Closed String Partition Functions and Doubled Geometries

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(Based on http://arxiv.org/abs/1403.4683)

- ❖ To examine the generality of an interesting fact : that the closed toroidal string partition function is the `**holomorphic square root**' of that of a T-duality covariant doubled sigma model.

  (i) Motivations and background

  (ii) General toroidal compactification

  (iii) Higher string loops

  (iv) Worldsheet supersymmetry

  (v) Orbifolds (translational)

- Doubling dimension of target space in string's sigma model
- Extra coordinates conjugate to string winding numbers
- Target space fields + its T-dual

$$|X, \tilde{X}\rangle = \sum_{n,w} e^{\frac{inX}{R}} e^{iw\tilde{X}R} |n, w\rangle$$

❖ Toroidal compactification

Eg. O(d,d;Z) – T-duality group for toroidal compactifications

$$\frac{1}{2}M^2 = (n\ w)H(n\ w)^T$$

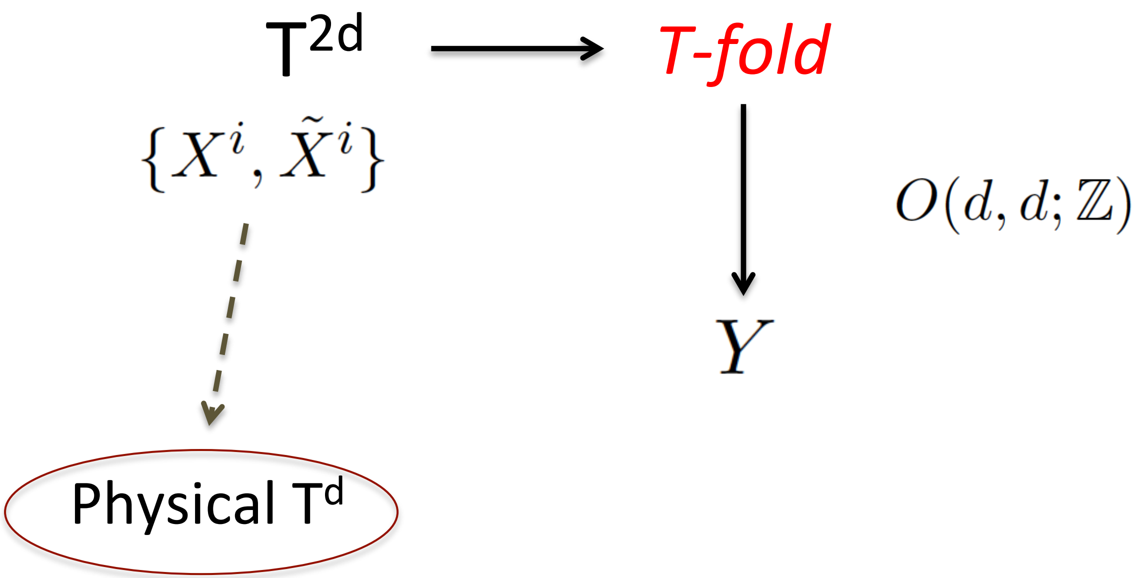
$$H \equiv \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

`*generalized metric*' ?

- `T-fold': closed strings twisted by T-duality
- Torus fibrations with T-duality valued transition functions

$$
\begin{array}{ccc}
T^{2d} & \longrightarrow & \textit{T-fold} \\
\{X^i, \tilde{X}^i\} & & \downarrow \\
 & & Y
\end{array}
$$

- Physical constraint: half of fields to be chiral and the other half to be anti-chiral (choice of polarization)



- Lagrangian formulation

$$\mathcal{L} = \frac{\pi}{2} H_{ij} \partial_a \mathbb{X}^i \partial^a \mathbb{X}^j + \pi \partial_b \mathbb{X}^i \left(A_{in} \partial^b Y^n + \bar{A}_{im} \epsilon^{bc} \partial_c Y^m \right) + \mathcal{L}_{\text{base}}(Y)$$

conserved current of T-duality transformations J

- Invariant under

$$\mathbb{X} \to M\mathbb{X}, \qquad H \to (M^{-1})^T H M^{-1}, \qquad J \to (M^{-1})^T J$$

T^{2d} $\{X^i, \tilde{X}^i\}$ ⟶ *T-fold*

↓ $M \in O(d,d;\mathbb{Z})$

Y

- Lagrangian formulation

$$\mathcal{L} = \frac{\pi}{2} H_{ij} \partial_a \mathbb{X}^i \partial^a \mathbb{X}^j + \pi \partial_b \mathbb{X}^i \underbrace{\left(A_{in} \partial^b Y^n + \bar{A}_{im} \epsilon^{bc} \partial_c Y^m \right)} + \mathcal{L}_{\text{base}}(Y)$$

