BMS and AdS

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March 20, 2018

Introduction

- This talk will be about analysing the asymptotics of both AdS and Ricci flat spacetimes.
- For AdS, the asymptotics play a crucial role in the holographic duality.
- For asymptotically flat spacetimes, the asymptotics are relevant to soft scattering theorems and gravitational memory effects.

Introduction

- For asymptotically flat spacetimes, much of the analysis is tied specifically to four dimensions e.g. use of two dimensional celestial sphere.
- Long standing question: soft scattering theorems exist in all dimensions, but they do not seem to be related to asymptotic symmetries except in $d = 4$.
- What is the asymptotic symmetry structure in $d > 4$?

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Introduction

- The asymptotic analysis for AdS and flat spacetimes seems very different:
  - Timelike versus null conformal boundaries;
  - Fefferman-Graham versus Bondi-Sachs parameterization;
  - Sources/expectation values of operators versus Bondi mass, news etc.
- How are these analyses related?

References

1. Federico Capone and Marika Taylor
"*Symmetries of asymptotically flat spacetimes*", 1804.xxxxx
2. Aaron Poole, Kostas Skenderis and Marika Taylor
"*A BMS approach to AdS_4*", 1804.xxxxx

1712.01204 by Pate, Raclariu and Stominger is complementary to 1.

Outline

- **Bondi-Sachs analysis for asymptotically flat 4d spacetimes**
- Asymptotic analysis for $d > 4$
- Analysis of asymptotically AdS spacetimes

Bondi gauge for asymptotically flat 4d spacetimes

- The 4d Minkowski metric can be written as

$$ds_M^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}$$

where $u = (t - r)$ is the retarded time and $\gamma_{z\bar{z}}$ is the round metric on S^2.

- An asymptotically (locally) flat metric can be expressed in Bondi gauge as

$$ds^2 = -Xdu^2 - 2e^{2\beta}dudr + h_{AB}\left(d\theta^A + U^A du\right)\left(d\theta^B + U^B du\right)$$

where $ds^2 \to ds_M^2$ (locally) as $r \to \infty$.

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Bondi gauge for asymptotically flat 4d spacetimes

- Imposing one gauge condition on h_{AB},

$$\det\left(\frac{h_{AB}}{r^2}\right) = 1$$

  there are six unknown functions in (X, β, h_{AB}, U^A).
- The Einstein equations split into “main" equations and “supplementary" equations.
- If the latter are satisfied on a constant u hypersurface, they are automatically satisfied everywhere.

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Axial symmetry

- To explain the integration scheme, for computational simplicity we impose axial symmetry and reflection symmetry under $\phi \to -\phi$.
- We can then rewrite the Bondi gauge metric as

$$ds^2 = \left(-\frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma} \right) du^2 - 2 e^{2\beta} du dr$$

$$-2 U r^2 e^{2\gamma} du d\theta + r^2 \left(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2 \right)$$

i.e. using $(V(u,r,\theta), U(u,r,\theta), \beta(u,r,\theta), \gamma(u,r,\theta))$.

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Nested form for main equations

- The four main equations take a nested form:
  - Given γ at constant $u = u_0$, we first determine β.
  - Next we determine U from (γ, β) and then V.
  - Finally, we determine the u evolution of γ.
- Iterating the process allows us to determine all metric functions in the future domain of dependence of the u_0 hypersurface.

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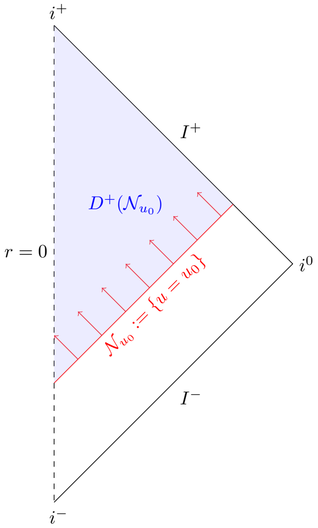
Bondi gauge for asymptotically flat spacetimes



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Asymptotic analysis

- To explore asymptotic structure, we need to analyse Einstein equations perturbatively as $r \to \infty$.
- For example, the first main equation is

$$\beta_r = \frac{1}{2} r \gamma_r^2$$

  which needs to be solved with appropriate asymptotically (locally) flat boundary conditions.
- Subtle question: what are appropriate boundary conditions?

