KINEMATICAL LIE ALGEBRAS & THEIR GEOMETRIES (MITP, 16/3/2018)

Based on several papers:

1711.05676 (Summary)
1711.06111 (D=3)
1711.07363 (D>3)
1802.04048 (D=2) w/ Tomasz Andrzejewski
180? ... (Spacetimes)
180? ... (Bargmann spacetimes) } w/ Stefan Prohazka

Outline

1. Statement of the problem
2. Methodology
3. Results (summary)

1. Kinematical Lie algebras and their spacetimes

By a kinematical Lie algebra (with D-dimensional space isotropy) we mean a real $\frac{1}{2}(D+1)(D+2)$-dimensional Lie algebra with generators $J_{ab} = -J_{ba}$ $1 \leq a,b \leq D$, B_a, P_a, H

$\operatorname{span}\{J_{ab}\} \cong \mathfrak{so}(D)$

B_a, P_a are $\mathfrak{so}(D)$-vectors

H is an $\mathfrak{so}(D)$-scalar

$\Rightarrow$

$$[J_{ab}, J_{cd}] = \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc}$$
$$[J_{ab}, B_c] = \delta_{bc} B_a - \delta_{ac} B_b$$
$$[J_{ab}, P_c] = \delta_{bc} P_a - \delta_{ac} P_b$$
$$[J_{ab}, H] = 0$$

Examples

- static algebra
- Galilean algebra $[B_a, H] = P_a$
- Poincaré algebra $[B_a, H] = P_a$ $[B_a, P_b] = \delta_{ab} H$ $[B_a, B_b] = -J_{ab}$
- AdS algebra $[B_a, H] = P_a$ $[B_a, P_b] = \delta_{ab} H$ $[B_a, B_b] = -J_{ab}$ $[H, P_a] = B_a$ $[P_a, P_b] = -J_{ab}$
- Carroll algebra $[B_a, P_b] = \delta_{ab} H$

These Lie algebras admit geometrical interpretations as "relativity algebras" on homogeneous spaces of dimension $D+1$.
A homogeneous space is a manifold M on which a Lie group G acts transitively (& smoothly). Fixing an "origin" $p \in M$, the subgroup $H \subset G$ which stabilises p is a closed subgroup and $M \cong G/H$, the space of right cosets of H in G. We can describe M infinitesimally by the Lie pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g} = \operatorname{Lie}(G)$ and $\mathfrak{h} = \operatorname{Lie}(H)$.

For $\mathfrak{g}$ a kinematical Lie algebra to admit a "physical" geometrical realisation, the subalgebra $\mathfrak{h}$ must take a particular form:
$\mathfrak{h}$ is spanned by J_{ab} and some linear combination $\alpha B_a + \beta P_a$. As a vector space, $\mathfrak{h} = \mathfrak{so}(D) \oplus W$ where W is a vectorial subspace under $\mathfrak{so}(D)$. In the above examples we can take $\mathfrak{h} = \mathfrak{so}(D) \oplus \operatorname{span}\{B_a\}$ and the corresponding spaces are, respectively, $\mathbb{A}^{D+1}$ (static), $\mathbb{A}^{D+1} \to \mathbb{A}$ (Newtonian), Minkowski spacetime, AdS spacetime, Carroll spacetime

Problem Classify Lie pairs $(\mathfrak{g},\mathfrak{h})$ where $\mathfrak{g}$ is a kinematical Lie algebra and $\mathfrak{h}$ is an admissible subalgebra, up to natural equivalence:
$$(\mathfrak{g}_1,\mathfrak{h}_1) \cong (\mathfrak{g}_2,\mathfrak{h}_2) \quad \text{if} \quad \varphi: \mathfrak{g}_1 \xrightarrow{\cong} \mathfrak{g}_2 \text{ is a LA isomorphism taking } \mathfrak{h}_1 \xrightarrow{\cong} \mathfrak{h}_2 .$$

This problem can be broken down into two:
1. classify KLAs up to isomorphism
2. For each KLA iso class $[\mathfrak{g}]$, determine Lie pairs $(\mathfrak{g},\mathfrak{h})$ up to $(\mathfrak{g},\mathfrak{h}_1) \cong (\mathfrak{g},\mathfrak{h}_2)$ for $\varphi \in \mathrm{Aut}(\mathfrak{g})$ taking $\mathfrak{h}_1 \xrightarrow{\cong} \mathfrak{h}_2$.

Prior art for 1

$D=0$ no rotations, $\mathfrak{g}$ one-dimensional

$D=1$ no rotations, any 3d RLA is kinematical $\Rightarrow$ Bianchi classification (1898)

$D=3$ Bacry + Lévy-Leblond (1968), Bacry + Nuyts (1986) $\leftarrow$ also pointed out existence of $(\mathfrak{g},\mathfrak{h})$ for every KLA in $D=3$

2. Methodology

Fact: every KLA is a deformation of the static KLA. This is the converse statement to the fact that any KLA contracts to the static KLA.