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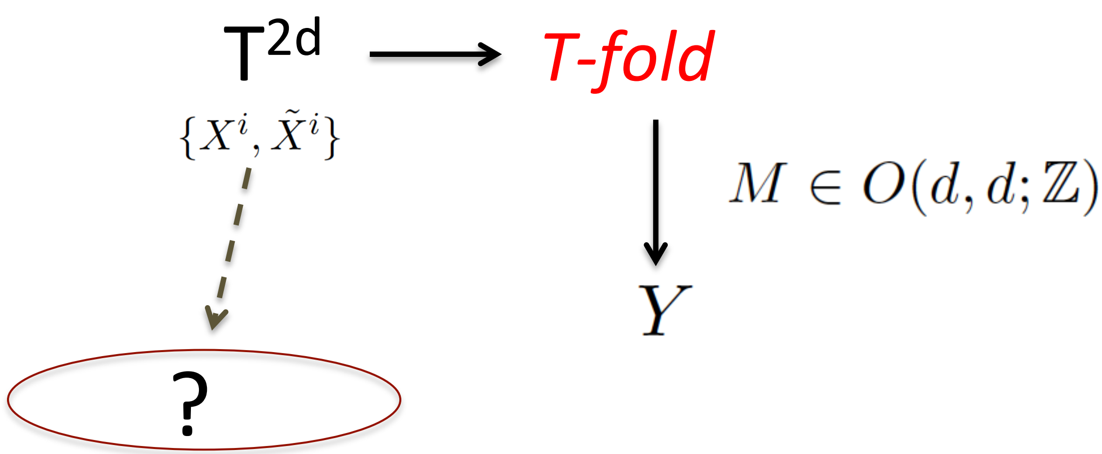
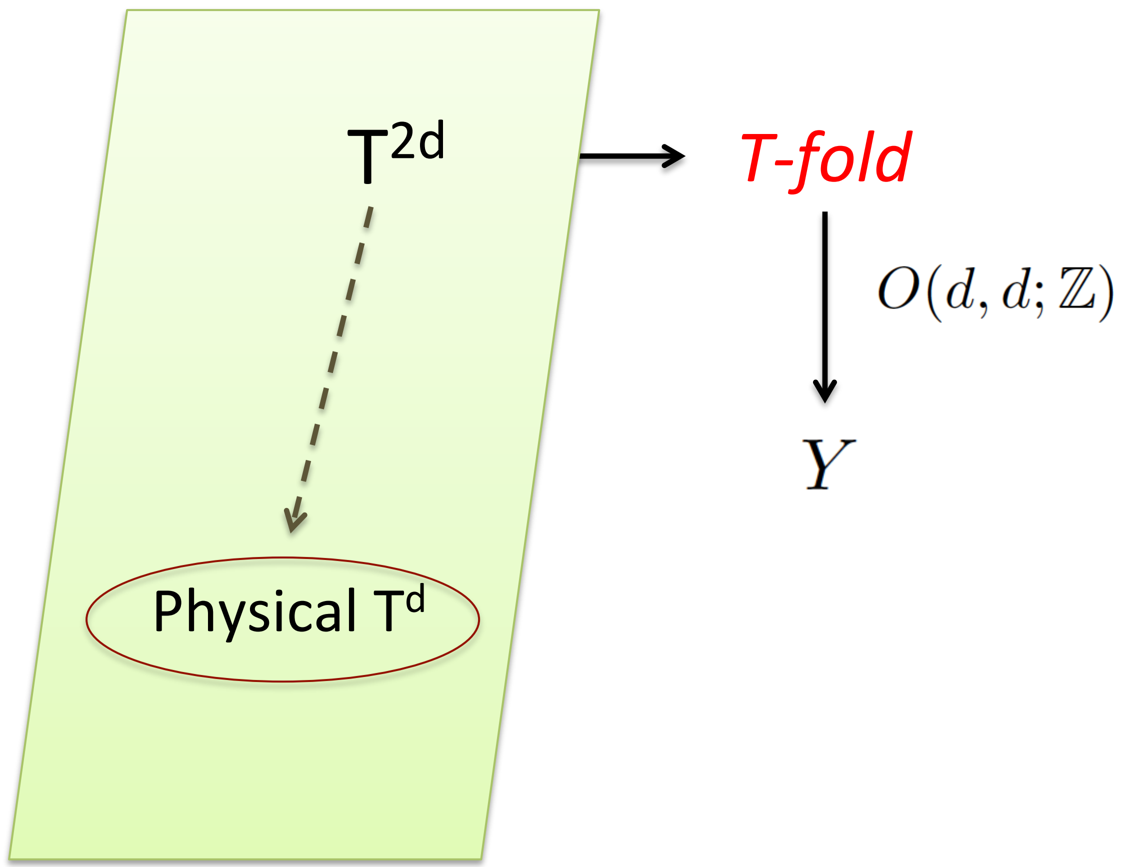


