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— Ludwig Boltzmann, *Populäre Schriften* (1905).

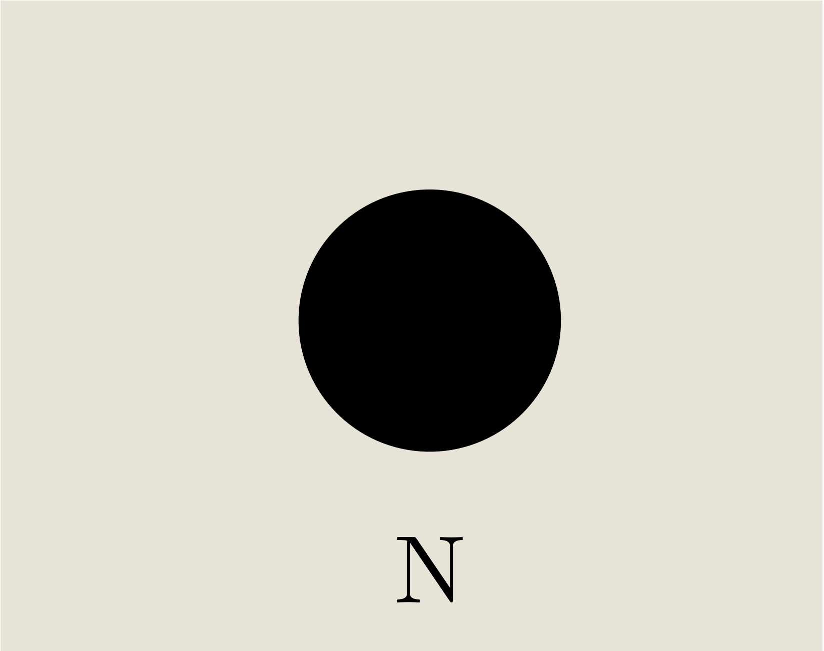
Exact Quantum Black hole entropy:
a macroscopic window into quantum gravity

Sameer Murthy
King's College London

The String Theory Universe
Mainz, Sep 25, 2014

Black holes in string theory

Macroscopic

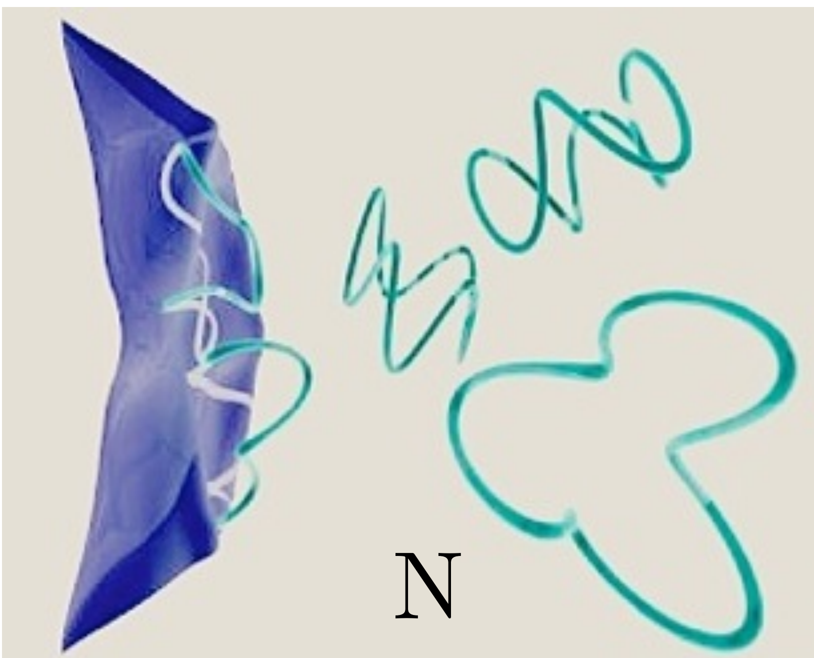


Bekenstein-Hawking ’74

$$S_{\rm BH}^{\rm class} = \frac{A_H}{4\ell_{\rm Pl}^2} = \pi\sqrt{N}$$

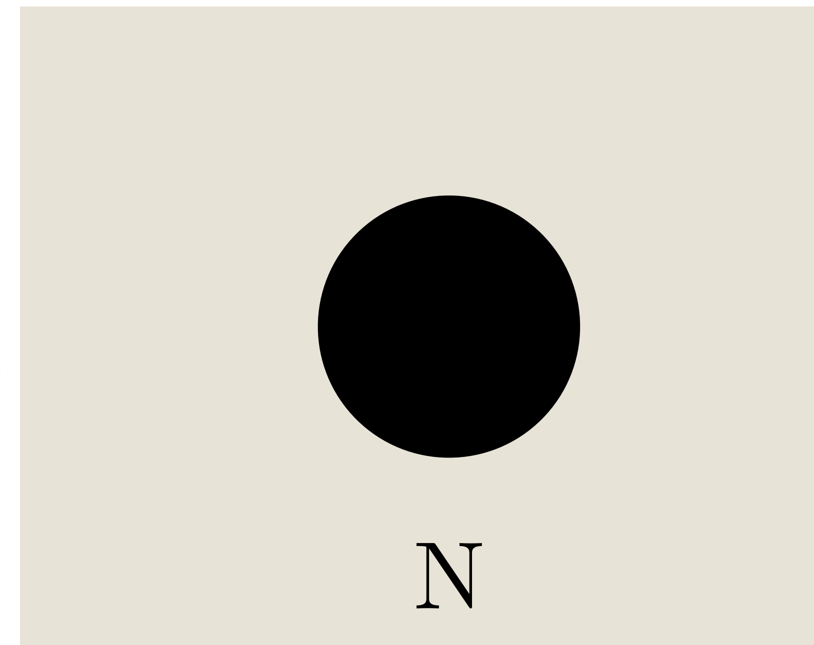
Black holes in string theory are ensembles of microscopic excitations