A real LA is a pair (V,μ) consisting of a real vector space V and $\mu: \Lambda^2 V \to V$ subject to the Jacobi identity.
If $\varphi \in GL(V)$, then $(V, \varphi^*\mu)$ is again a Lie algebra, where $(\varphi^*\mu)(X,Y) := \varphi\,\mu(\varphi^{-1}X, \varphi^{-1}Y)$. So $GL(V)$ acts on the space $\mathcal{L} \subset \Lambda^2 V^* \otimes V$ of Lie algebras on V. A Lie algebra (V,μ_∞) is a contraction of a Lie algebra (V,μ) if μ_∞ is in the closure of the $GL(V)$-orbit of μ.

Define $A^p := \Lambda^{p+1} V^* \otimes V$. Lie brackets on V belong to a variety $\mathcal{L} \subset A^1$. On $A^\bullet = \bigoplus_{p \geq -1}^{\dim V - 1} A^p$ there is a (graded) Lie superalgebra structure $[\![-,-]\!]: A^p \times A^q \longrightarrow A^{p+q}$. $\mu \in A^1$ defines a Lie bracket on V iff $[\![\mu,\mu]\!] = 0$.

Let (V,μ_0) be a Lie algebra on V. If (V,μ) is another Lie algebra on V, then $\varphi := \mu - \mu_0$ obeys the Maurer-Cartan equation $\partial\varphi = \frac{1}{2}[\![\varphi,\varphi]\!]$, where $\partial: A^p \longrightarrow A^{p+1}$ is defined by $\partial\varphi := -[\![\mu_0,\varphi]\!]$.
It follows from $[\![\mu_0,\mu_0]\!] = 0$ & the Jacobi identity of $[\![,]\!]$, that $\partial^2 = 0$. In fact, $H^p_\partial \cong H^{p+1}(\mathfrak{g}_0;\mathfrak{g}_0)$ where $\mathfrak{g}_0 = (V,\mu_0)$.
Deformation theory is "perturbation theory" for the MC equation:

H^1_∂: infinitesimal deformations $\quad H^2_\partial$: obstructions to integrating infinitesimal deformations.

This gives rise to the classification of KLAs for all D. There are "generic" KLAs which exist for all D, but there there are KLAs unique to $D=3$ and $D=2$. In $D=3$ they owe their existence to the $\mathfrak{so}(3)$-invariant vector product $\Lambda^2\mathbb{R}^3 \to \mathbb{R}^3$ which can give rise to brackets of the form $[B_a, B_b] = \epsilon_{abc} B_c$. In $D=2$ they are due to the $\mathfrak{so}(2)$-invariant symplectic structure $\Lambda^2\mathbb{R}^2 \to \mathbb{R}$ which can give rise to brackets of the form $[B_a, B_b] = \epsilon_{ab} H$.

The galilean algebra is the symmetry algebra of free Newtonian motion. It admits a universal central extension, which is the symmetry algebra of the free Schrödinger equation: the Bargmann algebra. It is generated by J_{ab}, B_a, P_a, H and Z, with the usual galilean brackets and, in addition, $[B_a, P_b] = \delta_{ab} Z$.

The Bargmann algebra is a deformation of the universal central extension of the static KLA, for $D \geq 3$. Indeed, for $D \geq 3$, the static KLA admits a 1d central extension with the above bracket: $[B_a, P_b] = \delta_{ab} Z$.
For $D=2$, there is a 4-parameter family of (kinematical) central extensions and for $D=1$ there is a 3-parameter family. The classification of deformations of the centrally-extended static KLA (and of the corresponding homogeneous spaces) has also been achieved for $D \geq 3$, using the same method. There are no special deformations in $D=3$ and, in fact, the results are uniform for all $D \geq 3$. There are 3 kinds of deformations:

1) nontrivial central extension of KLAs: static, Newton, galilean
2) trivial central extensions of KLAs: $\mathfrak{e}$, $\mathfrak{p}$, $\mathfrak{so}(D+1,1)$, $\mathfrak{so}(D,2)$, $\mathfrak{so}(D+2)$
3) noncentral extensions of KLAs.

We may now classify Lie pairs $(\mathfrak{g}, \mathfrak{h})$ with either ① $\mathfrak{g}$ KLA, $\mathfrak{h} = \mathrm{span}\{J, \alpha B + \beta P\}$
or ② $\mathfrak{g}$ gen Bargmann, $\mathfrak{h} = \mathrm{span}\{J, \alpha B + \beta P, \sigma H + \tau Z\}$
mod equivalence.

