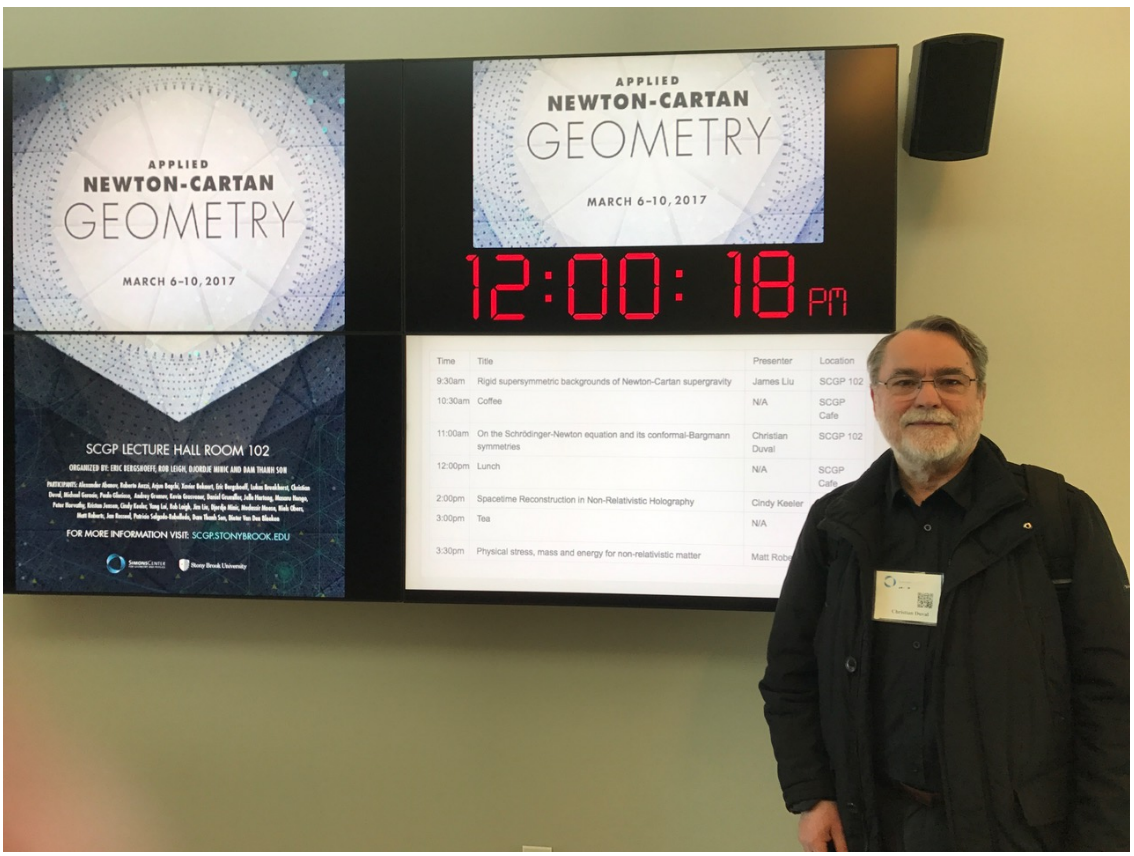
EISENHART-DUVAL LIFT:

UNIFIED FRAMEWORK

for

GALILEI & CARROLL SYMMETRY

C. Duval, G. Gibbons, P. Horvathy, P-M Zhang



Applied Newton-Cartan Workshop
Mainz, March 2018

Abstract: Whereas the usual Wigner-Inönü contraction $c \to \infty$ of the Poincaré group yields the Galilei group, another $c \to 0$ contraction yields the "Carroll group" of Lévy-Leblond. Both boost-invariant theories are conveniently unified within the "Eisenhart-Duval" framework. **Plane gravitational waves** carry a non-trivially implemented Carroll symmetry with broken rotations.

Based on:

- C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016 [arXiv:1402.0657 [gr-qc]]

- C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [arXiv:1702.08284 [gr-qc]]

- P.-M. Zhang, C. Duval and P. A. Horvathy *"Memory Effect for Impulsive Gravitational Waves,"* Class. Quant. Grav. **35** (2018) 065011 [arXiv:1709.02299 [gr-qc]]

Carroll group

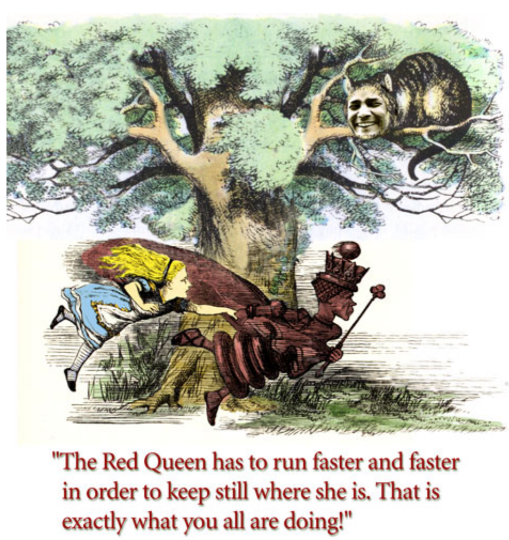
J. M. Lévy-Leblond

Carroll group *
constructed as novel type of contraction of Poincaré group

"Une nouvelle limite non-relativiste du group de Poincaré," Ann. Inst. H. Poincaré **3** (1965) 1

V. D. Sen Gupta, "On an Analogue of the Galileo Group," Il Nuovo Cimento **44** (1966) 512

no motion - *no* physics - mathematical curiosity



* Lewis Carroll *Through the Looking Glass and what Alice Found There* (1871).