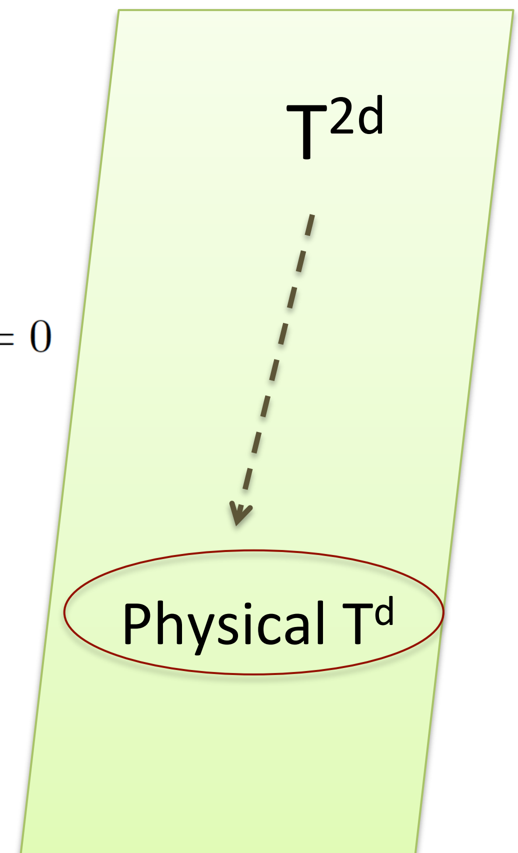
- Non-T-folds but a basic consistency check (quantum)
- A re-interpretation of closed strings on a toroidal background



- Ordinary toroidal compactification. Trivial bundle connection
- Constraint equations can be written very simply

$$* \, \partial \mathbb{X}^i = L^{ik} H_{kj} \partial \mathbb{X}^j + L^{im} J_m$$

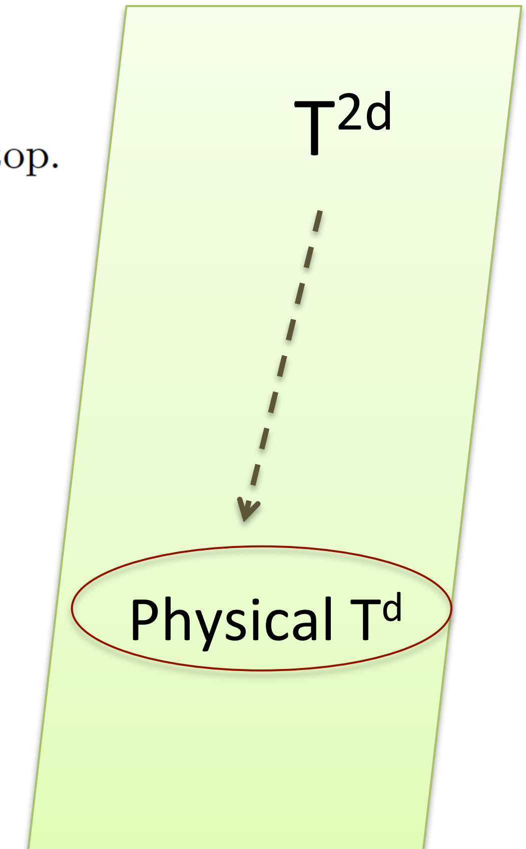
$$\left.\begin{aligned} P^a &= \frac{1}{\sqrt{2}} \left[\left(e_i^a - B_{ij} e^{ja} \right) dX^i + e^{ja} d\tilde{X}_j \right] , \\ Q^a &= \frac{1}{\sqrt{2}} \left[\left(e_i^a + B_{ij} e^{ja} \right) dX^i - e^{ja} d\tilde{X}_j \right] , \end{aligned}\right\} \partial_{\bar{z}} P^a = \partial_z Q_a = 0$$



- Starting point: doubled string sigma model obtained after setting O(d,d;Z) connection to be trivial
- Compute partition function and study how this relates to the physical partition function

$$L_{PQ} = \frac{\pi}{2}\eta_{ab} dP^a \wedge *dP^b + \frac{\pi}{2}\eta^{ab} dQ_a \wedge *dQ_b + \mathcal{L}_{\text{top.}}$$