For each Lie pair $(\mathfrak{g}, \mathfrak{h})$ we can determine the invariants in the linear isotropy representation $\mathfrak{g}/\mathfrak{h}$ and its tensors.
For applications to riemannian and/or NC geometries, we look at the invariants in $\underbrace{\mathfrak{g}/\mathfrak{h}}_{\text{vectors}}$, $\underbrace{(\mathfrak{g}/\mathfrak{h})^*}_{\text{one-forms}}$, $\underbrace{S^2(\mathfrak{g}/\mathfrak{h})}_{\text{sbfs on one-forms}}$, $\underbrace{S^2(\mathfrak{g}/\mathfrak{h})^*}_{\text{sbfs on vectors}}$, ...

$\mathfrak{h}$-invariance $\Rightarrow$ rotational invariance, so possible invariants are:

$$\eta \in (\mathfrak{g}/\mathfrak{h})^* \qquad H \in \mathfrak{g}/\mathfrak{h} \qquad \eta^2, \pi^2 \in S^2(\mathfrak{g}/\mathfrak{h})^* \qquad H^2, P^2 \in S^2(\mathfrak{g}/\mathfrak{h})$$

NC requires η, P^2 !

3. Summary of results

Classification of kinematical Lie algebras:

TABLE 1. Notation for Lie algebras

Notation	Name	Notation	Name
$\mathfrak{a}$	abelian	$\mathfrak{p}$	Poincaré
$\mathfrak{s}$	static	$\mathfrak{so}$	orthogonal
$\mathfrak{n}_+$	(euclidean) Newton	$\mathfrak{co}$	orthogonal + dilatation
$\mathfrak{n}_-$	(lorentzian) Newton	$\mathfrak{g}$	galilean
$\mathfrak{e}$	euclidean	$\mathfrak{c}$	Carroll

TABLE 2. Kinematical Lie algebras in $D = 1$

Bianchi	Nonzero Lie brackets			Comments	Metric?
I				$\mathfrak{a}\ (\cong \mathfrak{s})$	✓
II	$[H, B] = P$			$\mathfrak{g}\ (\cong \mathfrak{c})$	
III		$[H, P] = P$			
IV	$[H, B] = B + P$	$[H, P] = P$			
V	$[H, B] = B$	$[H, P] = P$			
VI_0	$[H, B] = -B$	$[H, P] = P$		$\mathfrak{n}_-\ (\cong \mathfrak{p})$	
VI_γ	$[H, B] = \gamma B$	$[H, P] = P$		$0 \neq \gamma \in (-1, 1)$	
VII_0	$[H, B] = P$	$[H, P] = -B$		$\mathfrak{n}_+\ (\cong \mathfrak{e})$	
VII_α	$[H, B] = \alpha B + P$	$[H, P] = \alpha P - B$		$\alpha > 0$	
VIII	$[H, B] = P$	$[H, P] = -B$	$[B, P] = -H$	$\mathfrak{so}(1, 2)$	✓
IX	$[H, B] = P$	$[H, P] = -B$	$[B, P] = H$	$\mathfrak{so}(3)$	✓

Table 3. Kinematical Lie algebras in $D = 2$ (complex form)

Nonzero Lie brackets					Comments	Metric?
					$\mathfrak{s}$	
$[H, \mathbf{B}] = \mathbf{P}$					$\mathfrak{g}$	
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$				$\mathfrak{n}_-$	
$[H, \mathbf{B}] = i\mathbf{B}$					$\mathfrak{n}_+$	
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = (\lambda + i\theta)\mathbf{P}$				$\lambda \in (-1, 1]$, $\theta \in \mathbb{R}$	
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = \mathbf{B} + \mathbf{P}$					
			$[\mathbf{B}, \bar{\mathbf{P}}] = H$		$\mathfrak{c}$	✓
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$		$[\mathbf{B}, \bar{\mathbf{P}}] = 2(R - iH)$		$\mathfrak{so}(3, 1)$	✓
	$[H, \mathbf{P}] = \mathbf{B}$		$[\mathbf{B}, \bar{\mathbf{P}}] = -2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = -2iR$	$\mathfrak{e}$	✓
	$[H, \mathbf{P}] = -\mathbf{B}$		$[\mathbf{B}, \bar{\mathbf{P}}] = -2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = 2iR$	$\mathfrak{p}$	✓
$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \bar{\mathbf{B}}] = -2iR$	$[\mathbf{B}, \bar{\mathbf{P}}] = -2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = -2iR$	$\mathfrak{so}(4)$	✓
$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \bar{\mathbf{B}}] = 2iR$	$[\mathbf{B}, \bar{\mathbf{P}}] = 2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = 2iR$	$\mathfrak{so}(2, 2)$	✓
		$[\mathbf{B}, \bar{\mathbf{B}}] = iH$		$[\mathbf{P}, \bar{\mathbf{P}}] = iH$		✓
$[H, \mathbf{B}] = i\mathbf{B}$		$[\mathbf{B}, \bar{\mathbf{B}}] = iH$		$[\mathbf{P}, \bar{\mathbf{P}}] = i(H + R)$		✓
		$[\mathbf{B}, \bar{\mathbf{B}}] = iH$				
$[H, \mathbf{B}] = \mathbf{P}$		$[\mathbf{B}, \bar{\mathbf{B}}] = iH$				
$[H, \mathbf{B}] = \pm i\mathbf{B}$		$[\mathbf{B}, \bar{\mathbf{B}}] = iH$				