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Asymptotically flat 4d metrics

- General form of asymptotic expansion is

$$X = 1 - \frac{2m_B}{r} + \cdots$$

$$h_{AB} = r^2\gamma_{AB} + rC_{AB} + \cdots$$

$$h_{AB}U^B = D^B C_{AB} + \frac{1}{r}\left(\frac{4}{3}(N_A + u\partial_u m_B)\right) - \frac{1}{4r}\partial_A(C_{BC}C^{BC}) + \cdots$$

- Here the highlighted quantities indicate integration functions in the asymptotic expansion.

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Asymptotically flat 4d metrics

- The asymptotic expansion of the metric is thus

$$ds^2 = ds_M^2 + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2$$

$$+D^z C_{zz}dudz + D^{\bar{z}} C_{\bar{z}\bar{z}}dud\bar{z}$$

$$+\frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)dudz + \cdots$$

where we use complex coordinates on the S^2.

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Asymptotic coefficients

- Bondi mass aspect $m_B(u, z, \bar{z})$; integrate over S^2 to get total Bondi mass M_B at time u.
- Traceless tensor $C_{AB}(u, z, \bar{z})$: captures gravitational memory effects, soft scattering theorems and gravitational waves.
- Angular momentum aspect $N^A(u, z, \bar{z})$; integrate over S^2 to get total angular momentum.

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Dynamical equations

- The Einstein equations give the evolution of the Bondi mass aspect

$$\partial_u m_B = \frac{1}{4}\left(D_A D_B(\partial_u C^{AB}) - \partial_u C^{AB}\partial_u C_{AB}\right)$$

with a similar equation for the angular momentum aspect.

- Hence a non-zero news $N_{AB} = \partial_u C_{AB}$ leads to mass non-conservation (gravitational waves).

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Superrotations and C_{AB}

- Superrotations act as meromorphic transformations on the S^2 coordinates i.e.

$$z \to \mathcal{Y}(z) \qquad \bar{z} \to \bar{\mathcal{Y}}(\bar{z})$$

- Such transformations change C_{AB}: ΔC_{zz} is expressed in terms of the Schwarzian derivative of $\mathcal{Y}$.
- Associated with the superrotations are (finite) superrotation charges. (Barnich and Troessart)

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Soft scattering theorems

- Consider an n-point scattering amplitude M_n, in the limit in which one leg becomes soft, i.e. very low energy.
- Then Weinberg (1965) realised that there is a kinematic relationship between M_n and M_{n-1}:

$$M_n = S_n\left(\frac{1}{t}\right) M_{n-1} + \mathcal{O}(t^0)$$

Here S_n is a principle part of a Laurent series (finite number of negative powers of t) and t is the momentum transfer.

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Superrotations and soft scattering

- Conservation of superrotation charges is associated with such soft scattering theorems for scattering amplitudes. (Strominger et al)
- Note that analytic transformations on the S^2 (supertranslations) do not give rise to non-trivial soft scattering theorems.

Analysis specific to 4d (2d celestial sphere plays essential role) but soft scattering theorems exist in all dimensions?

Outline

- Bondi-Sachs analysis for asymptotically flat 4d spacetimes
- **Asymptotic analysis for $d > 4$**
- Analysis of asymptotically AdS spacetimes

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Asymptotic symmetries in $d > 4$

- In d=4, integration functions were associated with asymptotic symmetries, and hence conservation laws.
- In d>4, the longstanding puzzle is that there do not seem to be analogues of such integration functions.
- However, people has mostly been looking for analogues of supertranslations (BMS) rather than superrotations.

Bondi gauge analysis in higher dimensions

- We can again parameterise the metric as

$$ds^2 = -Xdu^2 - 2e^{2\beta}dudr + h_{AB}\left(d\theta^A + U^A du\right)\left(d\theta^B + U^B du\right)$$

  in terms of metric functions (X, β, h_{AB}, U^A).
- The nested structure of the Einstein equations persists.
- The key question is again the boundary conditions for $r \to \infty$ for $d > 4$.

Outline

- Bondi-Sachs analysis of asymptotically flat spacetimes
- Analysis for $d > 4$
- **Analysis of asymptotically Anti-de Sitter spacetimes**

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Anti-de Sitter

- We now consider asymptotically AdS spacetimes.
- Key features:
  1. Timelike conformal boundary
  2. Relation to CFT in one less dimension
- CFT data allows the spacetime to be reconstructed asymptotically.