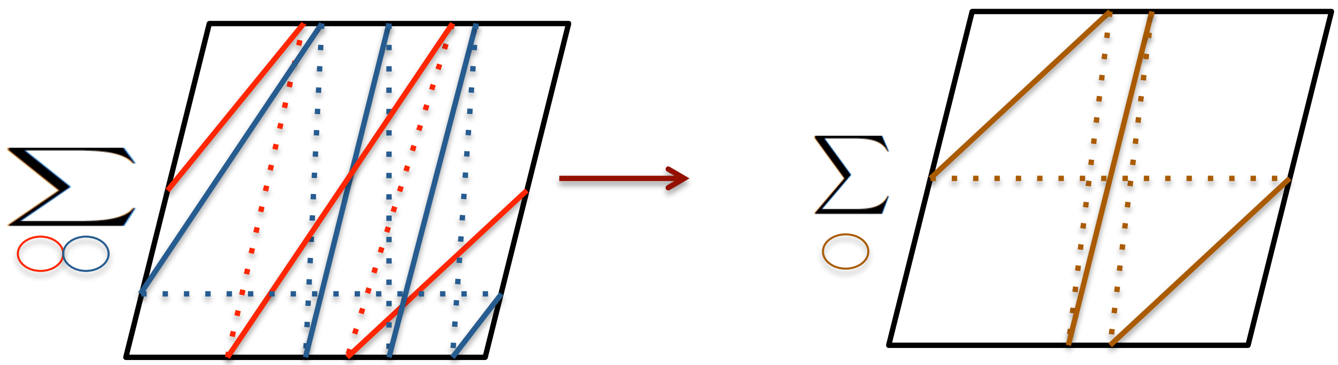
$$i\pi dX \wedge d\tilde{X}$$



- Literature background: doubled circle theory studied by Berman and Copland (2006), one-loop vacuum amplitude computed
- The physical partition function is the `**holomorphic square root**’ of the doubled one

$$S^1 \times S^1 \longrightarrow \text{Physical } S^1$$

- Computational technicality mostly involves separating the stringy zero modes
- For oscillators’ modes, the determinant needs to be of the form $|F(\tau)|^2$



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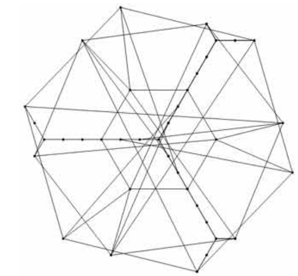
❖ General toroidal background



(i) Compute classical doubled theory's action

(ii) Perform Poisson resummations of a subset of zero modes

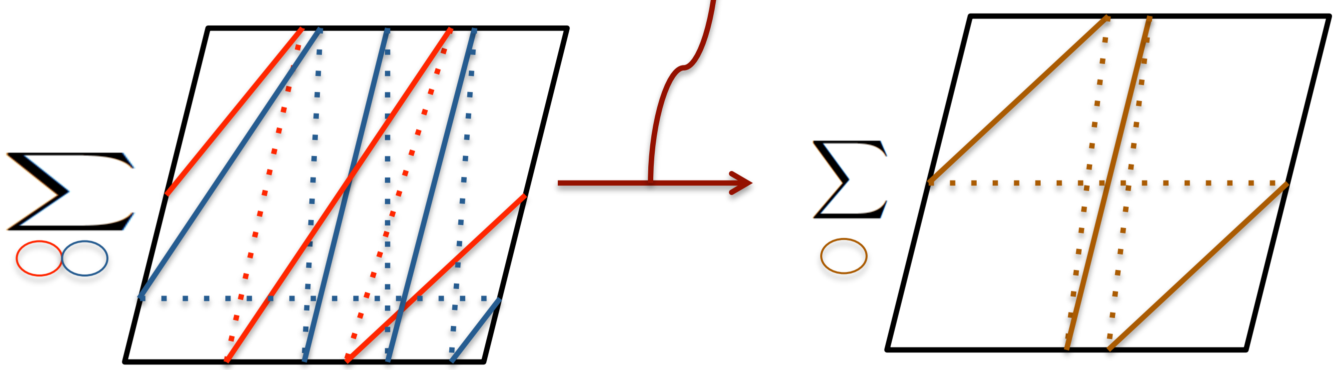
$$
\begin{aligned}
Z_{cl.} = \sum_{\{n,\bar{n},m,\bar{m}\}} \exp\Bigg[& -\frac{\pi|\tau|^2}{4\tau_2}\left[\left(e_i^a - B_{im}e^{ma}\right)n^i + e^{ja}\bar{n}_j\right]^2 - \frac{\pi}{4\tau_2}\left[\left(e_i^a - B_{im}e^{ma}\right)m^i + e^{ja}\bar{m}_j\right]^2 \\
& + \frac{\pi\tau_1}{2\tau_2}\left[\left(e_i^a - B_{im}e^{ma}\right)n^i + e^{ja}\bar{n}_j\right]\left[\left(e_i^a - B_{im}e^{ma}\right)m^i + e^{ja}\bar{m}_j\right] \\
& - \frac{\pi|\tau|^2}{4\tau_2}\left[\left(e_i^a + B_{im}e^{ma}\right)n^i - e^{ja}\bar{n}_j\right]^2 - \frac{\pi}{4\tau_2}\left[\left(e_i^a + B_{im}e^{ma}\right)m^i - e^{ja}\bar{m}_j\right]^2 \\
& + \frac{\pi\tau_1}{2\tau_2}\left[\left(e_i^a + B_{im}e^{ma}\right)n^i - e^{ja}\bar{n}_j\right]\left[\left(e_i^a + B_{im}e^{ma}\right)m^i - e^{ja}\bar{m}_j\right] \\
& + i\pi\left(n^j\bar{m}_j - \bar{n}_k m^k\right)\Bigg]
\end{aligned}
$$