Chern-Simons theories

Table 4. Kinematical Lie algebras in $D = 3$

Nonzero Lie brackets					Comments	Metric?
					$\mathfrak{s}$	
$[H, \mathbf{B}] = -\mathbf{P}$					$\mathfrak{g}$	
$[H, \mathbf{B}] = -\mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				$\mathfrak{n}_-$	
$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$				$\mathfrak{n}_+$	
$[H, \mathbf{B}] = \gamma\mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				$\gamma \in (-1, 1)$	
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$					
$[H, \mathbf{B}] = \alpha\mathbf{B} + \mathbf{P}$	$[H, \mathbf{P}] = \alpha\mathbf{P} - \mathbf{B}$				$\alpha > 0$	
$[H, \mathbf{B}] = \mathbf{B} + \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{P}$					
		$[\mathbf{B}, \mathbf{P}] = H$			$\mathfrak{c}$	
$[H, \mathbf{B}] = \mathbf{P}$		$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$		$\mathfrak{e}$	
$[H, \mathbf{B}] = -\mathbf{P}$		$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$		$\mathfrak{p}$	
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$	$[\mathbf{B}, \mathbf{P}] = H - \mathbf{R}$			$\mathfrak{so}(4,1)$	✓
$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{R}$	$\mathfrak{so}(5)$	✓
$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = -\mathbf{R}$	$\mathfrak{so}(3,2)$	✓
			$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{B} - \mathbf{R}$		✓
			$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{R} - \mathbf{B}$		✓
			$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$			✓
			$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$			✓
	$[H, \mathbf{P}] = \mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$			
$[H, \mathbf{B}] = -\mathbf{P}$			$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$			
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = 2\mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$			

TABLE 5. Kinematical Lie algebras in $D \geqslant 4$

Nonzero Lie brackets					Comments	Metric?
					$\mathfrak{s}$	
$[H, \mathbf{B}] = \mathbf{P}$					$\mathfrak{g}$	
$[H, \mathbf{B}] = -\mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				$\mathfrak{n}_-$	
$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$				$\mathfrak{n}_+$	
$[H, \mathbf{B}] = \gamma\mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				$\gamma \in (-1, 1]$	
$[H, \mathbf{B}] = \alpha\mathbf{B} + \mathbf{P}$	$[H, \mathbf{P}] = \alpha\mathbf{P} - \mathbf{B}$				$\alpha > 0$	
$[H, \mathbf{B}] = \mathbf{B} + \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{P}$					
		$[\mathbf{B}, \mathbf{P}] = H$			$\mathfrak{c}$	
$[H, \mathbf{B}] = \mathbf{P}$		$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$		$\mathfrak{e}$	
$[H, \mathbf{B}] = -\mathbf{P}$		$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$		$\mathfrak{p}$	
$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$	$[\mathbf{B}, \mathbf{P}] = H + \mathbf{R}$			$\mathfrak{so}(D+1, 1)$	✓
$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = \mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{R}$	$\mathfrak{so}(D+2)$	✓
$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \mathbf{P}] = H$	$[\mathbf{B}, \mathbf{B}] = -\mathbf{R}$	$[\mathbf{P}, \mathbf{P}] = -\mathbf{R}$	$\mathfrak{so}(D, 2)$	✓