NEWTON-CARTAN STRUCTURE

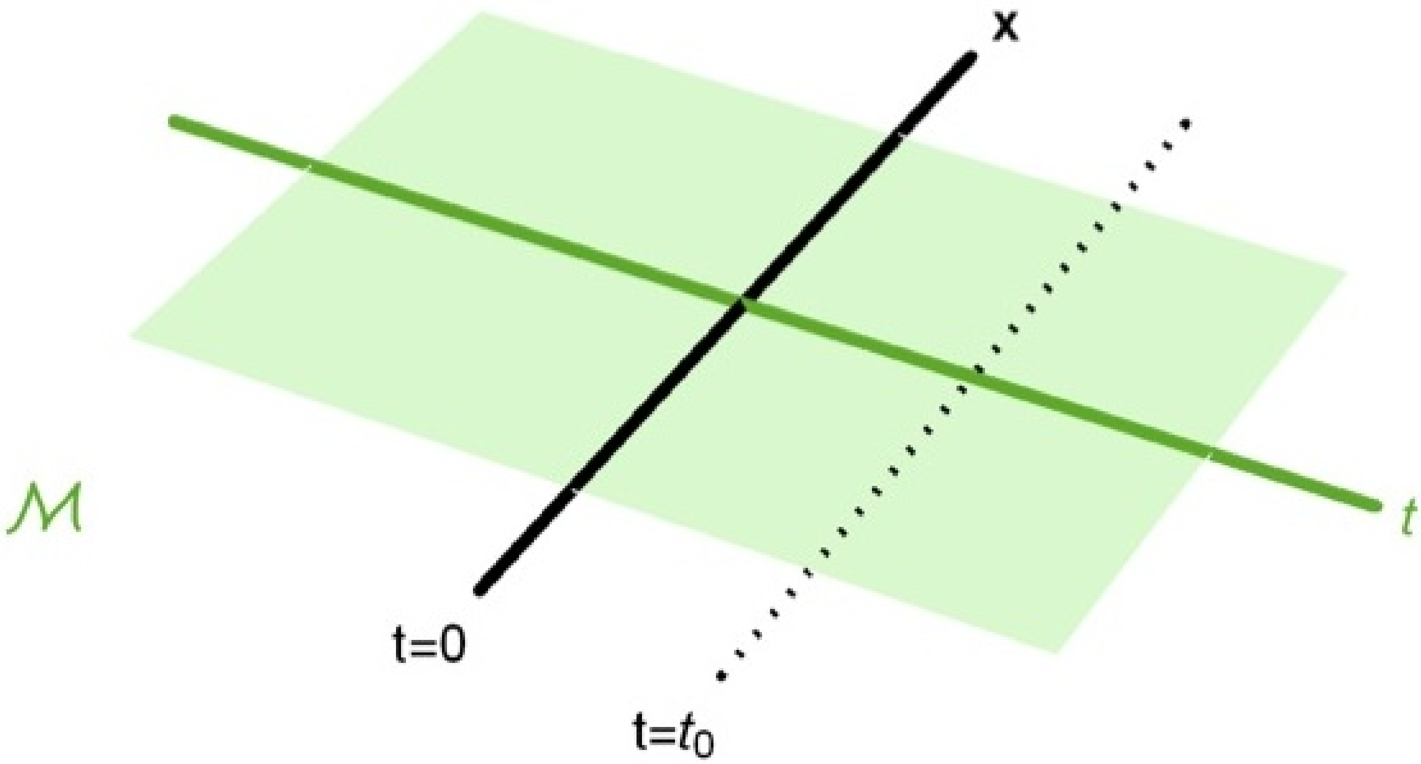


Fig. 1 : Galilean space-time, $\mathcal{M}$, described by $\begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix}$. Carries symmetric, contravariant non-negative [space-co-] "metric" tensor $\boldsymbol{\gamma}$, whose kernel is generated by dt. Projects onto absolute time.

Galilei boosts

$$\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} + \boldsymbol{b}\,t \\ t' &= t \end{cases} \tag{1}$$

act as

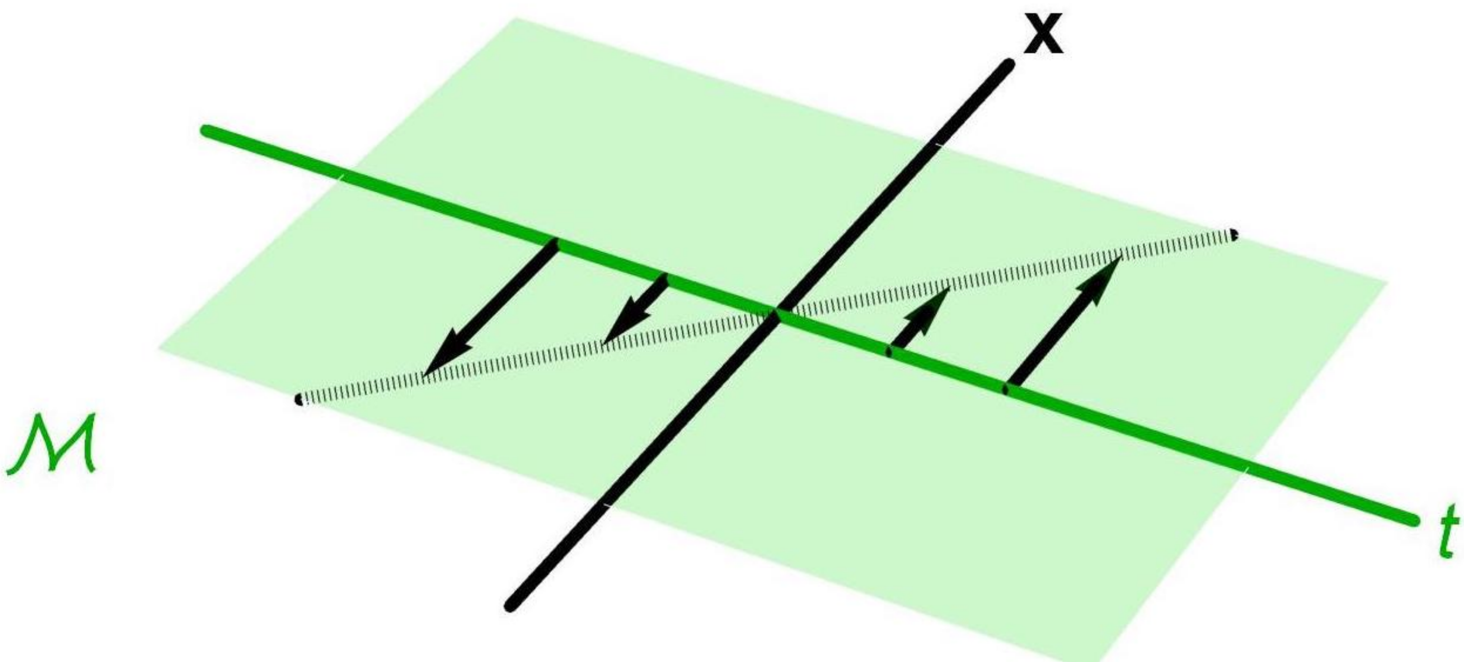


Fig.2 Galilei boost acting on Galilei space-time

CARROLL STRUCTURE

Lévy-Leblond 1965 :

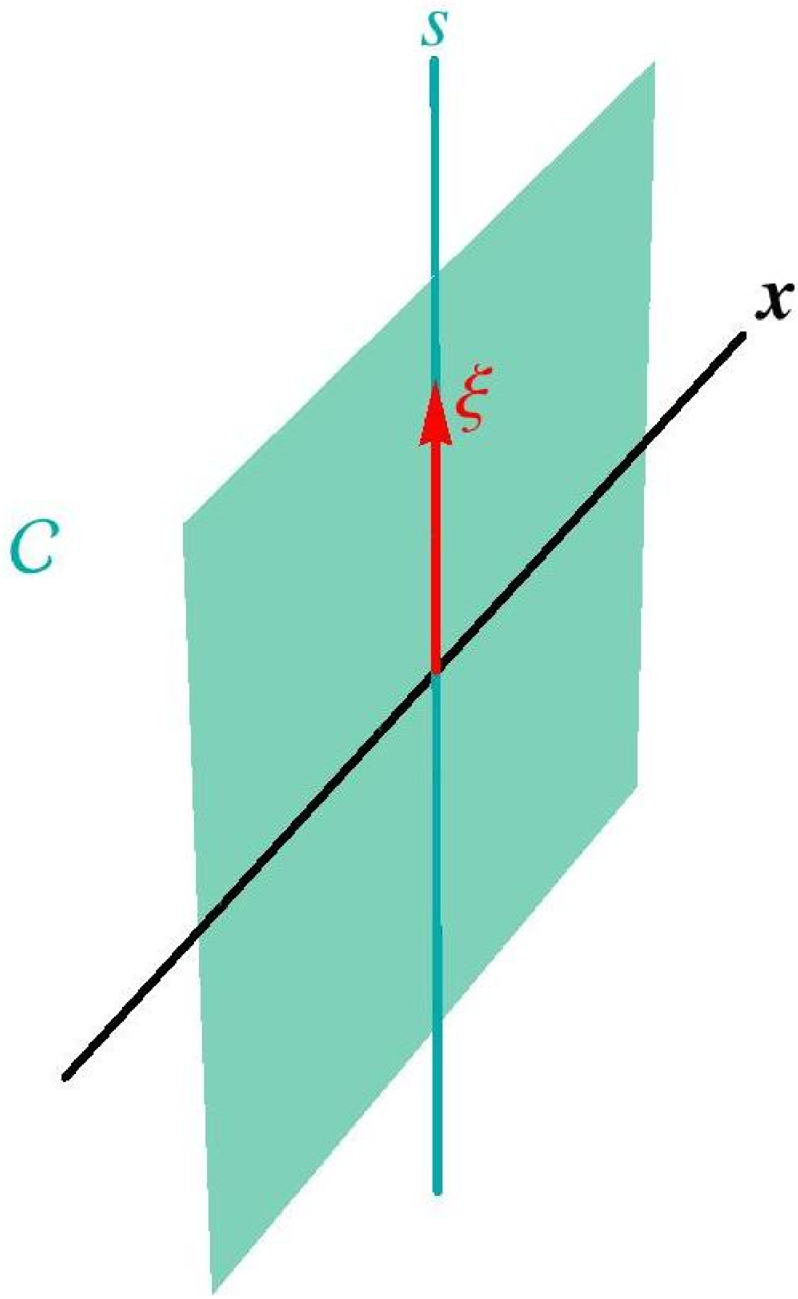


Fig.2 : Carroll space-time, $\mathcal{C}$ described by $\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix}$, is endowed with vector $\boldsymbol{\xi}$ which generates kernel of (singular) [space-] "metric" $\bar{G} = \delta_{AB}\, dx^A dx^B$.