❖ General toroidal background

(i) Compute classical doubled theory's action
(ii) Perform Poisson resummations of a subset of zero modes

$$Z = \frac{1}{|\eta|^{2d}} \sum_{p_{L,R},q_{L,R}} \exp\left[\frac{1}{2} i\pi\tau p_L^2 - \frac{1}{2} i\pi\bar{\tau} p_R^2\right] \exp\left[\frac{1}{2} i\pi\tau q_L^2 - \frac{1}{2} i\pi\bar{\tau} q_R^2\right]$$

$$\partial_{\bar{z}} P^a = \partial_z Q_a = 0$$



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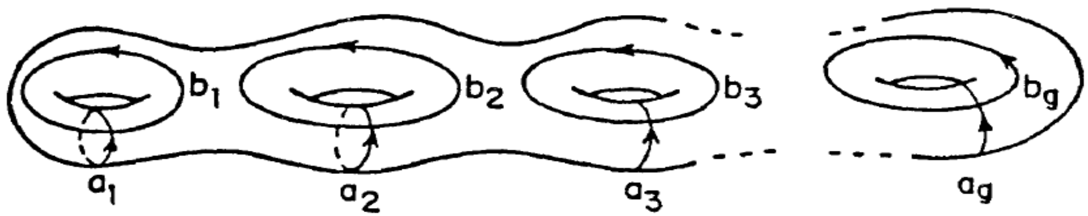
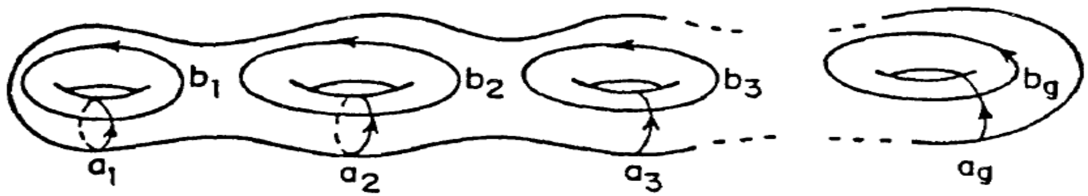
- Higher-genera worldsheets



- Period matrix characterizes complex structure of worldsheet

$$\tau_{\alpha\beta} = \int_{a_\alpha} \omega_\beta, \qquad \int_{b_\alpha} \omega_\beta = \delta_{\alpha\beta}, \qquad \int_{a_\alpha} dX^i = 2\pi n^i_\alpha, \qquad \int_{b_\alpha} dX^i = 2\pi m^i_\alpha$$

- Our previous result generalizes rather easily to higher-genera worldsheets (apart from a subtlety arising from the non-zero modes)!



- Instanton action: Poisson resummation (in ordinary torus theory)

$$Z_{cl.} = (\mathrm{Det}(\tau_2))^{\frac{d}{2}} (\mathrm{Det}(G))^{-\frac{g}{2}} \sum_{p_L, p_R} \exp\left[\frac{i\pi}{2}(p_L)_\alpha \tau_{\alpha\beta}(p_L)_\beta - \frac{i\pi}{2}(p_R)_\alpha \bar{\tau}_{\alpha\beta}(p_R)_\beta\right]$$

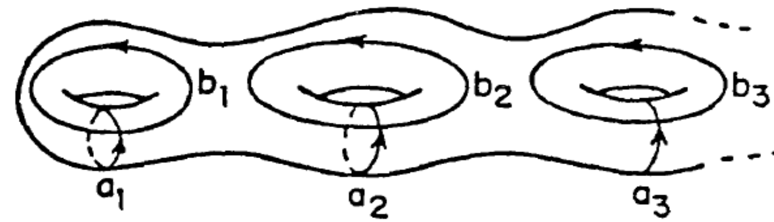
- We need to write the higher-loop determinants in a factorized form, for chiral factorization to work.

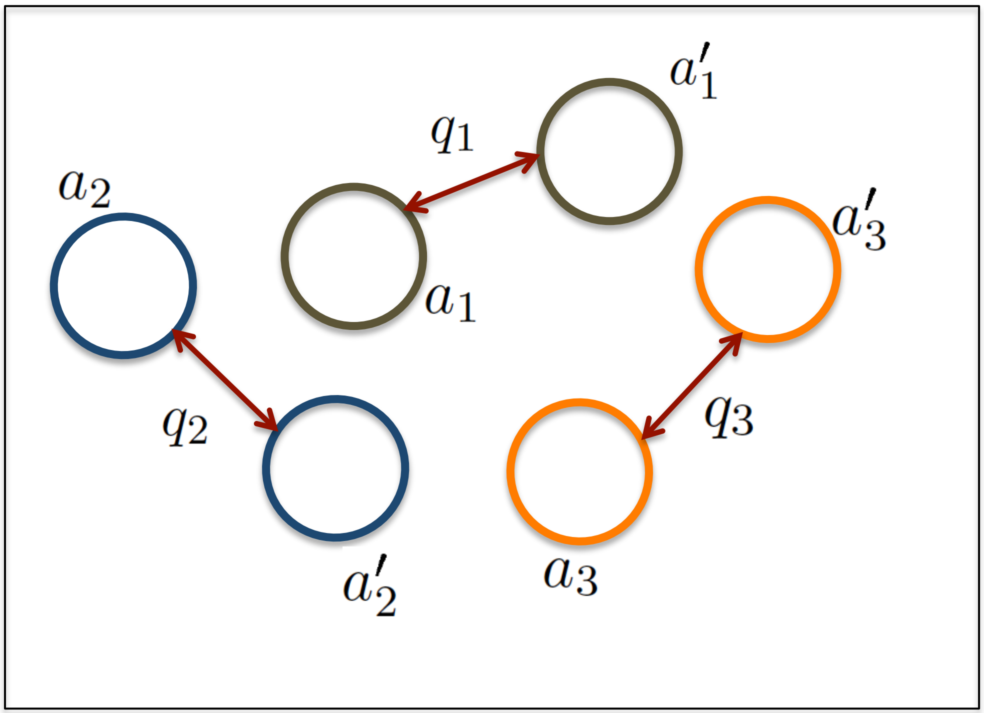
$$Z_{q.} = \frac{\sqrt{\mathrm{Det}(G)}}{(\mathrm{Det}(\tau_2))^{\frac{d}{2}}|\eta_g|^2}$$