TABLE 6. Deformations of $\hat{\mathfrak{s}}$ in $D \geqslant 3$

Nonzero Lie brackets						Comments	Metric?
$[\mathbf{B},\mathbf{P}] = Z$						$\hat{\mathfrak{s}}$	
$[\mathbf{B},\mathbf{P}] = Z$	$[H,\mathbf{B}] = \mathbf{B}$	$[H,\mathbf{P}] = -\mathbf{P}$				$\hat{\mathfrak{n}}_-$	
$[\mathbf{B},\mathbf{P}] = Z$	$[H,\mathbf{B}] = \mathbf{P}$	$[H,\mathbf{P}] = -\mathbf{B}$				$\hat{\mathfrak{n}}_+$	
$[\mathbf{B},\mathbf{P}] = Z$	$[H,\mathbf{B}] = -\mathbf{P}$					$\hat{\mathfrak{g}}$	
$[\mathbf{B},\mathbf{P}] = H$	$[H,\mathbf{B}] = \mathbf{P}$			$[\mathbf{B},\mathbf{B}] = \mathbf{R}$		$\mathfrak{e} \oplus \mathbb{R}$	
$[\mathbf{B},\mathbf{P}] = H$	$[H,\mathbf{B}] = -\mathbf{P}$			$[\mathbf{B},\mathbf{B}] = -\mathbf{R}$		$\mathfrak{p} \oplus \mathbb{R}$	
$[\mathbf{B},\mathbf{P}] = H + \mathbf{R}$	$[H,\mathbf{B}] = \mathbf{B}$	$[H,\mathbf{P}] = -\mathbf{P}$				$\mathfrak{so}(D+1,1) \oplus \mathbb{R}$	✓
$[\mathbf{B},\mathbf{P}] = H$	$[H,\mathbf{B}] = \mathbf{P}$	$[H,\mathbf{P}] = -\mathbf{B}$		$[\mathbf{B},\mathbf{B}] = \mathbf{R}$	$[\mathbf{P},\mathbf{P}] = \mathbf{R}$	$\mathfrak{so}(D+2) \oplus \mathbb{R}$	✓
$[\mathbf{B},\mathbf{P}] = H$	$[H,\mathbf{B}] = -\mathbf{P}$	$[H,\mathbf{P}] = \mathbf{B}$		$[\mathbf{B},\mathbf{B}] = -\mathbf{R}$	$[\mathbf{P},\mathbf{P}] = -\mathbf{R}$	$\mathfrak{so}(D,2) \oplus \mathbb{R}$	✓
$[\mathbf{B},\mathbf{P}] = Z$	$[H,\mathbf{B}] = \gamma\mathbf{B}$	$[H,\mathbf{P}] = \mathbf{P}$	$[H,Z] = (\gamma+1)Z$			$\gamma \in (-1,1]$	
$[\mathbf{B},\mathbf{P}] = Z$	$[H,\mathbf{B}] = \mathbf{B}$	$[H,\mathbf{P}] = \mathbf{B} + \mathbf{P}$	$[H,Z] = 2Z$				
$[\mathbf{B},\mathbf{P}] = Z$	$[H,\mathbf{B}] = \alpha\mathbf{B} + \mathbf{P}$	$[H,\mathbf{P}] = -\mathbf{B} + \alpha\mathbf{P}$	$[H,Z] = 2\alpha Z$			$\alpha > 0$	
$[\mathbf{B},\mathbf{P}] = Z$	$[Z,\mathbf{B}] = \mathbf{P}$	$[H,\mathbf{P}] = \mathbf{P}$	$[H,Z] = Z$	$[\mathbf{B},\mathbf{B}] = \mathbf{R}$		$\mathfrak{co}(D+1) \ltimes \mathbb{R}^{D+1}$	
$[\mathbf{B},\mathbf{P}] = Z$	$[Z,\mathbf{B}] = -\mathbf{P}$	$[H,\mathbf{P}] = \mathbf{P}$	$[H,Z] = Z$	$[\mathbf{B},\mathbf{B}] = -\mathbf{R}$		$\mathfrak{co}(D,1) \ltimes \mathbb{R}^{D,1}$	

Classification of homogeneous "spacetimes":

Table 6. Spacetimes for kinematical Lie algebras ($D \geqslant 3$)