Microscopic



$g_s \mathrm{N} \ll 1$ ← g_s → $g_s \mathrm{N} \gg 1$

Macroscopic



Strominger-Vafa '96

$$d_{\mathrm{micro}}(\mathrm{N}) = e^{\pi\sqrt{\mathrm{N}}} + \cdots \quad (\mathrm{N} \to \infty)$$

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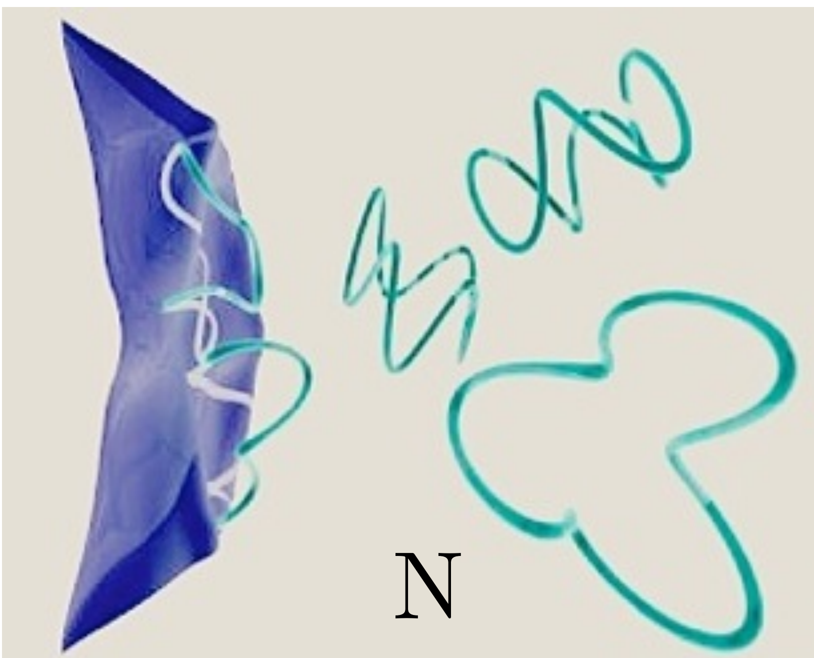

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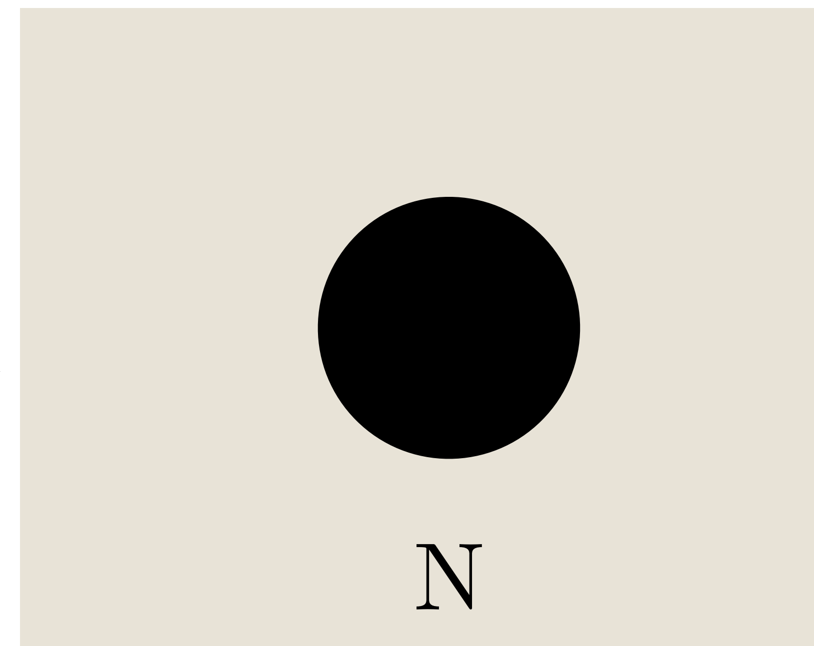

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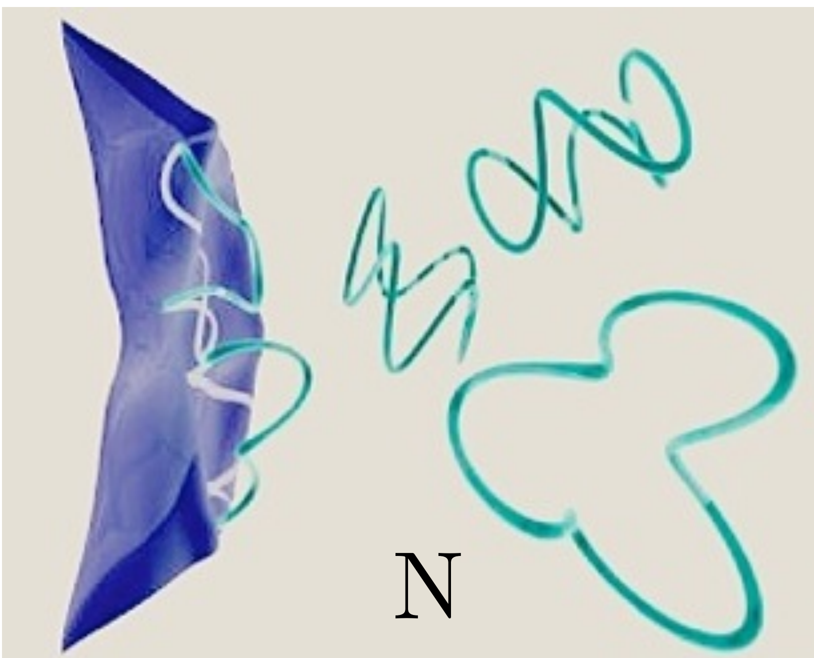
Microscopic | **Macroscopic**