Asymptotically AdS spacetimes

- A convenient parameterisation is Fefferman-Graham coordinates

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2} g_{ij}(x, \rho) dx^i dx^j$$

in the neighbourhood of the conformal boundary $\rho \to 0$.

- Einstein equations expressed in terms of derivatives of g

$$g^{ij} \partial_\rho^2 g_{ij} = 0$$

and so on.

Asymptotically AdS$_4$ spacetimes

- In Fefferman-Graham coordinates:

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\left(g_{(0)ij} + g_{(2)ij}\rho^2 + g_{(3)ij}\rho^3 + \cdots\right) dx^i dx^j$$

- Near boundary expansion reconstructed from $g_{(0)ij}$ (background metric for dual theory) and $g_{(3)ij}$ (stress energy tensor for dual theory).
- All other terms ($g_{(2)}$ etc) are expressed in terms of curvatures of this data (de Haro, Solodukhin, Skenderis, 2000).

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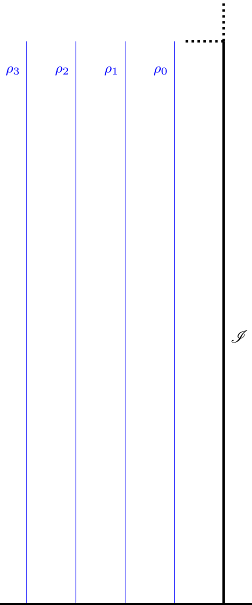
Fefferman-Graham reconstruction



- In the Fefferman-Graham coordinate system, the neighbourhood of the conformal boundary is foliated by constant ρ hypersurfaces.
- The defining boundary data $(g_{(0)}, g_{(3)})$ allow the metric to be reconstructed.

Bondi gauge

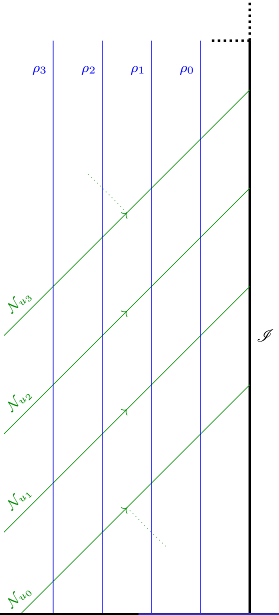
- We can also parameterise an asymptotically locally AdS_4 spacetime in Bondi gauge as

$$ds^2 = -Xdu^2 - 2e^{2\beta}dudr + h_{AB}\left(d\theta^A + U^A du\right)\left(d\theta^B + U^B du\right)$$

as $r \to \infty$.

- Now the spacetime is foliated by hypersurfaces of constant u in the vicinity of the conformal boundary.

Bondi gauge



ρ_3
ρ_2
ρ_1
ρ_0
$\mathscr{I}$
$\mathcal{N}_{u_3}$
$\mathcal{N}_{u_2}$
$\mathcal{N}_{u_1}$
$\mathcal{N}_{u_0}$

Asymptotic analysis in Bondi gauge

- For simplicity let us impose axi-symmetry and reflection symmetry in ϕ i.e.

$$ds^2 = -Wr^2 e^{2\beta} du^2 - 2e^{2\beta} dudr + r^2 e^{2\gamma}(d\theta - Udu)^2$$
$$+r^2 e^{-2\gamma} \sin^2\theta d\phi^2$$

- The metric is thus defined by four functions: $(W(u,r,\theta), \gamma(u,r,\theta), \beta(u,r,\theta), U(u,r,\theta))$.
- Note that the determinant of the S^2 is gauge fixed, as is usual in Bondi gauge.

Why use Bondi gauge?

- The nested structure of the Einstein equations persists for $\Lambda \neq 0$ (although is typically broken by non-zero matter).
- Bondi gauge is natural choice for numerical AdS simulations involving horizons. (See e.g. Chesler and Yaffe)
- Relativists work with characteristic null initial problems, Bondi mass aspect etc.

BUT: the relation of asymptotic data in Bondi gauge to the dual QFT is unclear.

Einstein equations in Bondi gauge

- Analytic structure is manifest expressing $z = 1/r$:

$$\beta_{,z} + \frac{1}{2} z \gamma_{,z}^2 = 0.$$

$$0 = 2z\beta_{,z\theta} - 2z\gamma_{,z\theta} + zU_{,zz}e^{2(\gamma-\beta)} - 2U_{,z}e^{2(\gamma-\beta)}$$
$$+2zU_{,z}e^{2(\gamma-\beta)}(\gamma_{,z} - \beta_{,z}) + 4\beta_{,\theta} + 4z\gamma_{,z}\gamma_{,\theta} - 4z\cot\theta\gamma_{,z}$$

plus two more main equations.