ST#	LA#	Nonzero Lie brackets in addition to $[\mathbf{J},\mathbf{J}]=\mathbf{J}$, $[\mathbf{J},\mathbf{B}]=\mathbf{B}$, $[\mathbf{J},\mathbf{P}]=\mathbf{P}$					Comments
1	1						static
2	2	$[H,\mathbf{B}]=\mathbf{P}$					galilean
3	2		$[H,\mathbf{P}]=\mathbf{B}$				para-galilean
4_γ	3_γ	$[H,\mathbf{B}]=\gamma\mathbf{B}$	$[H,\mathbf{P}]=\mathbf{P}$				$\gamma\in(-1,1)$
5_γ	3_γ	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=\gamma\mathbf{P}$				$\gamma\in(-1,1)$
6_γ	3_γ	$[H,\mathbf{B}]=\mathbf{P}$	$[H,\mathbf{P}]=-\gamma\mathbf{B}+(1+\gamma)\mathbf{P}$				$\gamma\in(-1,1)$
7	4	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=\mathbf{P}$				
8	5	$[H,\mathbf{B}]=-\mathbf{B}$	$[H,\mathbf{P}]=\mathbf{P}$				
9	5	$[H,\mathbf{B}]=\mathbf{P}$	$[H,\mathbf{P}]=\mathbf{B}$				dS galilean
NR 10	6	$[H,\mathbf{B}]=\mathbf{B}+\mathbf{P}$	$[H,\mathbf{P}]=\mathbf{P}$				
11	6	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=\mathbf{B}+\mathbf{P}$				
NR 12_θ	7_θ	$[H,\mathbf{B}]=\theta\mathbf{B}+\mathbf{P}$	$[H,\mathbf{P}]=\theta\mathbf{P}-\mathbf{B}$				$\theta>0$
13	8	$[H,\mathbf{B}]=\mathbf{P}$	$[H,\mathbf{P}]=-\mathbf{B}$				adS galilean
14	9				$[\mathbf{B},\mathbf{P}]=H$		Carroll
15_{+1}	10_{+1}	$[H,\mathbf{B}]=\mathbf{P}$		$[\mathbf{B},\mathbf{B}]=\mathbf{J}$	$[\mathbf{B},\mathbf{P}]=H$		euclidean
15_{-1}	10_{-1}	$[H,\mathbf{B}]=-\mathbf{P}$		$[\mathbf{B},\mathbf{B}]=-\mathbf{J}$	$[\mathbf{B},\mathbf{P}]=H$		Poincaré
16_{+1}	10_{+1}		$[H,\mathbf{P}]=-\mathbf{B}$		$[\mathbf{B},\mathbf{P}]=H$	$[\mathbf{P},\mathbf{P}]=\mathbf{J}$	para-euclidean
16_{-1}	10_{-1}		$[H,\mathbf{P}]=\mathbf{B}$		$[\mathbf{B},\mathbf{P}]=H$	$[\mathbf{P},\mathbf{P}]=-\mathbf{J}$	para-Poincaré
17	11	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=-\mathbf{P}$		$[\mathbf{B},\mathbf{P}]=H-\mathbf{J}$		
18_{+1}	11	$[H,\mathbf{B}]=\mathbf{P}$	$[H,\mathbf{P}]=\mathbf{B}$	$[\mathbf{B},\mathbf{B}]=\mathbf{J}$	$[\mathbf{B},\mathbf{P}]=H$	$[\mathbf{P},\mathbf{P}]=-\mathbf{J}$	hyperbolic
18_{-1}	11	$[H,\mathbf{B}]=-\mathbf{P}$	$[H,\mathbf{P}]=-\mathbf{B}$	$[\mathbf{B},\mathbf{B}]=-\mathbf{J}$	$[\mathbf{B},\mathbf{P}]=H$	$[\mathbf{P},\mathbf{P}]=\mathbf{J}$	de Sitter
19_{+1}	12_{+1}	$[H,\mathbf{B}]=\mathbf{P}$	$[H,\mathbf{P}]=-\mathbf{B}$	$[\mathbf{B},\mathbf{B}]=\mathbf{J}$	$[\mathbf{B},\mathbf{P}]=H$	$[\mathbf{P},\mathbf{P}]=\mathbf{J}$	sphere
19_{-1}	12_{-1}	$[H,\mathbf{B}]=-\mathbf{P}$	$[H,\mathbf{P}]=\mathbf{B}$	$[\mathbf{B},\mathbf{B}]=-\mathbf{J}$	$[\mathbf{B},\mathbf{P}]=H$	$[\mathbf{P},\mathbf{P}]=-\mathbf{J}$	anti de Sitter

= boosts act trivially   = Newton-Cartan   NR = non-reductive

Table 7. Spacetimes for kinematical Lie algebras unique to $D = 3$

ST#	LA#	Nonzero Lie brackets in addition to $[\mathbf{J},\mathbf{J}] = \mathbf{J}$, $[\mathbf{J},\mathbf{B}] = \mathbf{B}$, $[\mathbf{J},\mathbf{P}] = \mathbf{P}$				Comments
20_ε	13_ε			$[\mathbf{B},\mathbf{B}] = \mathbf{B}$	$[\mathbf{P},\mathbf{P}] = \varepsilon(\mathbf{B} - \mathbf{J})$	$\varepsilon = \pm 1$
21	14			$[\mathbf{B},\mathbf{B}] = \mathbf{B}$		
22	14				$[\mathbf{P},\mathbf{P}] = \mathbf{P}$	
23	15				$[\mathbf{P},\mathbf{P}] = \mathbf{B}$	
24	16		$[H,\mathbf{P}] = \mathbf{P}$	$[\mathbf{B},\mathbf{B}] = \mathbf{B}$		
25	16	$[H,\mathbf{B}] = \mathbf{P}$			$[\mathbf{P},\mathbf{P}] = \mathbf{P}$	
26	17		$[H,\mathbf{P}] = -\mathbf{B}$		$[\mathbf{P},\mathbf{P}] = \mathbf{B}$	
27	18	$[H,\mathbf{B}] = 2\mathbf{B}$	$[H,\mathbf{P}] = \mathbf{P}$		$[\mathbf{P},\mathbf{P}] = \mathbf{B}$	

TABLE 8. Spacetimes for kinematical Lie algebras ($D = 2$)