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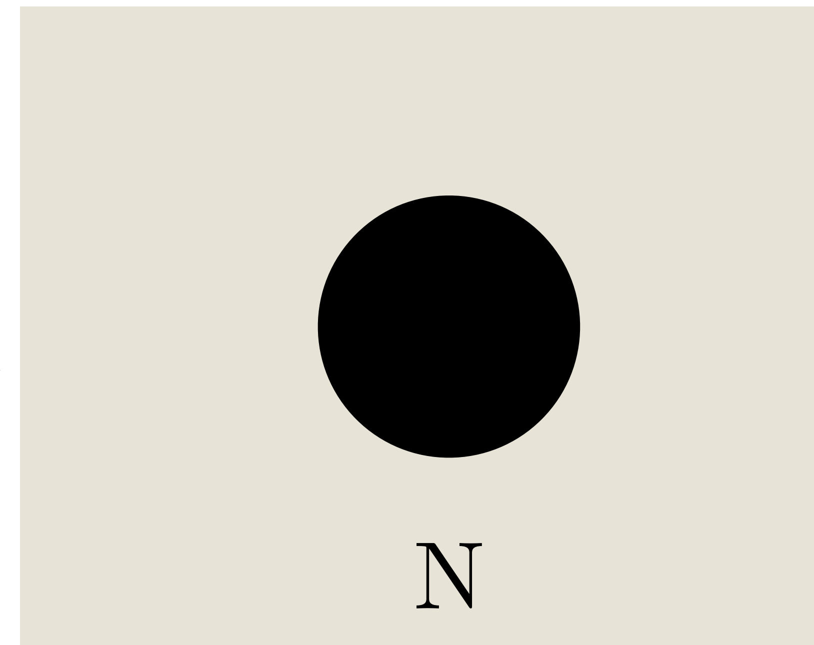

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$$\log d_{\mathrm{micro}} = S_{\mathrm{BH}}^{\mathrm{class}} + \cdots \quad \rightarrow \quad S_{\mathrm{BH}}^{\mathrm{quant}}$$

(finite N)

What is new? Finite size quantum effects!

$$S_{\mathrm{BH}}^{\mathrm{quant}} = \frac{1}{4}A + a_0 \log(A) + a_1 \frac{1}{A} + a_2 \frac{1}{A^2} + \cdots$$
$$+ b_1(A) e^{-A} + \cdots$$

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1. What is the physics of these corrections?
2. How to compute them in a concrete model?
3. Can we compare them to a similar expansion in the microscopic theory?

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Exact AdS/CFT

Supersymmetric Localization

Mock modular forms

Finite size corrections arise from quantum fluctuations in the black hole

Wald Entropy formalism (c.f. talk of J. Camps)

- Obeys the first law of thermodynamics
- Extends Bekenstein-Hawking area law in GR
- Applicable to any *local* effective action of gravity
- Successfully applied to BH models in supergravity

(Cardoso, de Wit, Mohaupt ’99)

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(Cardoso, de Wit, Mohaupt '99)

We still need a good formalism to study Quantum BH entropy including non-analytic and non-local terms.

"If you want to study quantum gravity, then you better study $\mathrm{AdS/CFT}$."

— R. Emparan, at lunch on Monday.

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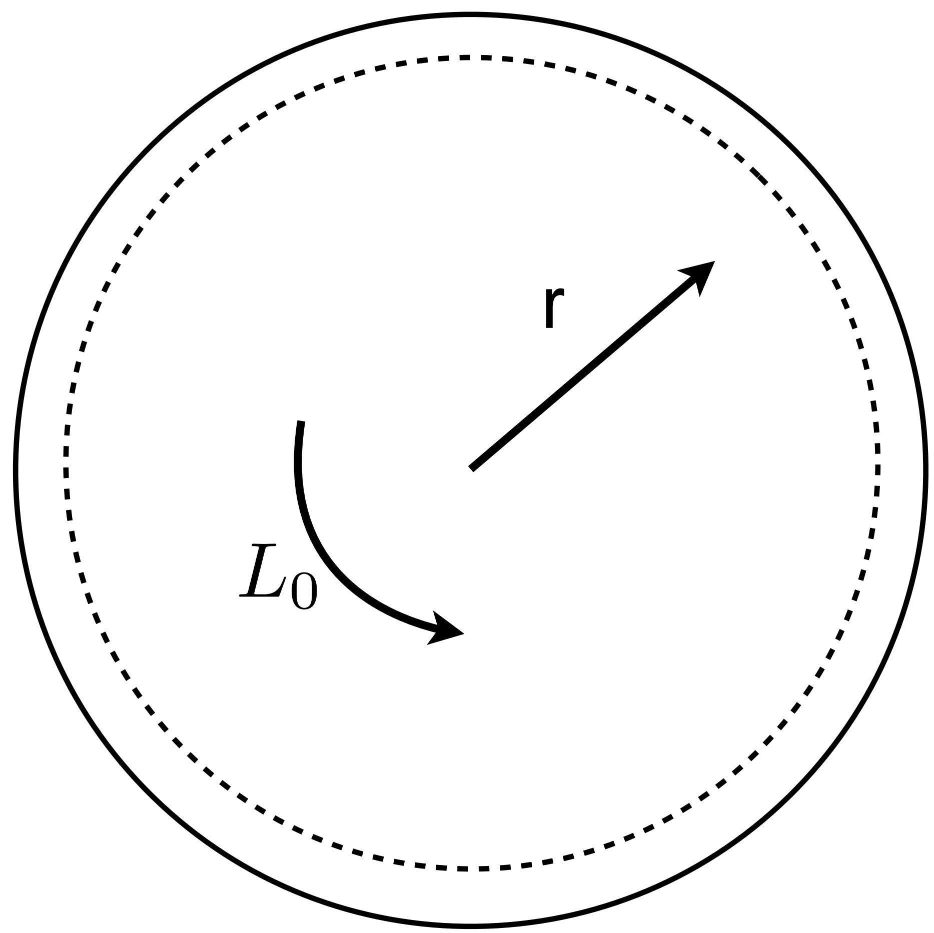
Note!