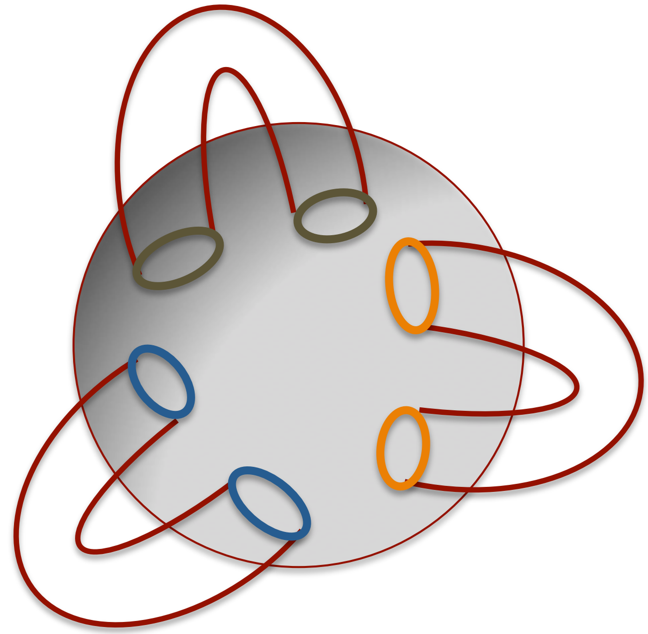
This turns out to rely on a description of Riemann surfaces by quotient of the sphere using discrete subgroups of SL(2,C), or Schottky uniformization

$$\mathrm{Det}'\nabla^2 = (\mathrm{Det}\tau_2)\exp\left(-\frac{S_L}{12\pi}\right)|F|^2, \qquad F = \prod_{\{\gamma\}}\prod_{m=0}^{\infty}\left(1 - q_\gamma^{1+m}\right)$$

On Schottky Uniformization







$$\mathrm{Det}'\nabla^2 = (\mathrm{Det}\tau_2)\exp\left(-\frac{S_L}{12\pi}\right)|F|^2, \qquad F = \prod_{\{\gamma\}}\prod_{m=0}^{\infty}\left(1-q_\gamma^{1+m}\right)$$

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- Doubled worldsheet fermions from worldsheet in superspace

$$S = \frac{1}{2}\int d^2\sigma d^2\theta \; \frac{1}{4} H_{IJ} C^{rs} \left[D_r \mathbb{X}^I + \mathcal{A}^I_m D_r Y^m\right] \left[D_s \mathbb{X}^J + \mathcal{A}^J_n D_s Y^n\right] - \frac{1}{2}\gamma^{rs} L_{IJ} \left[D_r \mathbb{X}^I + \mathcal{A}^I_m D_r Y^m\right] \mathcal{A}^J_s + \mathcal{L}(Y).$$

- Supersymmetrizing the constraint

$$\left[D_s \mathbb{X}^J + \mathcal{A}^J_n D_s Y^n\right] = S\left(\gamma_3\right)_{sr} \left[D_r \mathbb{X}^J + \mathcal{A}^J_n D_r Y^n\right]$$

↓

$$\mathcal{L} = H_{IJ}\left(\partial^a \mathbb{X}^I \partial_a \mathbb{X}^J + i\bar{\psi}^I \rho^m \partial_m \psi^J + f^I f^J\right) + i\pi \Omega_{IJ}\left[\partial_t \mathbb{X}^I \partial_\sigma \mathbb{X}^J + i\psi^I_2(\partial_t - \partial_\sigma)\psi^J_2 + i\psi^J_1(\partial_t + \partial_\sigma)\psi^I_1\right]$$

❖ Constraints on worldsheet fermions look simpler in a certain basis

$$\varphi^a = \left(e_i^a + B_{ij}e^{ja}\right)\psi^i + e^{ja}\tilde{\psi}_j$$
$$\eta^a = \left(e_i^a - B_{ij}e^{ja}\right)\psi^i - e^{ja}\tilde{\psi}_j,$$

$$\mathcal{L}_{\text{fermions}} = \frac{1}{2}\varphi_2^a\partial_z\varphi_2^a + \frac{1}{2}\varphi_1^a\partial_{\bar{z}}\varphi_1^a + \frac{1}{2}\eta_2^a\partial_z\eta_2^a + \frac{1}{2}\eta_1^a\partial_{\bar{z}}\eta_1^a$$

set to zero!