Carroll group Carr$(d+1)$ $\sim$ *"Carrollian boosts"*

$$\boxed{\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} \\ s' &= s - \boldsymbol{b}\cdot\boldsymbol{x} \end{cases}} \qquad (2)$$

NB: In NR QM wave fct transforms according to:

$$\boxed{\psi'(x,t) = e^{i(\mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}^2 t)}\psi(\boldsymbol{x} - \mathbf{b}t, t)} \qquad (3)$$

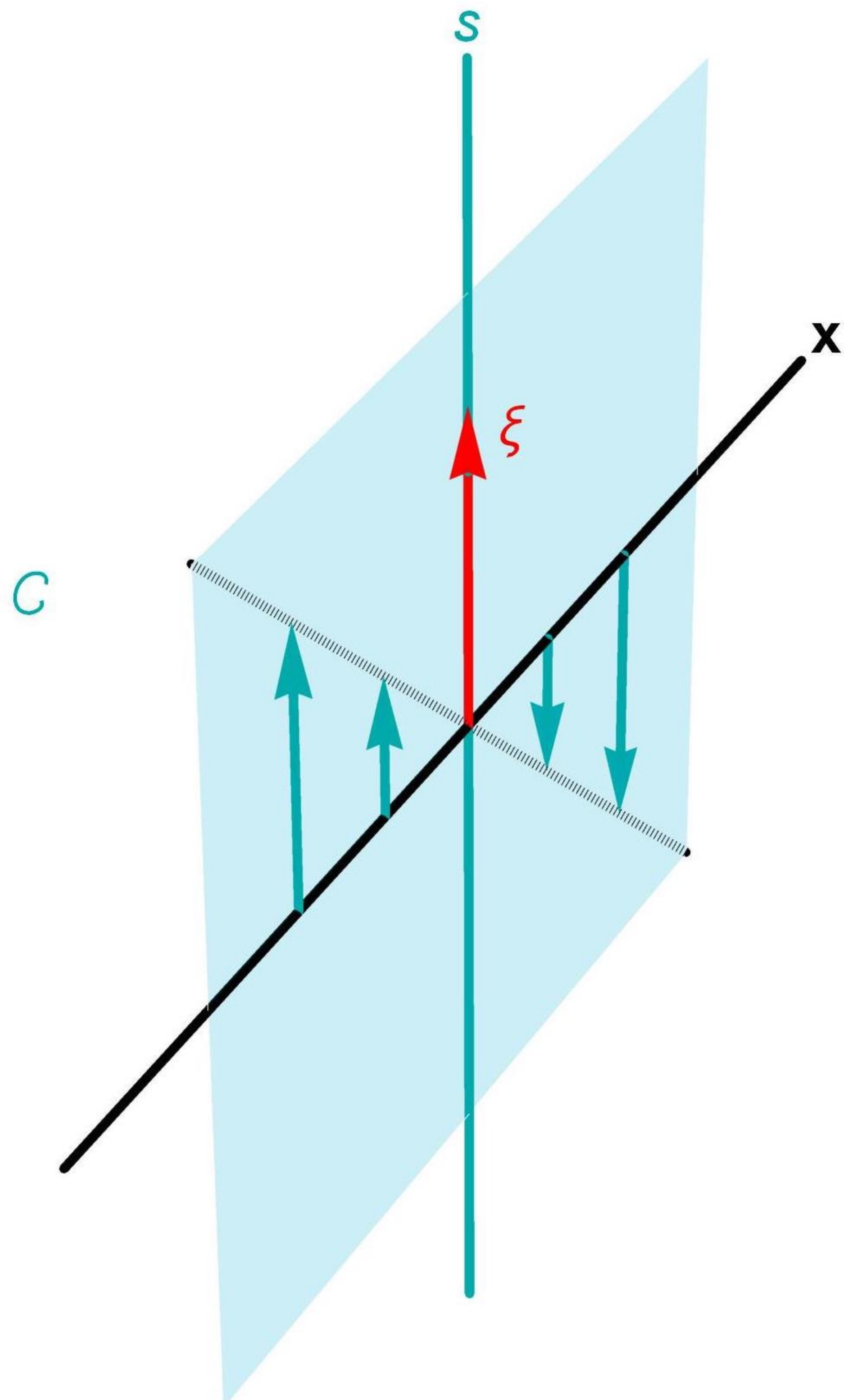


Fig.3 Carroll boost acts on flat **Carroll** space-time

Carroll group represented by matrices

$$\begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $f \in \mathbb{R}$. Acts on $\begin{pmatrix} \boldsymbol{x} \\ s \\ 1 \end{pmatrix}$ affinely by matrix action. Carroll Lie algebra $\boxed{\mathfrak{carr}(d+1)}$

$$Z = \begin{pmatrix} \omega & 0 & \gamma \\ -\boldsymbol{\beta}^T & 0 & \varphi \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

$\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$, and $\varphi \in \mathbb{R}$ acts on Carroll space-time as

$$X = (\omega^A_B\, x^B + \gamma^A)\frac{\partial}{\partial x^A} + \left(\varphi\ \boxed{-\beta_A\, x^A}\right)\frac{\partial}{\partial s}\,, \tag{6}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$, and $\varphi \in \mathbb{R}$.

N.B. : *Galilei Lie algebra* $\boxed{\mathfrak{gal} \equiv \mathfrak{gal}(d+1)}$

$$X = (\omega^A_B\, x^B + \boxed{\beta^A t} + \gamma^A)\frac{\partial}{\partial x^A} + \epsilon\frac{\partial}{\partial t} \tag{7}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}$.

Unification: Bargmann manifolds

A *Bargmann manifold** is

(i) a $(d+2)$-dim manif B

(ii) endowed with metric G of signature $(d+1,1)$

(iii) carries nowhere vanishing, complete, null "vertical" vector ξ, parallel-transported by Levi-Civita connection, ∇.

L. P. Eisenhart, "Dynamical trajectories and geodesics", Annals. Math. **30** 591-606 (1928).

J. Gomis and J. M. Pons, "Poincare Transformations and Galilei Transformations," Phys. Lett. A **66** (1978) 463.

C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, "Bargmann Structures and Newton-Cartan Theory," Phys. Rev. D **31** (1985) 1841.

* Introduced by Duval et al as geometrical structure underlying Bargmann [$\equiv$ centrally extended Galilei] group.