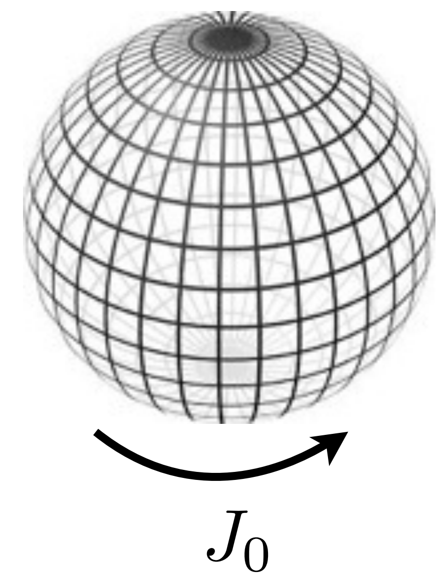
Quantum gravity = 1/N effects.

Supersymmetric black holes develop a near-horizon $\mathrm{AdS_2}$ factor

4d extremal Reissner-Nordstrom solution

$\Rightarrow$ near-horizon geometry $AdS_2 \times S^2$.





$\mathbf{Euclidean\ AdS_2 \times S^2}$

Bekenstein-Hawking-Wald entropy recast as a minimization problem. (Sen ’05, c.f. attractor mechanism Ferrara, Kallosh, Strominger ’95)

Quantum BH entropy is a functional integral over $\mathrm{AdS_2}$ configurations (Sen '08)

$$\exp(S^{\mathrm{qu}}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \left\langle \exp\left[-i\, q_I \oint A^I\right] \right\rangle^{\mathrm{reg}}_{\mathrm{AdS_2}}$$

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- Boundary conditions fixed by classical BH configuration
- Saddle point evaluation $\Longrightarrow$ classical Wald entropy
- Logarithmic one-loop corrections can be computed.

(Sen + Banerjee, Gupta, Mandal, '10-'14, c.f. talk of R. Gupta)

Dual theory for BPS BH is a collection of supersymmetric ground states

Dual $\mathbf{CFT_1}$ obtained as IR limit of brane configuration that makes up the black hole.

In d=0+1, no space for long-wavelength fluctuations.

$$Z_{\mathrm{CFT}_1}(q) = \mathrm{Tr}_{\mathcal{H}(q)}\, 1 = d_{\mathrm{micro}}(q)\,.$$

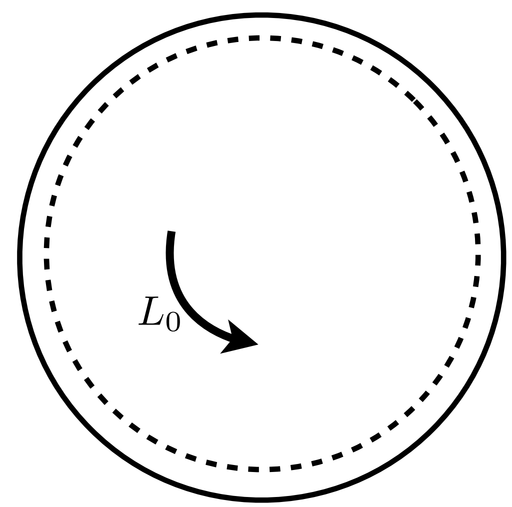
AdS/CFT correspondence

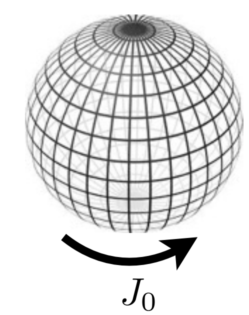
$$\Rightarrow Z_{\mathrm{AdS}_2}(q) = d_{\mathrm{micro}}(q)$$

Set up for the QE functional integral

$$\exp(S^{\text{qu}}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \Big\langle \exp\big[- i\, q_I \oint A^I \big] \Big\rangle^{\text{reg}}_{\text{AdS}_2} \,.$$

Supercharge Q with $Q^2 = L_0 - J_0\,.$





Euclidean $\mathbf{AdS_2 \times S^2}$

$\mathcal{M}$: Field space of supergravity.

$d\mu$: Measure on this field space.

$\mathcal{O}$: Wilson line.

$\mathcal{S}$: Action of graviton and other massless fields.

Localization

Witten '88, Duistermaat-Heckmann '82, Atiyah-Bott '84, Pestun '07

An integral of a $Q-$invariant operator $\mathcal{O}$

$$I := \int_{\mathcal{M}} d\mu \, \mathcal{O} \, e^{-\mathcal{S}} \, .$$

localizes onto the submanifold $\mathcal{M}_Q$ of solutions of the off-shell BPS equations $Q\,\Psi = 0$

$$I = \int_{\mathcal{M}_Q} d\mu_Q \, \mathcal{O} \, e^{-\mathcal{S}} \, .$$

How to compute the functional integral

(A.Dabholkar, J.Gomes, S.M. ’10, ’11, ’14)

1. Formalism: N=2 off-shell supergravity. (de Wit, van Holten, Van Proeyen ’80)

2. Find all solutions of localization equations $Q\,\Psi = 0$, subject to $AdS_2 \times S^2$ boundary conditions. (R.Gupta, S.M. ’12)

3. Evaluate action on these solutions (including all higher derivative terms). Compute the measure.

4. Only chiral-superspace integrals in the action contribute. These are exactly known in string theory. (V.Reys, S.M. ’13)

Prototype: N=8 string theory in 4d (macro)

(Cremmer, Julia '78)

Macroscopic description: d=4 supergravity coupled to 28 U(1) gauge fields + superpartner scalars + fermions.