- Key point: lack of covariance obscures asymptotic structure.

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Results of asymptotic analysis

- The metric functions admit analytic expansions in $z = 1/r$

$$\gamma = \sum_{n=0}^{\infty} \frac{\gamma_n(u,\theta)}{r^n} \qquad \beta = \sum_{n=0}^{\infty} \frac{\beta_n(u,\theta)}{r^n}$$

$$W = \sum_{n=0}^{\infty} \frac{W_n(u,\theta)}{r^n} \qquad U = \sum_{n=0}^{\infty} \frac{U_n(u,\theta)}{r^n}$$

- The entire expansion can be determined algebraically from knowledge of (γ_0, β_0, U_0) and (γ_3, U_3, W_3).

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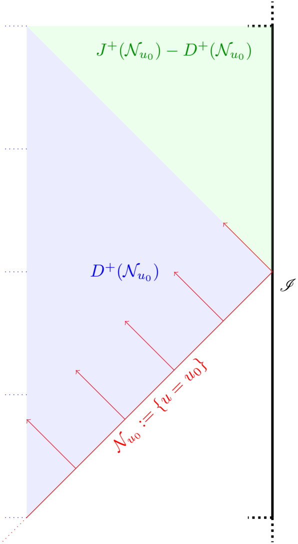
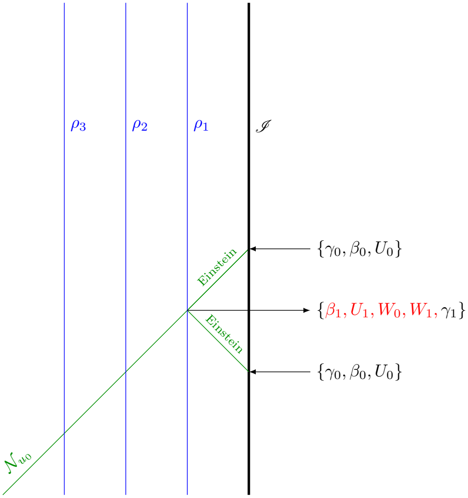
Interpretation

1. The cosmological constant (as expected) changes the structure of the asymptotic expansions.
2. The integration functions (γ_0, β_0, U_0) correspond to the (constrained) background metric for the 3d QFT.
3. The other integration functions (γ_3, U_3, W_3) would be termed "Bondi mass aspect" and "Bondi angular momentum aspect" by relativists but are related algebraically to the dual QFT stress tensor.

1. Integration scheme



- Naively, one would think that in the Bondi gauge data from constant u hypersurfaces is propagated forwards in time.
- Such propagation could only include the future domain of dependence of the u hypersurface.



- In practice, the integration scheme takes boundary data (γ_0, β_0, U_0) at different values of time and generates the subleading terms $(\beta_n, U_n, W_n, \gamma_n)$.
- Thus, in the Bondi gauge for AdS, the evolution is effectively radially inwards.

2. Boundary metric

- We can write the boundary metric $g_{(0)}$ in terms of the Bondi data:

  $$ds^2 = (e^{2\gamma_0} U_0^2 - e^{4\beta_0}) dt^2 - 2e^{2\gamma_0} U_0 dt d\theta + e^{2\gamma_0} d\theta^2 + e^{-2\gamma_0} \sin^2\theta d\phi^2$$

  where t is the time.
- Asymptotically AdS (boundary metric $R_t \times S^2$) corresponds to $\beta_0 = \gamma_0 = U_0 = 0$.

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2. Boundary metric

- Non-trivial $(\gamma_0(t,\theta), \beta_0(t,\theta), U_0(t,\theta))$ corresponds to a curved, time dependent background metric for the CFT.
- From an AdS/CFT perspective, the determinant restriction on the S^2 is very unnatural: excludes " breathing" modes for the sphere, that are e.g. relevant for quark gluon plasma simulations.