❖Spin structures imposed by hand in Type II and Heterotic strings (similarly for chiral bosons)

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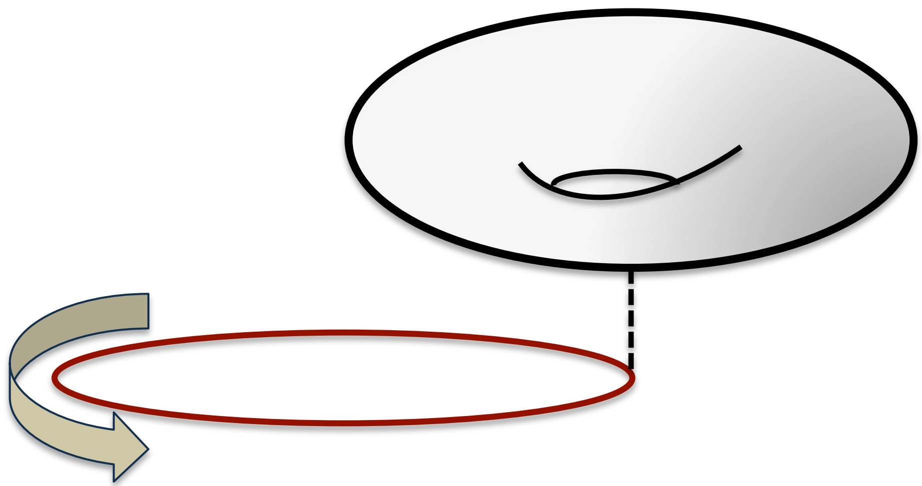
- Symmetric $S^1/\mathbb{Z}_N$ twists

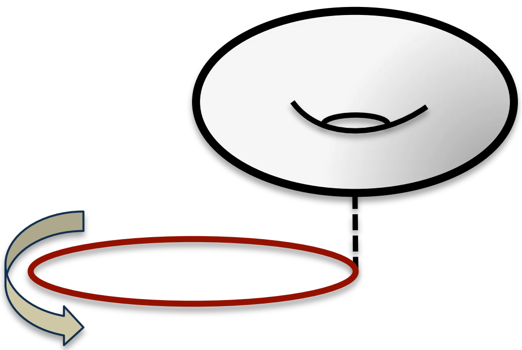
$$Z_{S^1/\hat{s}} = \frac{1}{N} \sum_{g,h=1}^{\hat{s}^{N-1}} Z^g{}_h$$

$$\begin{aligned} X(\sigma_1+1,\sigma_2) &= X(\sigma_1,\sigma_2) + \delta'' \qquad (\mathrm{mod}\ 1) \\ X(\sigma_1,\sigma_2+1) &= X(\sigma_1,\sigma_2) + \delta' \qquad (\mathrm{mod}\ 1). \end{aligned}$$



❖ Symmetric $S^1/\mathbb{Z}_N$ twists

$$Z^g{}_h(\tau) = \frac{R}{\sqrt{\tau_2}|\eta(\tau)|^2} \sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|\tau(n+\delta')-(m+\delta'')|^2}$$



(Poisson resummation)

$$Z^g{}_h(\tau) = \mathrm{Tr}_h\left(g q^{\frac{1}{4}p_L^2} \bar{q}^{\frac{1}{4}p_R^2}\right) = \frac{1}{|\eta|^2} \sum_{n,w} e^{-2\pi i \delta'' w} q^{\frac{1}{4}\left(\frac{w}{R}+R(n+\delta')\right)^2} \bar{q}^{\frac{1}{4}\left(\frac{w}{R}-R(n+\delta')\right)^2}.$$

- Asymmetric Shift Orbifolds – could act as base manifolds for T-folds

$$X \to X + a\frac{1}{N}, \qquad \tilde{X} \to \tilde{X} + b\frac{1}{N}$$

$$Z^{(\delta'',\bar{\delta}'')}_{(\delta',\bar{\delta}')} = \sum_{w,n} e^{-2\pi i\delta''(w+\bar{\delta}')-2\pi i\bar{\delta}''(n+\delta')} q^{\frac{1}{4}\left(\frac{w+\bar{\delta}'}{R}+R(n+\delta')\right)^2} \bar{q}^{\frac{1}{4}\left(\frac{w+\bar{\delta}'}{R}-R(n+\delta')\right)^2}.$$

(winding number shifts)

(c.f. symmetric orbifold)

$$Z^g{}_h(\tau) = \mathrm{Tr}_h\left(g q^{\frac{1}{4}p_L^2}\bar{q}^{\frac{1}{4}p_R^2}\right) = \frac{1}{|\eta|^2}\sum_{n,w} e^{-2\pi i\delta'' w} q^{\frac{1}{4}\left(\frac{w}{R}+R(n+\delta')\right)^2}\bar{q}^{\frac{1}{4}\left(\frac{w}{R}-R(n+\delta')\right)^2}.$$

❖ How does doubled sigma model see the winding number shift?