1/8 BPS dyonic BH solutions. (Cvetic, Youm '96)

BH Charges (q_I, p^I) Quartic U-duality invariant $N(q,p)$

Classical BH Entropy $S_{BH} = \pi\sqrt{N} + \cdots$

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Microscopic degeneracies $d_{\mathrm{micro}}(\mathrm{N})$ computed using representation as D1-D5-P-K system in Type II string theory. (Maldacena, Moore, Strominger '99)

Evaluation of the functional integral

- Truncation of N=8 to N=2 theory with 7 vector multiplets.
- QG path integral reduces to an 8-dimensional integral.
- 7 of the integrals are Gaussian.

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- 7 of the integrals are Gaussian.

$$e^{S_{BH}^{\mathrm{qu}}}(N) = \int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \pi^2 N/4\sigma\right) = \widetilde{I}_{7/2}(\pi\sqrt{N})$$

A quantitative test

(Classical entropy)

N	$d_{\rm micro}(\rm N)$		$\exp(S^{\rm cl}(\rm N))$
3	8		230.76
4	12		535.49
7	39		4071.93
8	56		7228.35
11	152		33506.14
12	208		53252.29
15	513		192400.81
...	...		...
10^5	exp(295.7)		exp(314.2)

$$\log(d_{\rm micro}) \xrightarrow{\Delta\to\infty} S^{\rm cl}_{BH} \,.$$

A quantitative test

(A.Dabholkar, J.Gomes, S.M. '11)

N	$d_{\rm micro}(\rm N)$	$\exp\big(S^{\rm qu}(\rm N)\big)$	$\exp\big(S^{\rm cl}(\rm N)\big)$
3	8	7.97	230.76
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...	...	...	...
10^5	exp(295.7)	exp(295.7)	exp(314.2)

$$d_{\rm micro}(\Delta) = e^{S_{BH}^{\rm qu}(\Delta)}\big(1 + O(e^{-\pi\sqrt{\Delta}/2})\big)$$

For the experts..

- D-terms, one-loop determinants (partial progress).
- Addition of hypermultiplets, gravitini mutiplets,
- Sub-leading saddle points (partial progress).

- Extension to higher dimensional black holes (partial progress).
- Extension to higher dimensional AdS/CFT (partial progress).
- Other observables in quantum gravity.

Conclusions and outlook

- Finite size effects in BH thermodynamics can be computed.
- Localization methods give us convergent perturbation expansions for the quantum gravity partition function.
- Emergence of quantum structure from continuum gravity, inclusion of sub-leading saddle points are important.
- Effective low-energy theory provides strong constraints on quantum theory of gravity.

Why does this work so well?

The Fourier series of the microscopic degeneracies

$$Z(\tau) \equiv \sum_N d_{\mathrm{micro}}(N)\, e^{2\pi i N\tau} = \theta(\tau)/\eta(\tau)^6$$

is a modular form.

Strong-weak coupling symmetry: $Z(-1/\tau) = \tau^{5/2} Z(\tau)$

$$\tau \to \tau + 1$$
$$\tau \to -1/\tau\,.$$

$\Rightarrow$ $SL_2(\mathbb{Z})$ Modular symmetry group

Highly constraining

Exact formula for degeneracies

Hardy-Ramanujan-Rademacher expansion

$$d_{\rm micro}(N) = \sum_{c=1}^{\infty} c^{-9/2}\, K_c(N)\, \widetilde{I}_{7/2}\Big(\frac{\pi\sqrt{N}}{c}\Big)$$

$$= \widetilde{I}_{7/2}(\pi\sqrt{N}) + O(e^{-\pi\sqrt{N}/2})$$

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$$= e^{\pi\sqrt{N}}\Big(1 - \frac{15}{4}\log N + O(\frac{1}{N})\Big)\,.$$

Bekenstein-Hawking

One-loop corrections

Exact formula for degeneracies

Hardy-Ramanujan-Rademacher expansion

$$
\begin{aligned}
d_{\text{micro}}(N) &= \sum_{c=1}^{\infty} c^{-9/2}\, K_c(N)\, \widetilde{I}_{7/2}\Big(\frac{\pi\sqrt{N}}{c}\Big) \\
&= \widetilde{I}_{7/2}(\pi\sqrt{N}) + O(e^{-\pi\sqrt{N}/2}) \\
&= e^{\pi\sqrt{N}}\Big(1 - \frac{15}{4}\log N + O(\frac{1}{N})\Big)\,.
\end{aligned}
$$