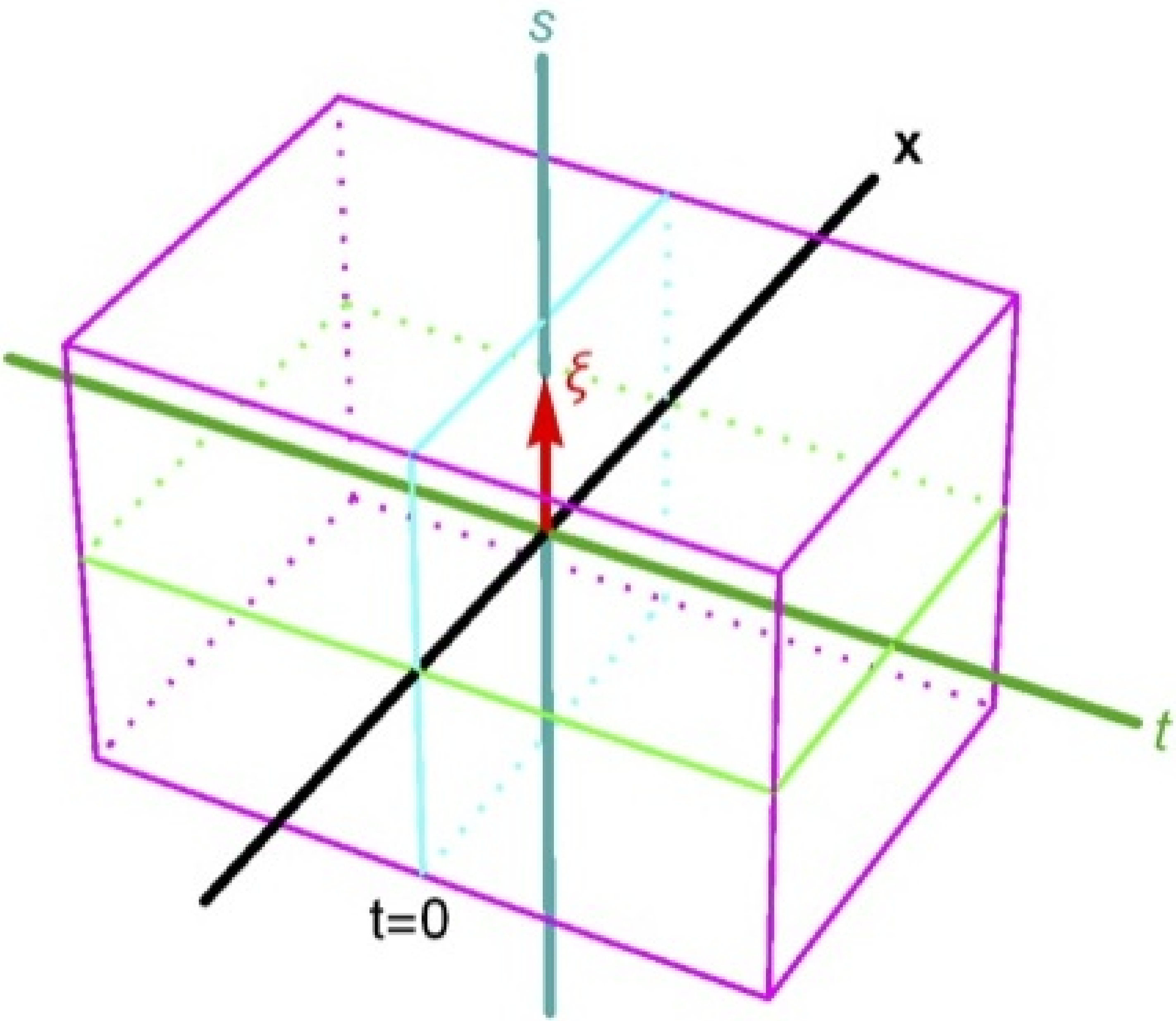


Fig. 4 : Bargmann space : $(d+1,1)$ dim manifold with Lorentz metric & coordinates $(\boldsymbol{x}, t, s)$, endowed with covariantly constant null vector $\xi = \partial_s$.

Flat Bargmann structure $\sim$ Minkowski space :

$$B = \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ t \\ s \end{pmatrix} \right\}, \tag{8}$$

$$G = \delta_{AB}\, dx^A dx^B + 2dtds, \tag{9}$$

$$\boldsymbol{\xi} = \partial_s\,. \tag{10}$$

Both s & t light-cone (null), coords. t has dimension of time, coordinate s has that of action/mass.

• Factoring out "vertical" translations along ξ, $(d+1)$-dim quotient acquires Newton-Cartan structure

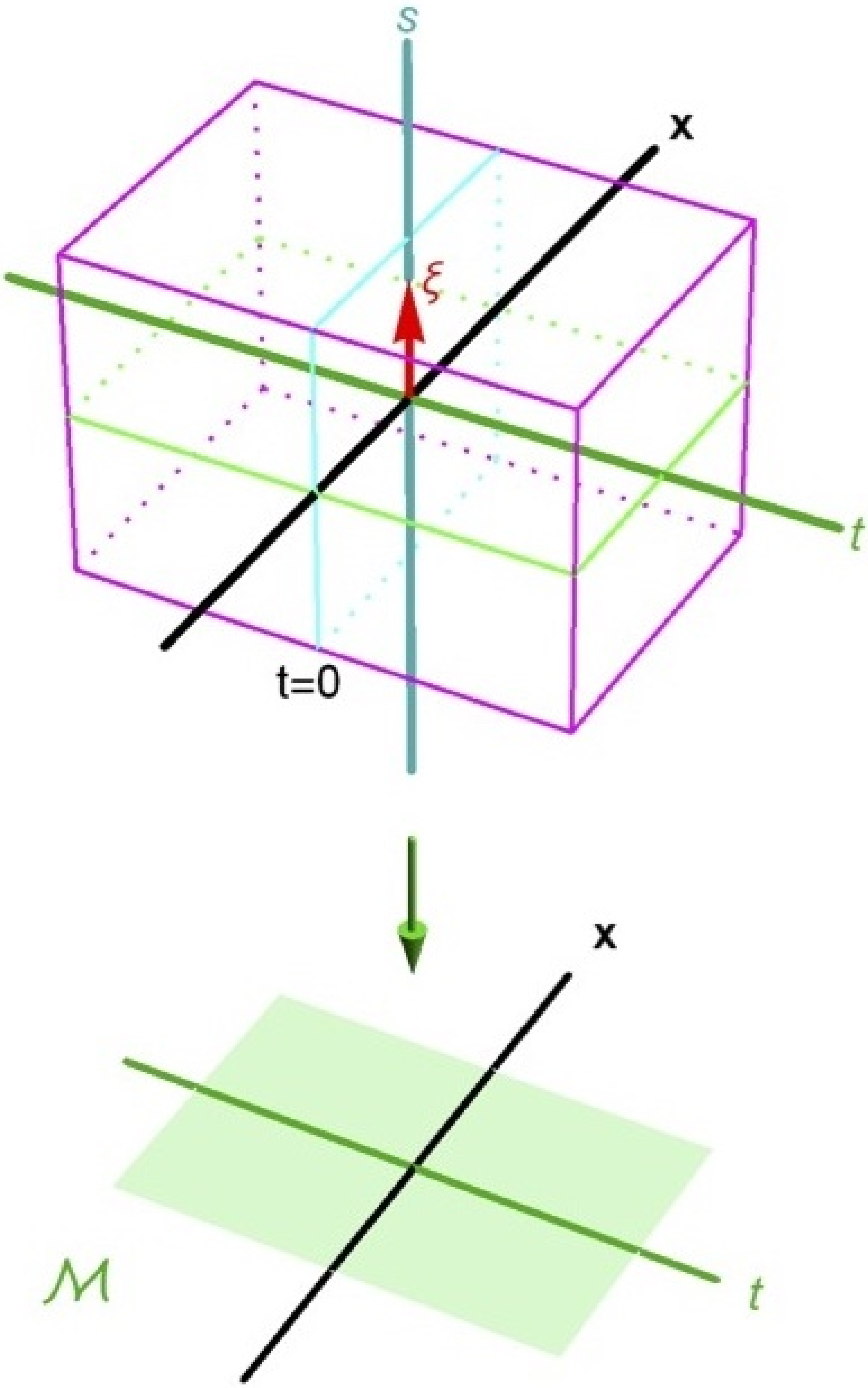


Fig.5 : Bargmann sp projects to Galilean space-time.

• Restriction to $t = \text{const}$ turns off $dtds$ in metric (9), leaving singular "metric" $\delta_{AB}\, dx^A dx^B$ $\rightsquigarrow$ admits flat Carroll structure embedded into Bargmann space $\mathcal{C} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ 0 \\ s \end{pmatrix} \right\}$