ST#	LA#	Nonzero Lie brackets in addition to $[J, \mathbf{B}] = \mathbf{B}$, $[J, \mathbf{P}] = \mathbf{P}$					Comments
28	19						static
29	20	$[H, \mathbf{B}] = -\mathbf{P}$					galilean
30	20	$[H, \mathbf{P}] = -\mathbf{B}$					para-galilean
31	21	$[H, \mathbf{B}] = \mathbf{B} + \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{P}$				
32	21	$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = \mathbf{B} + \mathbf{P}$				
33	22_1	$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				
34	23	$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$				
35	23	$[H, \mathbf{B}] = -\mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				
36	23	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$				
$37_{\lambda+i\theta}$	$22_{\lambda+i\theta}$	$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = (\lambda + i\theta)\mathbf{P}$				$\lambda + i\theta \neq \pm 1$
$38_{\lambda+i\theta}$	$22_{\lambda+i\theta}$	$[H, \mathbf{B}] = (\lambda + i\theta)\mathbf{B}$	$[H, \mathbf{P}] = \mathbf{P}$				$\lambda + i\theta \neq \pm 1$
$39_{\lambda+i\theta}$	$22_{\lambda+i\theta}$	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{P} - (\lambda + i\theta)\mathbf{B}$				$\lambda + i\theta \neq \pm 1$
40	24	$[H, \mathbf{B}] = i\mathbf{B}$					
41	24	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = i\mathbf{P}$				
42	25				$[\mathbf{B}, \bar{\mathbf{P}}] = H$		Carroll
43	26_{+1}		$[H, \mathbf{P}] = \mathbf{B}$		$[\mathbf{B}, \bar{\mathbf{P}}] = 2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = 2iJ$	euclidean
44	26_{-1}		$[H, \mathbf{P}] = -\mathbf{B}$		$[\mathbf{B}, \bar{\mathbf{P}}] = 2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = -2iJ$	Poincaré
45	26_{+1}	$[H, \mathbf{B}] = -\mathbf{P}$		$[\mathbf{B}, \bar{\mathbf{B}}] = 2iJ$	$[\mathbf{B}, \bar{\mathbf{P}}] = 2H$		para-euclidean
46	26_{-1}	$[H, \mathbf{B}] = \mathbf{P}$		$[\mathbf{B}, \bar{\mathbf{B}}] = -2iJ$	$[\mathbf{B}, \bar{\mathbf{P}}] = 2H$		para-Poincaré
47	27	$[H, \mathbf{B}] = \mathbf{B}$	$[H, \mathbf{P}] = -\mathbf{P}$		$[\mathbf{B}, \bar{\mathbf{P}}] = 2(J - iH)$		
48	27	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \bar{\mathbf{B}}] = -iJ$	$[\mathbf{B}, \bar{\mathbf{P}}] = H$	$[\mathbf{P}, \bar{\mathbf{P}}] = iJ$	hyperbolic
49	27	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \bar{\mathbf{B}}] = iJ$	$[\mathbf{B}, \bar{\mathbf{P}}] = H$	$[\mathbf{P}, \bar{\mathbf{P}}] = -iJ$	de Sitter
50	28_{+1}	$[H, \mathbf{B}] = -\mathbf{P}$	$[H, \mathbf{P}] = \mathbf{B}$	$[\mathbf{B}, \bar{\mathbf{B}}] = 2iJ$	$[\mathbf{B}, \bar{\mathbf{P}}] = 2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = 2iJ$	sphere
51	28_{-1}	$[H, \mathbf{B}] = \mathbf{P}$	$[H, \mathbf{P}] = -\mathbf{B}$	$[\mathbf{B}, \bar{\mathbf{B}}] = -2iJ$	$[\mathbf{B}, \bar{\mathbf{P}}] = 2H$	$[\mathbf{P}, \bar{\mathbf{P}}] = -2iJ$	anti de Sitter
52	31					$[\mathbf{P}, \bar{\mathbf{P}}] = iH$	
53	32		$[H, \mathbf{P}] = \mathbf{B}$			$[\mathbf{P}, \bar{\mathbf{P}}] = iH$	
54	33		$[H, \mathbf{P}] = i\mathbf{P}$			$[\mathbf{P}, \bar{\mathbf{P}}] = iH$	
55	33		$[H, \mathbf{P}] = -i\mathbf{P}$			$[\mathbf{P}, \bar{\mathbf{P}}] = iH$	

Table 9. Spacetimes for kinematical Lie algebras ($D = 1$)