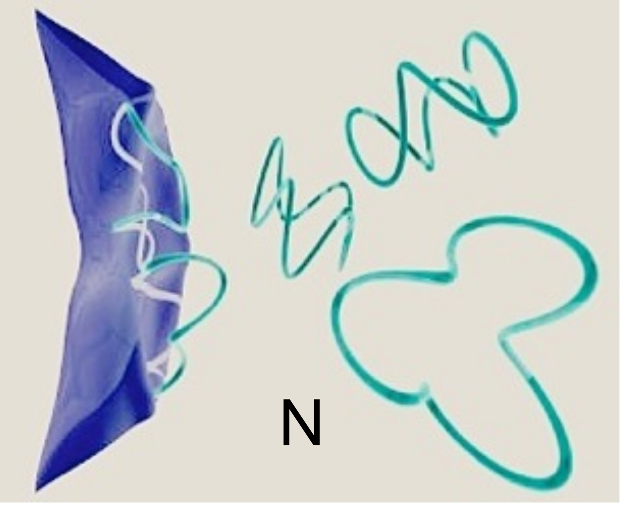
Orbifolds of AdS_2

(A.Dabholkar, J.Gomes, S.M. arXiv:1404.0033

Bekenstein-Hawking

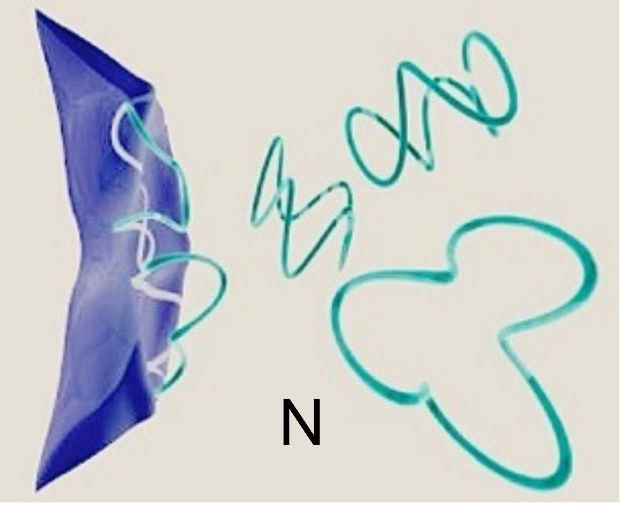
One-loop corrections

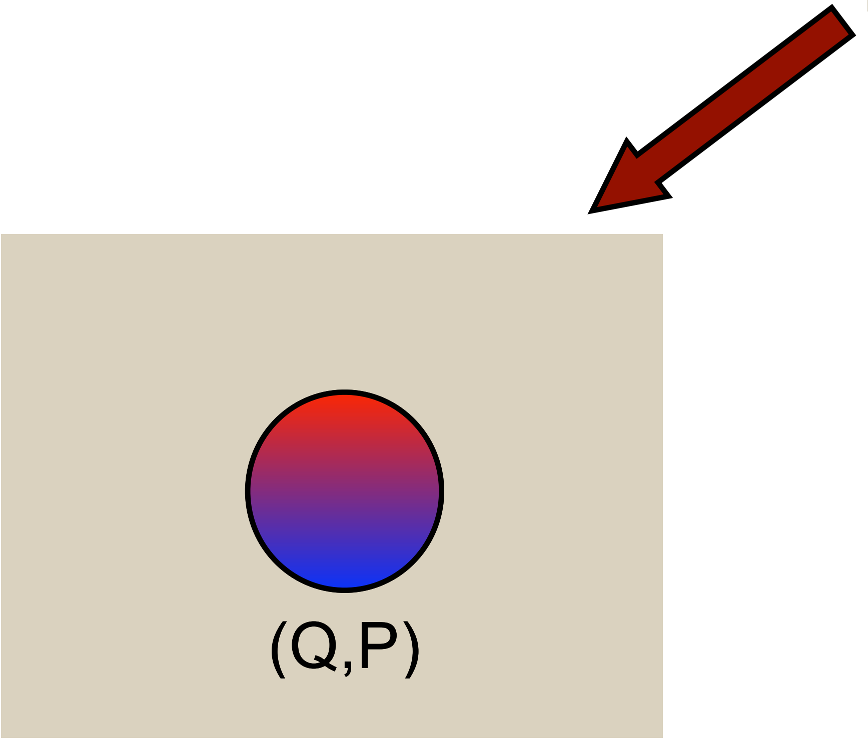
Wall-crossing and BH phase transitions



Wall-crossing and BH phase transitions

Phase I





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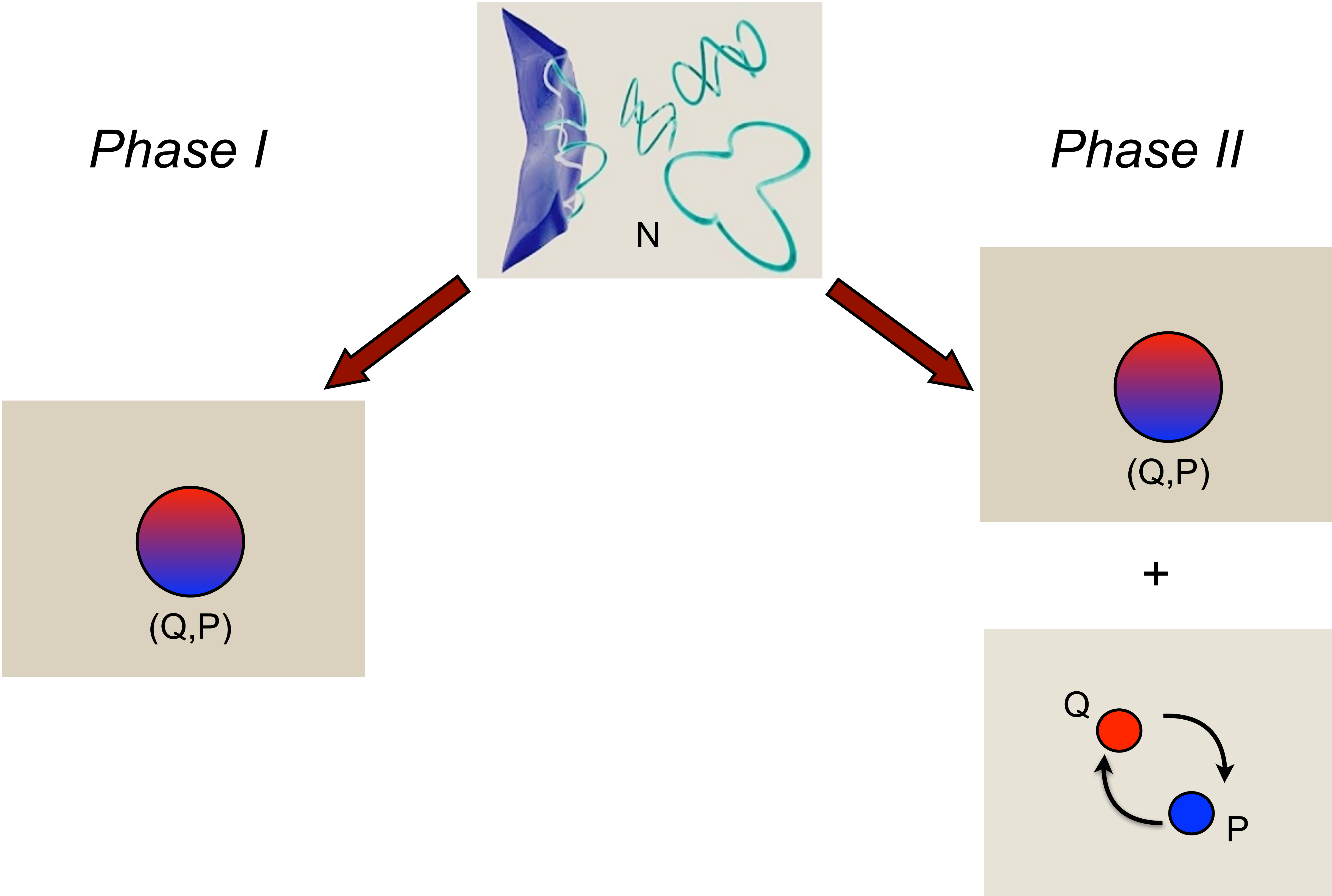