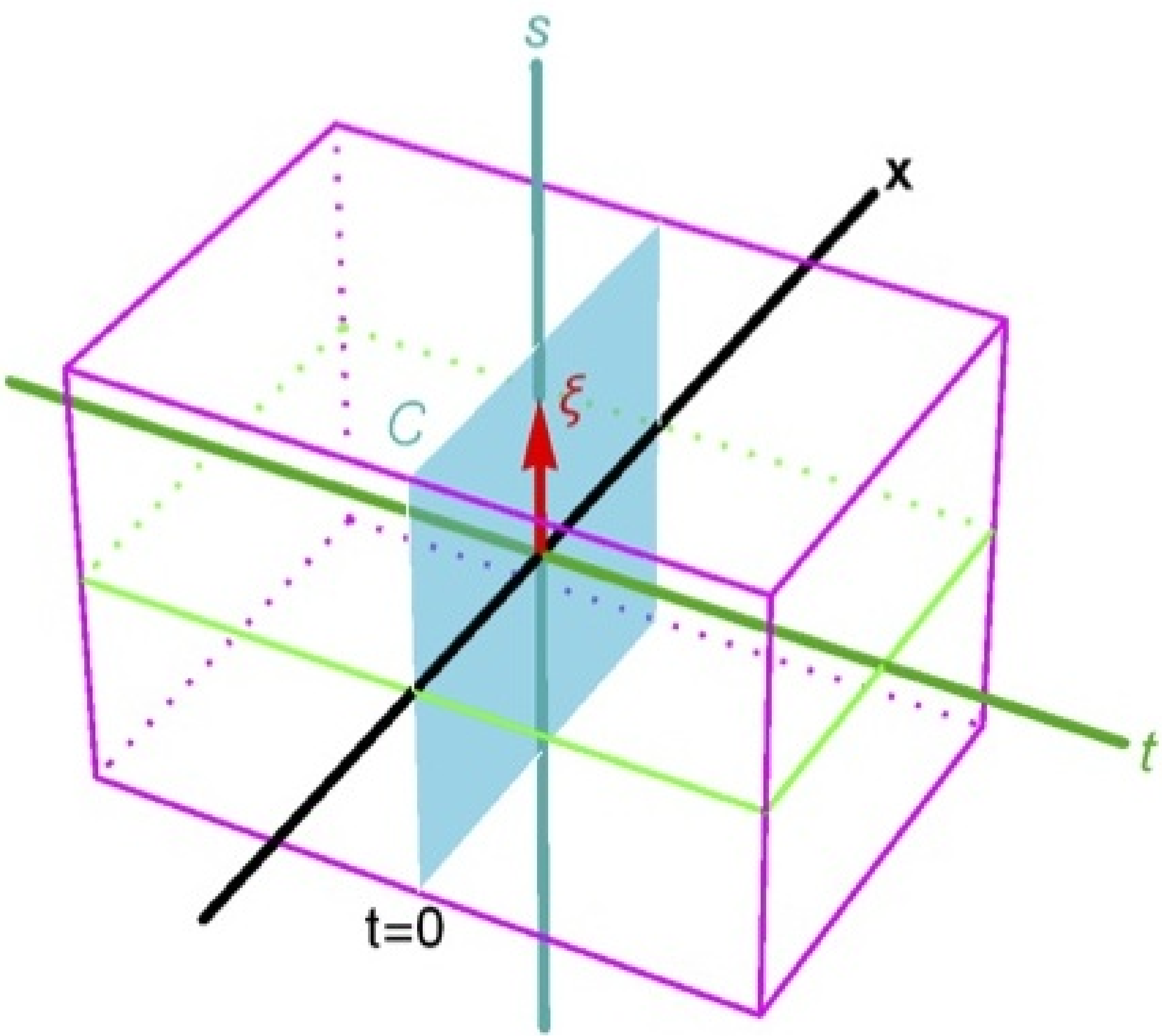


Fig.6 : $t = \text{const}$ slice is "Carroll space-time" $\mathcal{C}$ embedded into Bargmann space.

Symmetries

ξ-preserving isometries of Bargmann :

$$a = \begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & e \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, and $e, f \in \mathbb{R}$ form centrally extended Galilei [$\equiv$ Bargmann] group Barg $\equiv$ Barg$(d+1)$. Boost :

$$\begin{pmatrix} \boldsymbol{x} \\ t \\ s \end{pmatrix} \to \begin{pmatrix} \boldsymbol{x} + \mathbf{b}t \\ t \\ s - \mathbf{b} \cdot \boldsymbol{x} + \frac{1}{2}\mathbf{b}^2 t \end{pmatrix} \tag{12}$$

N.B. : lifting ordinary wave fct to equivariant ($\equiv \partial_s \Psi = im\Psi$) on B-space, Galilei boost action (3) is Bargmann action. Affine action on $\begin{pmatrix} \boldsymbol{x} \\ t \\ s \\ 1 \end{pmatrix} \rightsquigarrow$ Bargmann algebra $\mathfrak{barg} \equiv \mathfrak{barg}(d+1)$

$$\boxed{(\omega^A_B\, x^B + \beta^A\, t + \gamma^A)\frac{\partial}{\partial x^A} + \varepsilon \frac{\partial}{\partial t} + (\varphi - \beta_A\, x^A)\,\frac{\partial}{\partial s}} \tag{13}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\varepsilon, \varphi \in \mathbb{R}$.

Seen before: restriction of Bargmann space to $t=0$ is Carroll manifold $\mathcal{C}$ left invariant by restriction of Bargmann action (13) with $e=0 \rightsquigarrow$ action of Carr, embedded into Bargmann group,

$$\begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \hookrightarrow \begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & 0 \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $f \in \mathbb{R}$.

Carr$(d+1)$: $e=0$ subgroup of Barg(d+1).

Infinitesimally:

$$(\omega^A_B\, x^B + \gamma^A)\frac{\partial}{\partial x^A} + (\varphi - \beta_A\, x^A)\,\frac{\partial}{\partial s} \tag{15}$$

$\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\varphi \in \mathbb{R}$ (seen before).

N.B. : for $t=t_0$ Carroll boost acts as

$$v \to v - \mathbf{b}.\boldsymbol{x} - \tfrac{1}{2}\mathbf{b}^2\, t_0 \tag{16}$$

