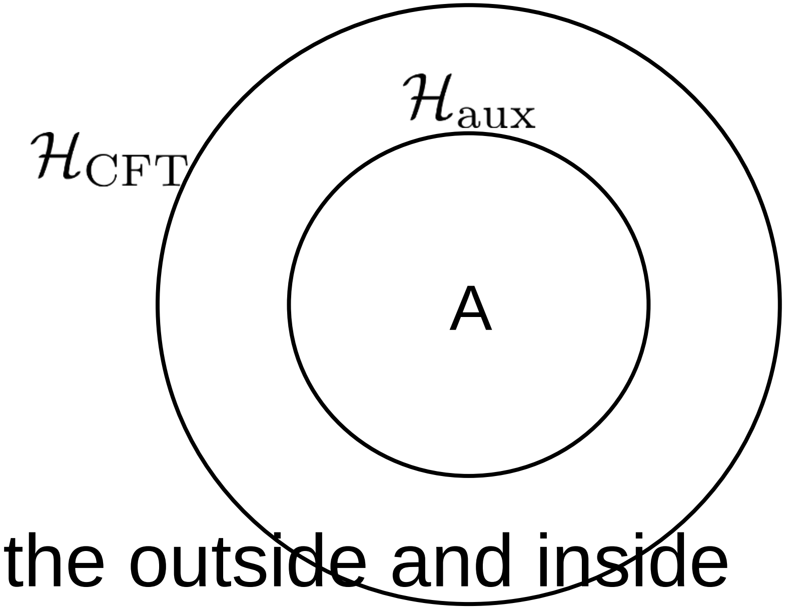
$$\mathcal{H}_{\mathrm{CFT}} \otimes \mathcal{H}_{\mathrm{aux}}$$

and to the inside region a state in

$$\mathcal{H}_{\mathrm{aux}}$$

and that gluing the spacetimes together involves taking an obvious product over $\mathcal{H}_{\mathrm{aux}}$.

Now if we write



$$\rho = \sum_i a_i |i\rangle\langle i|$$

$$|0\rangle_{\mathrm{CFT}} = \sum b_i |i\rangle$$

then it is natural to associate to the outside and inside regions the pure states

$$\psi_{\mathrm{out}} = \sum_i \sqrt{a_i}|i\rangle_{\mathrm{CFT}} \otimes |\tilde{i}\rangle_{\mathrm{aux}}$$

$$\psi_{\mathrm{in}} = \sum_i \frac{b_i}{\sqrt{a_i}}|\tilde{i}\rangle_{\mathrm{aux}}$$

Tracing ψ_{out} over $\mathcal{H}_{\mathrm{aux}}$ then yields back ρ.
Gluing ψ_{out} and ψ_{in} together reproduces the vacuum state. Consistent picture!!!