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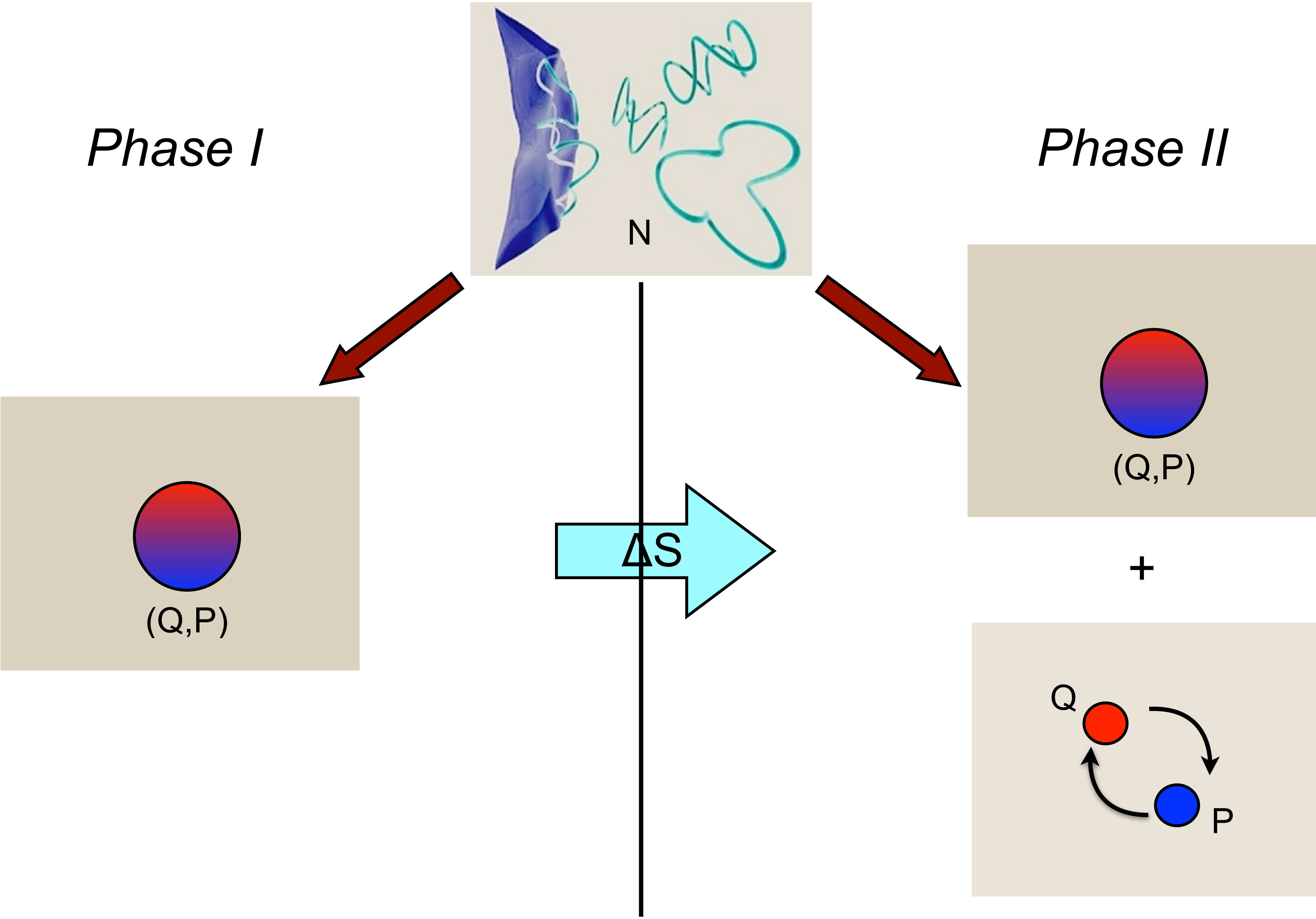