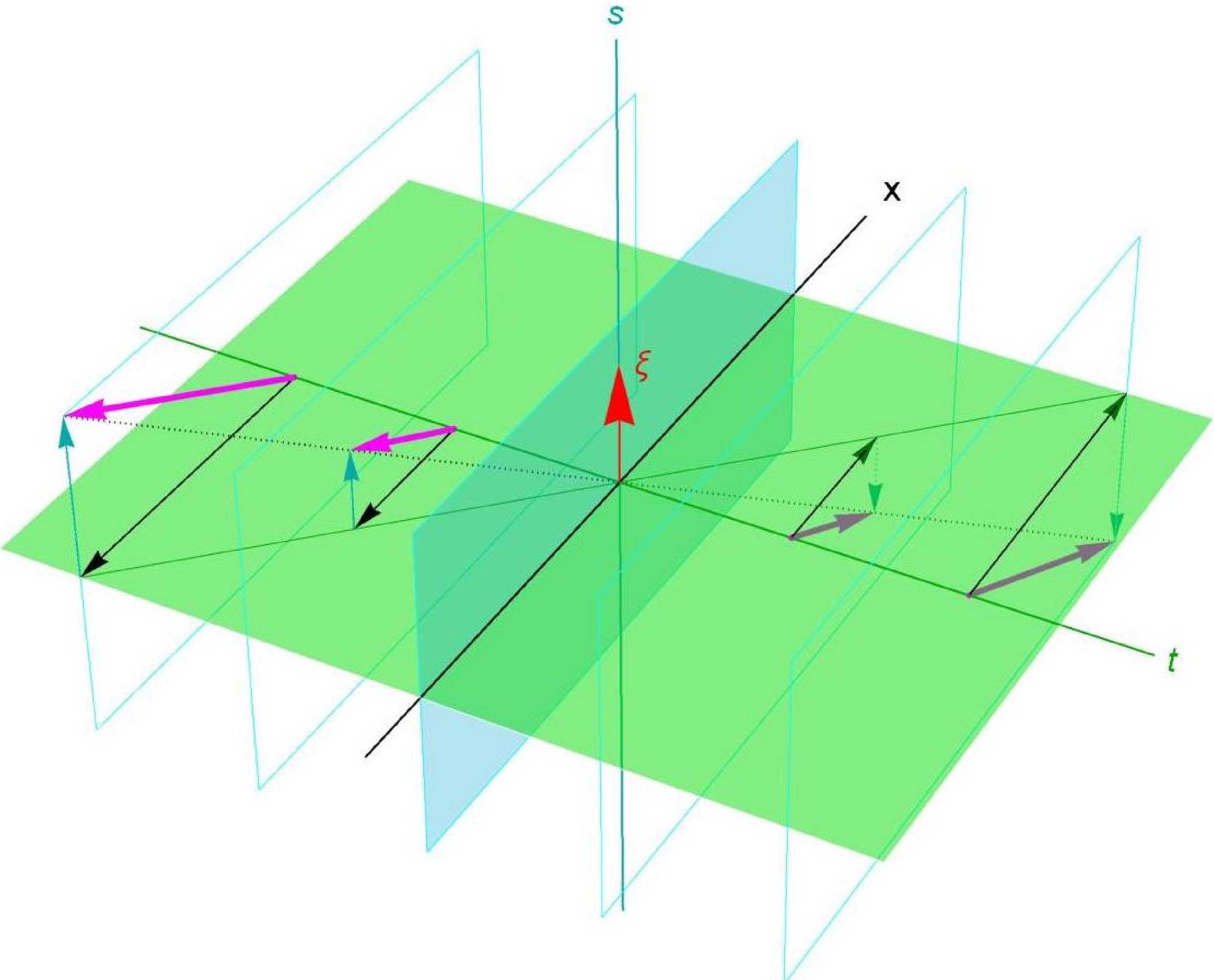


Fig.7 Boost acting on flat Bargmann space

Plane gravitational waves

In Brinkmann coordinates

$$ds^2 = d\boldsymbol{X}^2 + 2dUdV - K(U,\boldsymbol{X})\,dU^2 \qquad (17)$$

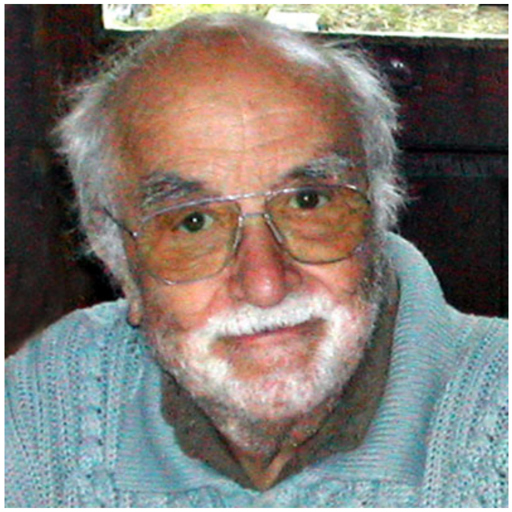
U and V light-cone coords, $\boldsymbol{X} = (X_1, X_2) \sim$ transverse plane. Vacuum Einstein eqn satisfied with

$$K(U,\boldsymbol{X}) = \mathcal{A}(U)\Big(X_1^2 - X_2^2\Big) + 2\mathcal{B}(U)X_1X_2. \qquad (18)$$

Clue: (17) Bargmann space $\sim$ anisotropic oscillator.

P. M. Zhang, P. A. Horvathy, K. Andrzejewski, J. Gonera and P. Kosinski, "Newton-Hooke type symmetry of anisotropic oscillators," Ann. Phys. **333** (2013) 335 [`arXiv:1207.2875 [hep-th]`].

Isometries: Bondi et al 1959. 5-parameters. 3 translations + 2 MYSTERIOUS (**not** written explicitely). Torre 2006: solution of Sturm-Liouville eqn.

Souriau 1973  metric in BJR (Baldwin-Jeffery-Rosen) coords :

$$\boxed{ds^2 = \boxed{a_{ij}(u)}\, dx^i dx^j + 2du dv.} \qquad (19)$$

Isometries : $u \to u$, completed with

$$\boxed{\boldsymbol{x} \to \boldsymbol{x} + H(u)\mathbf{b} + \mathbf{c},} \qquad (20a)$$

$$\boxed{v \to v - \mathbf{b} \cdot \boldsymbol{x} - \tfrac{1}{2}\mathbf{b} \cdot H(u)\mathbf{b} + f} \qquad (20b)$$

where $H = (H_{ij})$ is 2×2 matrix

$$\boxed{H(u) = \int_{u_0}^{u} a^{-1}(w) dw.} \qquad (21)$$

$\mathbf{c} \in \mathbb{R}^2 \sim$ transverse-space transl, $f \sim$ null translat along v coord.

Group composition law: that of $\boxed{\text{Carroll group}}$ with no rotations. $\mathbf{b} \in \mathbb{R}^2$ generates Carroll boost, implemented as in (20).

Flat case: $a_{ij} = \delta_{ij} \quad \Rightarrow$

$$H(u) = (u - u_0)\,\mathrm{Id} \tag{22}$$

choosing $u_0 = 0$

$$\boldsymbol{x} \to \boldsymbol{x} + u\,\mathbf{b}, \tag{23a}$$

$$u \to u, \tag{23b}$$

$$v \to v - \mathbf{b} \cdot \boldsymbol{x} - \tfrac{1}{2}\mathbf{b}^2\,u \tag{23c}$$

Galilei boosts lifted to flat Bargmann space.

(See again at the end)

Relation with Brinkmann-coords ?

1. Given B-profile $K(U)$, solve Sturm-Liouville

$$\boxed{\ddot{P}_{kj} = K_{kr} P_{rj}} \tag{24}$$

for U-dept 2×2 matrix $P_{kj}(U)$.

2. Putting

$$X^i = P_{ij}\, x^j \qquad U = u \tag{25a}$$

$$a_{ij}(u) = P_{ri} P_{rj}, \quad V = v - \frac{1}{4} \frac{da_{ij}}{du}\, x^i x^j \tag{25b}$$

allows to present metric (17) in BJR form

$$ds^2 = \boxed{a_{ij}(u)}\, dx^i dx^j + 2du dv$$

cf. (19) *provided* also $P^\dagger \dot{P} = \dot{P}^\dagger P$.

Quadratic "scalar potential" in B, $K_{ij} X^i X^j \, dU^2$ in (17), traded for "time"-dependent" transverse metric $a_{ij}(u)$ (while leaving $U = u$ unchanged).

EXAMPLES

0. Restriction of flat Minkowski space

$$d\boldsymbol{r}^2 + 2dt\,ds$$

to $t = 0$ is Carroll manifold, upon which restriction $e = 0$ of Bargmann group acts consistently with Carroll action.