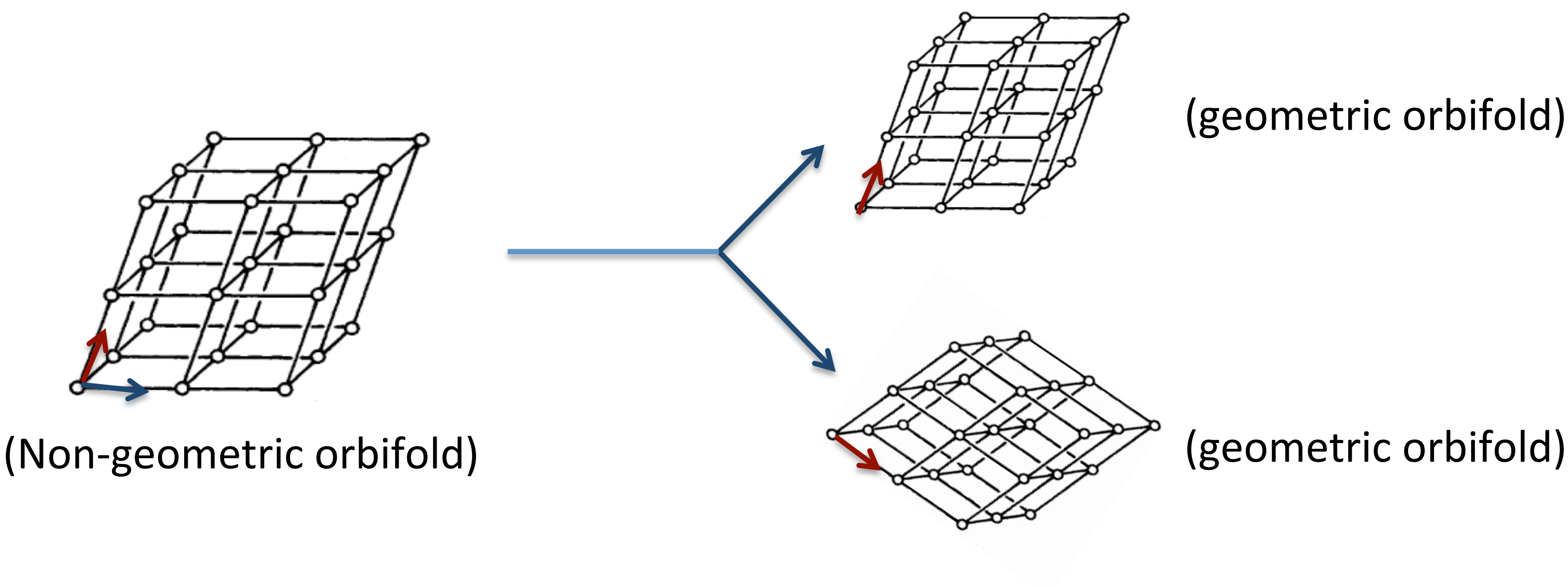
$$L = dX + (n+\delta')\alpha_2 + (m+\delta'')\alpha_1,$$
$$\tilde{L} = d\tilde{X} + (\bar{n}+\bar{\delta}')\alpha_2 + (\bar{m}+\bar{\delta}'')\alpha_1.$$

❖ Partition function of doubled sigma model

$$Z = \frac{1}{|\eta|^2} \sum_{n,\bar{n},w,\bar{w}} e^{-2\pi i\left[\delta''(w+\bar{\delta}')+\bar{\delta}''(\bar{w}+n+\delta')\right]} q^{\frac{1}{4}(p_L^2+q_L^2)} \bar{q}^{\frac{1}{4}(p_R^2+q_R^2)}$$

$$p_L = (n+\bar{w}+\delta')R + \frac{(w+\bar{\delta}')}{R}, \qquad p_R = \frac{\bar{n}-w}{R} + \bar{w}R$$
$$q_L = \frac{\bar{n}-w}{R} - \bar{w}R, \qquad q_R = (n+\bar{w}+\delta')R - \frac{(w+\bar{\delta}')}{R},$$

- Doubled Sigma Model and Asymmetric Shift Orbifolds



- More general picture: sewing chiral and anti-chiral blocks together

$$Z_G(\tau,\bar{\tau}) = \frac{1}{|G|}\sum_{g,h}\sum_{p_{L,R}} K(p_L,p_R,h_L,h_R,g_L,g_R)\mathcal{F}^{g_L}_{h_L}(p_L;\tau)\overline{\mathcal{F}^{g_R}_{h_R}(p_R;\tau)}$$

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