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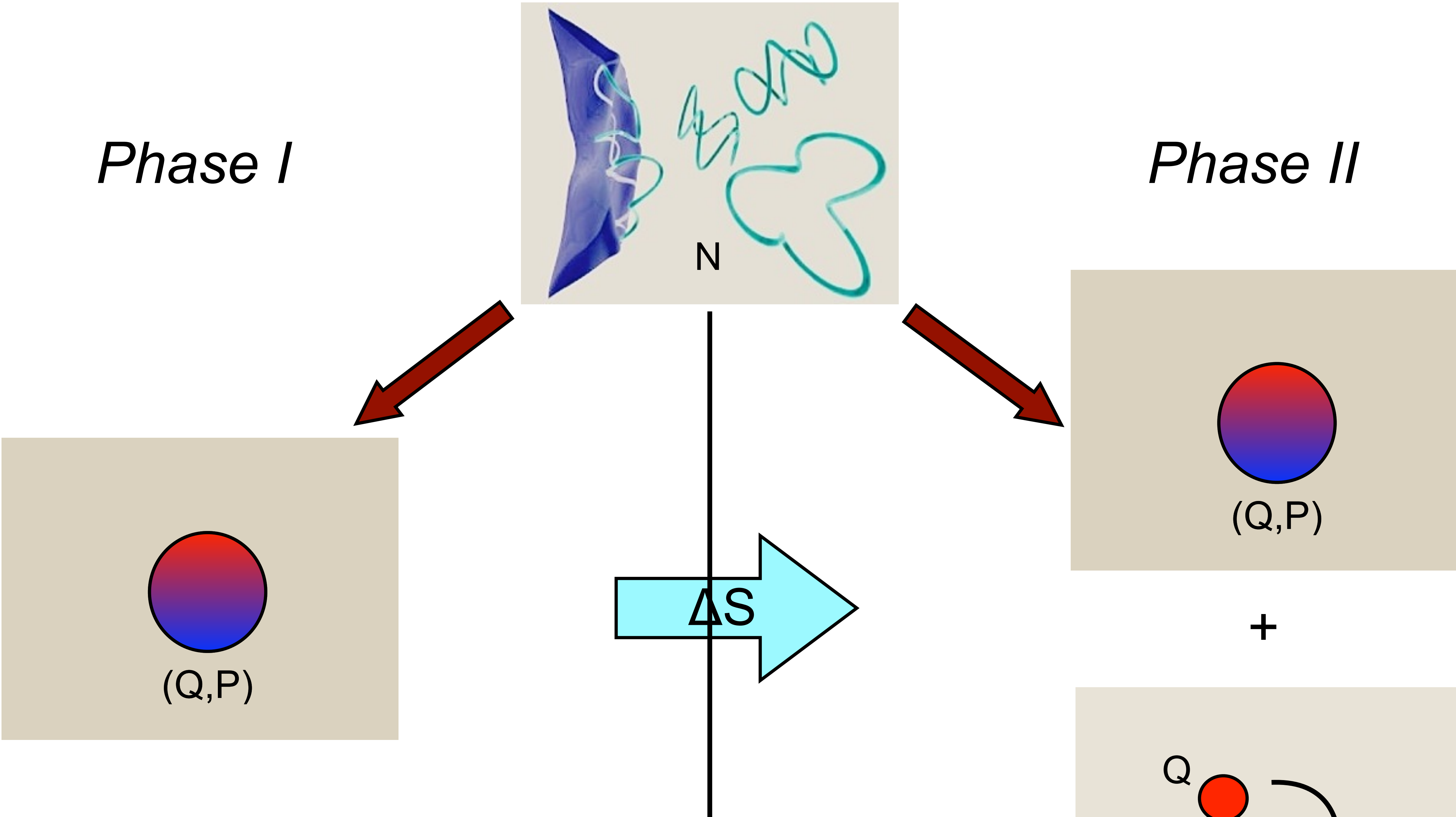

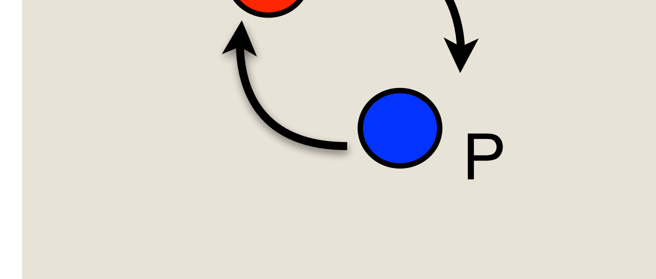

Serious problem: throwing out multi-centered BHs destroys the modular symmetry. (Denef-Moore 2007)

Mock modular forms provide the answer

(A.Dabholkar, S.M., D.Zagier '12)

These functions were described by Ramanujan, who gave a list of examples, but did not give a definition!

Their definition and structural properties were finally understood by S. Zwegers in 2000.

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Surprisingly, this is exactly what we need to solve the BH wall-crossing problem.

For the N=4 theory, we could solve it fully (based on formula due to Dijkgraaf, Verlinde, Verlinde '96), and explicitly compute the partition function of a single BH as a function of its charges.

What is the partition function of a single-centered black hole?

We have a canonical decomposition of the partition function:

$$Z_{\mathrm{micro}}(\tau) = Z_{\mathrm{BH}}(\tau) + Z_{\mathrm{multi}}(\tau)$$

- $Z_{\mathrm{multi}}(\tau)$ contains all the wall-crossing information.
- $Z_{\mathrm{BH}}(\tau)$ is the partition function of the single centered BH. It is a mock modular form.

One can now use modular symmetry to make Rademacher expansions as before. (e.g. Manschot, Bringmann '13).

Many new explorations have opened up as a result.
e.g. Large discrete symmetry groups (moonshine) of BHs in string theory (J. Harvey, S.M. '13)