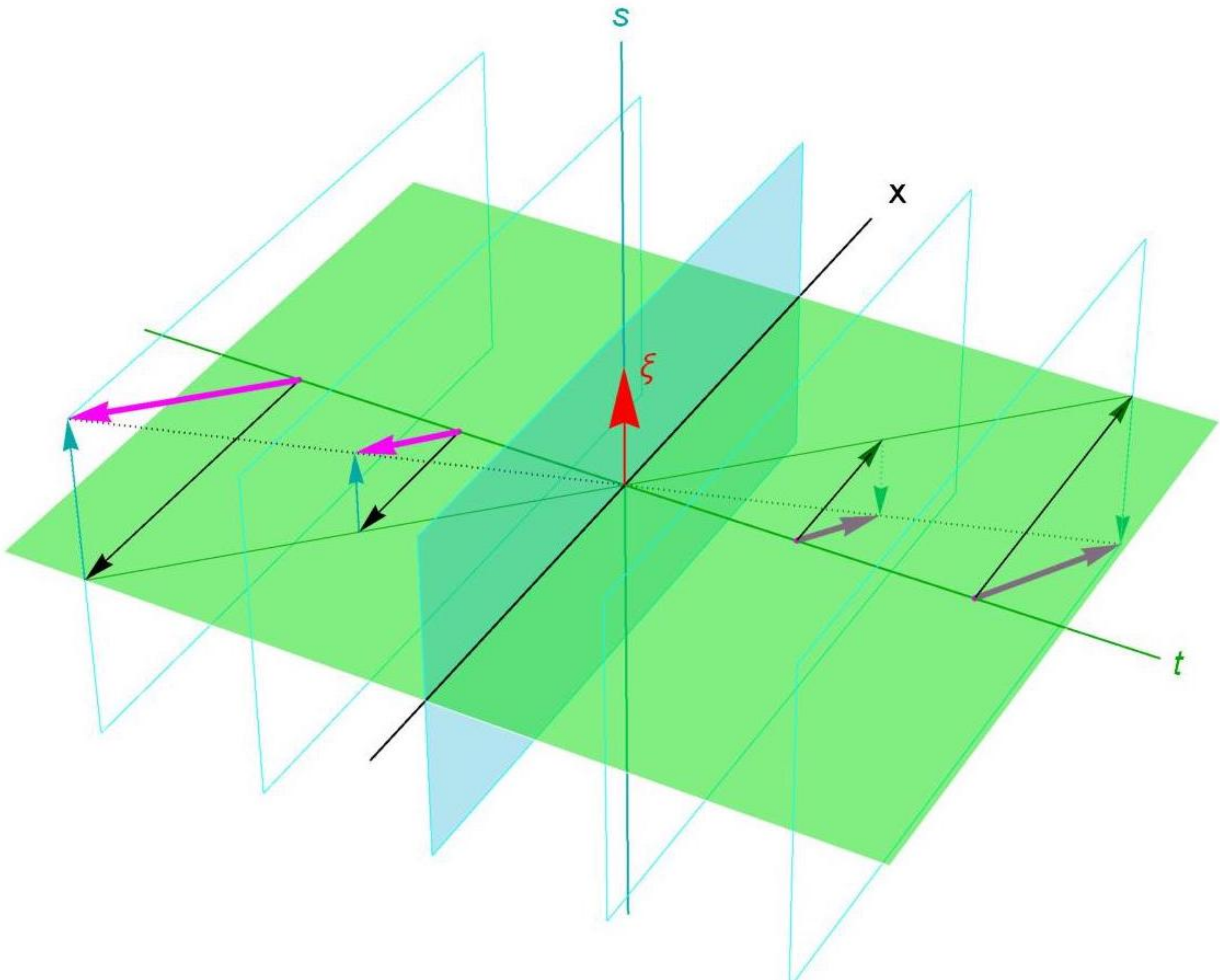


Fig.7bis Boost acting on flat Bargmann space

Linearly polarized "sudden burst" $\sim$ Gaussian profile ($\sim$ anisotropic oscillator with time-dependent frequency)

$$K_{ij}(u)X^iX^j = \frac{e^{-u^2}}{\sqrt{\pi}}\Big((X^1)^2 - (X^2)^2\Big). \qquad (26)$$

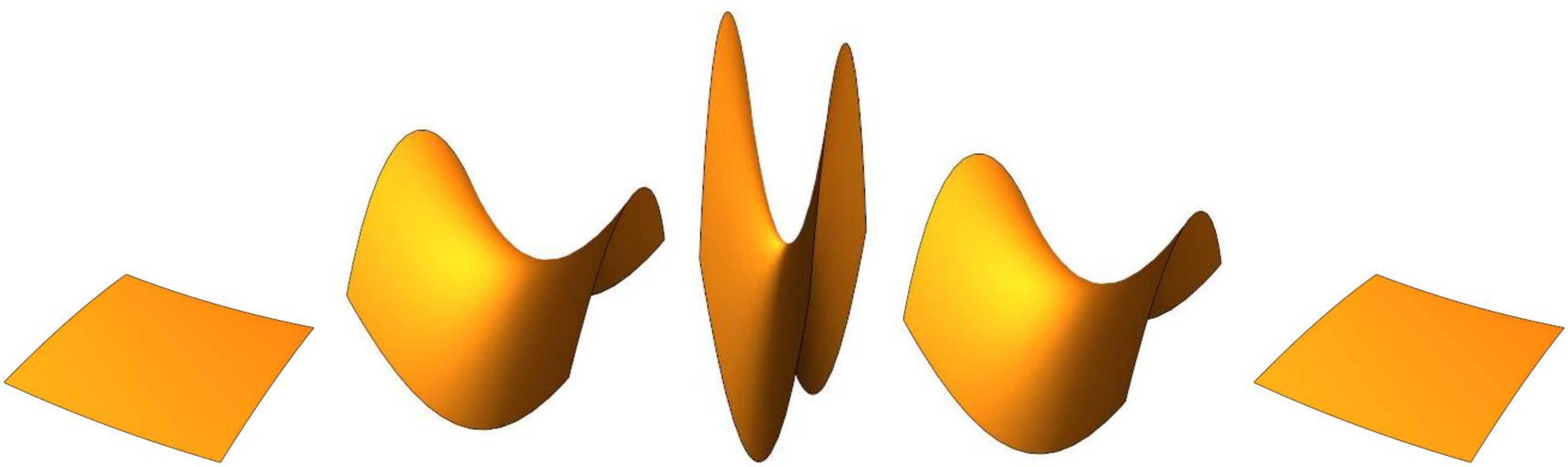


Fig.8 "Time" evolution of wave for "sudden burst" with Gaussian profile $\mathcal{A}(u) = \exp[-u^2]$.

Sandwich wave: $K(u) \neq 0$ only in "wave zone" $U_i < U < U_f$. Assumption : metric Minkowski in "before-zone" $U < U_i$ and flat in "after-zone" $U_f < U$.

Impulsive wave :

$$\mathcal{A}(U) = 2k\,\delta(U) \tag{27}$$

$k \in \mathbb{R}$. Wave zone suppressed, $U_i = U_f = 0$. SL eqn. (24) solved by

$$\boxed{P(u) = 1 + u\,\theta(u)\,c_0} \tag{28}$$

where $\theta(u)$ Heaviside, $c_0 = \frac{1}{2}\dot{a}(0+)$ initial "speed" of transverse metric. Can be chosen $c_0 = k\,\mathrm{diag}(1,-1)$.

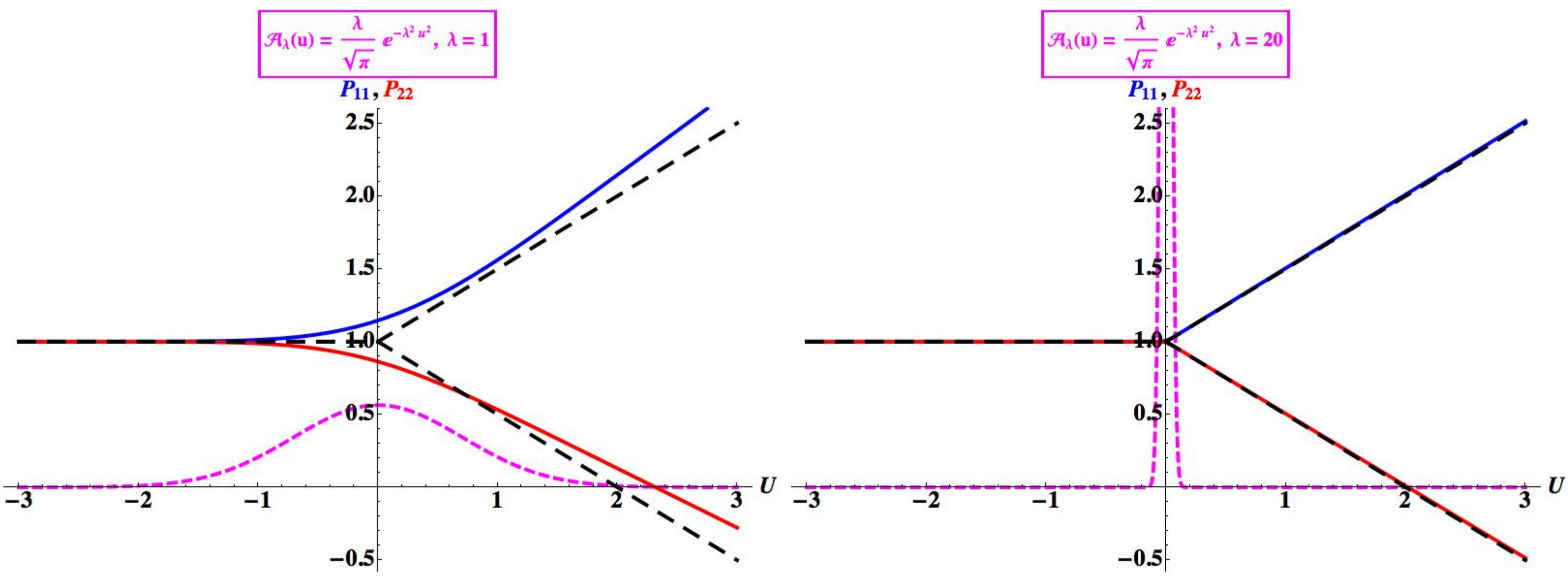


Fig.9. Numerical solution of S-L eqn (24) for profile $\mathcal{A}_\lambda(U) = (\lambda/\sqrt{\pi})\,e^{-\lambda^2 U^2}$ shows that components of diagonal matrix $P_\lambda(U)$ approach, for large λ, those of impulsive wave [in ***dashed black****].*

$$a(u) = \begin{cases} 1 & \text{for} \quad u \leq 0, \\ (1 + u\,c_0)^2 & \text{for} \quad u \geq 0. \end{cases} \tag{29}$$

More generally

$$\mathcal{A}_\lambda(U) = \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 U^2}. \tag{30}$$

Squeezing Gaussians to Dirac δ by letting $\lambda \to \infty$, components of $P_\lambda(u)$ and of transverse metric $a_\lambda(u) = P_\lambda^T(u) P_\lambda(u)$ tend to those of impulsive wave.