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3. Bondi aspects v QFT data

- Relativists define a Bondi mass aspect $m_B(u, z, \bar{z}) \sim W_3(u, z, \bar{z})$, in analogy to asymptotically flat spacetimes.
- From an AdS/CFT perspective, the dual stress energy tensor of the CFT

$$T_{ij} = g_{(3)ij}$$

(with $g_{(3)ij}$ the Fefferman-Graham coefficient) is the natural observable.
- T_{ij} is relevant to positivity theorems, entanglement entropy, entropy bounds etc.

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3. Bondi aspects v QFT data

- For asymptotically AdS_4

$$T_{tt} \sim m_B$$

i.e. the Bondi mass aspect is simply related to the CFT stress energy tensor.

3. Bondi aspects v QFT data

- For asymptotically locally AdS_4 the relation between m_B and the dual stress energy tensor is in general (very) complicated:

$$T_{tt} \sim m_B + \mathcal{T}(\gamma_0, \beta_0, U_0)$$

- Lesson: Bondi mass aspect (as defined in the literature) is not natural from holographic perspective!

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$$
\begin{aligned}
\mathcal{T} = &- e^{-2(\beta_0+\gamma_0)}(2\gamma_{0,u} - U_0(\cot(\theta) - 2\gamma_{0,\theta}) + U_{0,\theta})(4e^{4\beta_0}(\beta_{0,\theta})^2 \\
&- e^{2\gamma_0} U_0(-2\gamma_{0,u\theta} + 4\gamma_{0,u}(\gamma_{0,\theta} - \cot(\theta)) + U_0(4(\gamma_{0,\theta})^2 \\
&- 2\gamma_{0,\theta\theta} - 6\cot(\theta)\gamma_{0,\theta} + \cot^2(\theta) - 1) - U_{0,\theta\theta} - \cot(\theta)U_{0,\theta})) \\
&+ e^{-2\gamma_0}(2e^{4\gamma_0} U_0 U_3 - 2e^{2\beta_0}\beta_{0,\theta}(-2\gamma_{0,u\theta} + 4\gamma_{0,u}(\gamma_{0,\theta} - \cot(\theta)) \\
&+ U_0(4(\gamma_{0,\theta})^2 - 2\gamma_{0,\theta\theta} - 6\cot(\theta)\gamma_{0,\theta} + \cot^2(\theta) - 1) - U_{0,\theta\theta} \\
&+ \frac{1}{3}e^{2\gamma_0} U_0^2 \left[6\gamma_3 + \frac{1}{2}e^{-6\beta_0}(-2\gamma_{0,u} + U_0(\cot(\theta) - 2\gamma_{0,\theta}) - U_{0,\theta})^3\right] \\
&+ 2e^{-2\beta_0}\beta_{0,\theta} U_0(2\gamma_{0,u} - U_0(\cot(\theta) - 2\gamma_{0,\theta}) + U_{0,\theta})^2 \\
&+ \frac{1}{8}e^{-2\beta_0}(U_{0,\theta} + \cot(\theta)U_0)(2\gamma_{0,u} - U_0(\cot(\theta) - 2\gamma_{0,\theta}) + U_{0,\theta})^2
\end{aligned}
\tag{1}
$$

Applications of Bondi → Fefferman-Graham map

- For simulations carried out in Bondi gauge, one can now read off directly the QFT data.
- Our analysis is relevant for AdS Robinson-Trautmann metrics

$$ds^2 = F(r, u, z, \bar{z})du^2 - 2dudr + 2r^2 e^{\phi(u,z,\bar{z})} dz d\bar{z}$$

(Admit geodesic congruence with zero shear, twist and non-vanishing divergence.)

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Applications of Bondi analysis

- Preliminary holographic analysis and discussion of Bondi mass in Bakas, Skenderis and Withers[1].
- System relaxes to black hole at late times; describes QFT thermalization process analytically.
- Outgoing modes pass through the conformal boundary.

[1] See also de Freitas and Reall; Ciambelli, Marteau, Petkou, Petropoulous, Siampos for discussions of AdS-Robinson-Trautman.

Summary and conclusions

- Asymptotic analysis for asymptotically flat spacetimes: integration functions associated with analogues of superrotations exist in all dimensions.
- These occur at a different order in the asymptotic expansion to gravitational waves.
- Do associated symmetry charges lead to soft scattering theorems?

Summary and conclusions

- For asymptotically AdS, naively Bondi gauge analysis propagates data via null hypersurfaces.
- Integration scheme in practice propagates data via constant radius hypersurfaces.
- Bondi mass aspect as defined previously rather unnatural from QFT perspective.

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