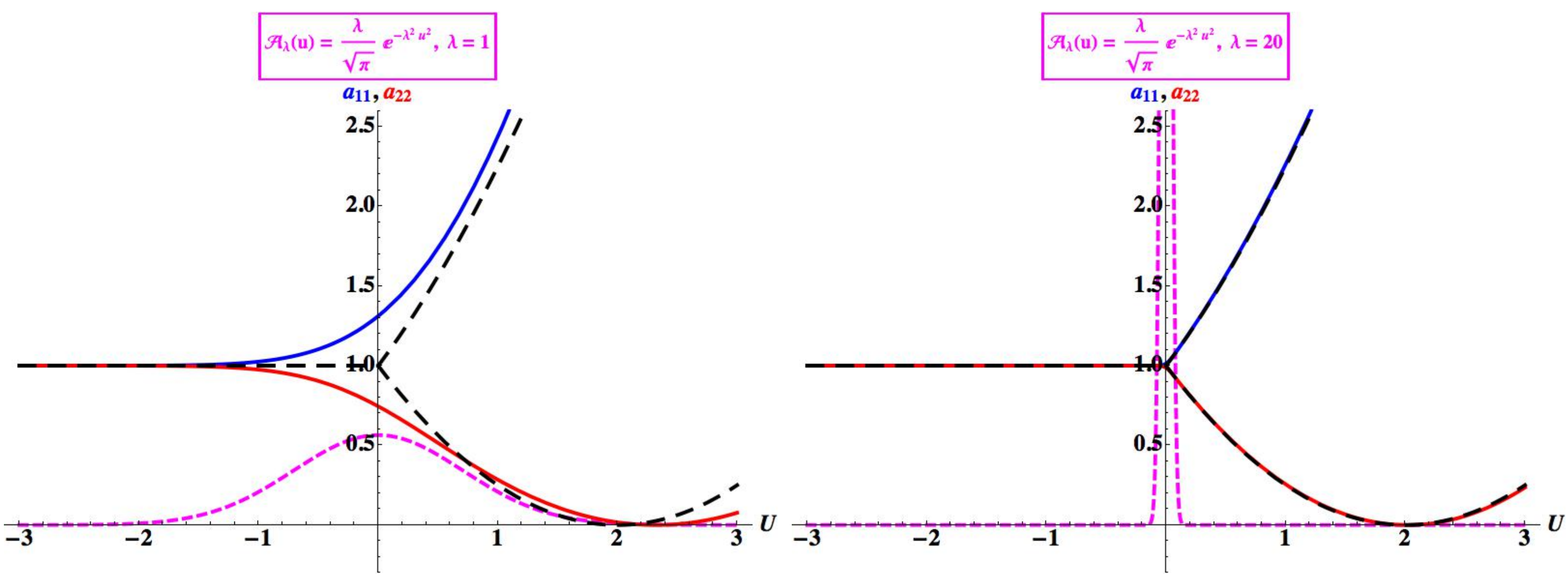


Fig.10. Squeezing Gaussians $\mathcal{A}_\lambda$ to Dirac δ, transverse metrics $a_\lambda(u)$ (in **red** *and* **blue***) tend to that of impulsive wave in BJR coordinates, depicted in* **dashed black** *lines.*

Carroll boost for impulsive GW

Boost implemented as $\boldsymbol{x} \to \boldsymbol{x} + H(u)\mathbf{b}$, $v \to v - \mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}\cdot H(u)\mathbf{b}$ cf. (20). For impulsive

$$
\begin{aligned}
H(u) &= u\,P^{-1}(u) && (31)\\
P &= \begin{cases} 1 & u \leq 0 \\ \mathrm{diag}(1+u/2, 1-u/2) & u \geq 0 \end{cases} && (32)
\end{aligned}
$$

$$
H = \mathrm{diag}(H_+, H_-) = \begin{pmatrix} \frac{u}{1+u/2} & \\ & \frac{u}{1-u/2} \end{pmatrix} \; u \geq 0. \tag{33}
$$

Boost with $\mathbf{b} = (b_+, b_-)$ implemented as,

$$
x_1 \to x_1 + \frac{u}{1+u/2}\,b_+ \tag{34a}
$$

$$
x_2 \to x_2 + \frac{u}{1-u/2}\,b_- \tag{34b}
$$

$$
\begin{aligned}
v \to v - (x_1 b_+ + x_2 b_-) - \\
\tfrac{1}{2}\left(\frac{u}{1+u/2}\,b_+^2 + \frac{u}{1-u/2}\,b_-^2\right)
\end{aligned} \tag{34c}
$$

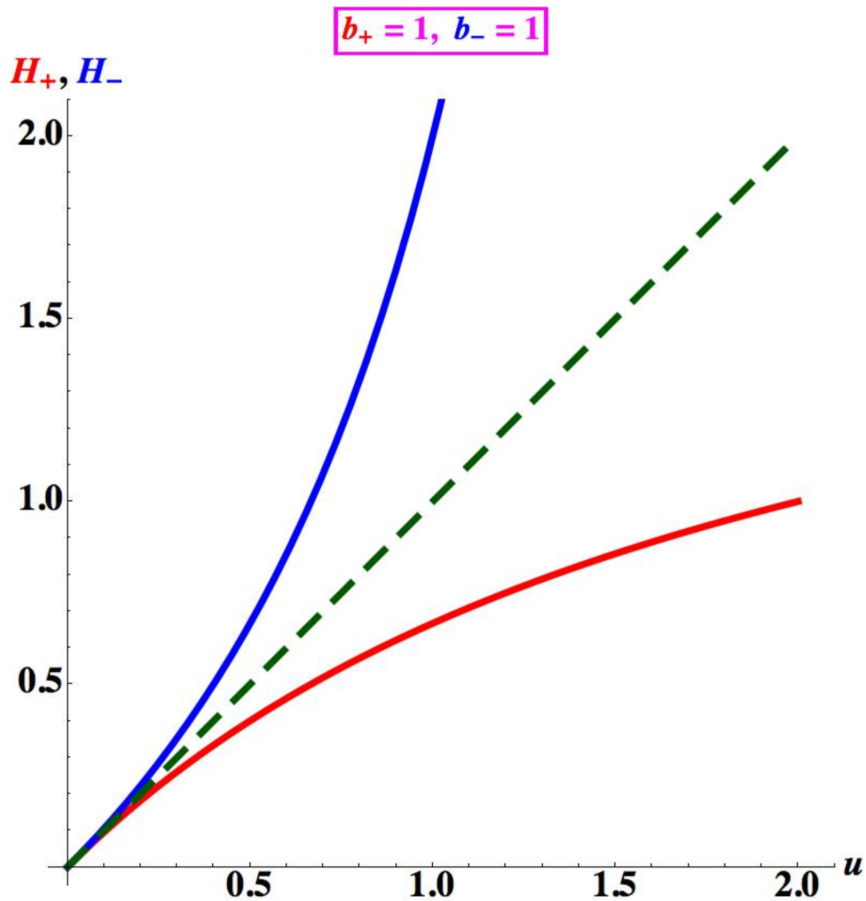


Fig.11 Boost acts on impulsive space-time according to $\boldsymbol{x} \to \boldsymbol{x} + H\mathbf{b}$, $v \to v - \mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}\cdot H(u)\mathbf{b}$. $H = \mathrm{diag}(H_+, H_-)$ but components differ considerably from usual Galilei implementation $H_\pm = u\,\mathrm{Id}$.