#	Bianchi	Nonzero Lie brackets			Comments
56	I				static
57	II			$[P, H] = B$	para-galilean
58	II	$[B, H] = P$			galilean
59	IV	$[B, H] = B$		$[P, H] = B + P$	
60	IV	$[B, H] = P$		$[P, H] = 2P - B$	
61	V	$[B, H] = B$		$[P, H] = P$	
62	VI_0	$[B, H] = -B$		$[P, H] = P$	
63	VI_0	$[B, H] = P$		$[P, H] = B$	
64	$VI_{c>0}$	$[B, H] = (c-1)B$		$[P, H] = (1+c)P$	
65	$VI_{c>0}$	$[B, H] = (c+1)B$		$[P, H] = (c-1)P$	
66	$VI_{c>0}$	$[B, H] = cB + P$		$[P, H] = cP + B$	
67	VII_0	$[B, H] = -P$	$[B, P] = H$		euclidean
68	VII_0	$[B, H] = -P$		$[P, H] = B$	
69	$VII_{c>0}$	$[B, H] = cB - P$		$[P, H] = B + cP$	
70	VIII	$[B, H] = -P$	$[B, P] = -H$	$[P, H] = B$	(anti) de Sitter
71	VIII	$[B, H] = -P$	$[B, P] = H$	$[P, H] = -B$	hyperbolic
72	IX	$[B, H] = -P$	$[B, P] = H$	$[P, H] = B$	sphere
73	IV	$[B, H] = -P - H$	$[B, P] = -P$		
74	V	$[B, H] = -H$	$[B, P] = -P$		
75	VI_0	$[B, H] = -H$	$[B, P] = P$		Poincaré
76	$VI_{c>0}$	$[B, H] = -(1+c)H$	$[B, P] = (1-c)P$		
77	$VII_{c>0}$	$[B, H] = -P - cH$	$[B, P] = H - cP$		
NR → 78	VIII	$[B, H] = 2P$	$[B, P] = -B$	$[P, H] = -H$	

Table 2. Generalised Bargmann spacetimes ($D \geqslant 3$)

	ST#	LA#	Nonzero Lie brackets in addition to $[\mathbf{J},\mathbf{J}]=\mathbf{J}$, $[\mathbf{J},\mathbf{B}]=\mathbf{B}$, $[\mathbf{J},\mathbf{P}]=\mathbf{P}$					Comments
	1	1	$[\mathbf{B},\mathbf{P}]=Z$					
	2	1	$[\mathbf{B},\mathbf{P}]=H$					
	3	2	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=-\mathbf{P}$			
	4	2	$[\mathbf{B},\mathbf{P}]=H$	$[Z,\mathbf{B}]=\mathbf{B}$	$[Z,\mathbf{P}]=-\mathbf{P}$			
	5	2	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\mathbf{P}$	$[H,\mathbf{P}]=\mathbf{B}$			
NR	6	3	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\mathbf{P}$	$[H,\mathbf{P}]=-\mathbf{B}$			
	7	4	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=-\mathbf{P}$				
	8	4	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{P}]=-\mathbf{B}$				
	9	4	$[\mathbf{B},\mathbf{P}]=H$	$[Z,\mathbf{P}]=\mathbf{B}$				
	10	5	$[\mathbf{B},\mathbf{P}]=H+\mathbf{J}$	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=-\mathbf{P}$			
	11	5	$[\mathbf{B},\mathbf{P}]=Z-H+\mathbf{J}$	$[Z,\mathbf{B}]=\mathbf{B}$	$[Z,\mathbf{P}]=-\mathbf{P}$			
	12_ε	6_ε	$[\mathbf{B},\mathbf{P}]=H$	$[H,\mathbf{B}]=\varepsilon\mathbf{P}$	$[H,\mathbf{P}]=-\varepsilon\mathbf{B}$	$[\mathbf{B},\mathbf{B}]=\varepsilon\mathbf{J}$	$[\mathbf{P},\mathbf{P}]=\varepsilon\mathbf{J}$	$\varepsilon=\pm1$
	13_ε	7_ε	$[\mathbf{B},\mathbf{P}]=H$	$[H,\mathbf{B}]=\varepsilon\mathbf{P}$		$[\mathbf{B},\mathbf{B}]=\varepsilon\mathbf{J}$		$\varepsilon=\pm1$
	14_ε	7_ε	$[\mathbf{B},\mathbf{P}]=-H$		$[H,\mathbf{P}]=\varepsilon\mathbf{B}$		$[\mathbf{P},\mathbf{P}]=\varepsilon\mathbf{J}$	$\varepsilon=\pm1$
	15_ε	7_ε	$[\mathbf{B},\mathbf{P}]=-Z$		$[Z,\mathbf{P}]=\varepsilon\mathbf{B}$		$[\mathbf{P},\mathbf{P}]=\varepsilon\mathbf{J}$	$\varepsilon=\pm1$
NR $\rightarrow$	16_ε	7_ε	$[\mathbf{B},\mathbf{P}]=H-Z$		$[Z,\mathbf{P}]=\varepsilon\mathbf{B}$		$[\mathbf{P},\mathbf{P}]=\varepsilon\mathbf{J}$	$\varepsilon=\pm1$
	17	8	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=\mathbf{P}$	$[H,Z]=2Z$		
	18	8	$[\mathbf{B},\mathbf{P}]=H$	$[Z,\mathbf{B}]=\mathbf{B}$	$[Z,\mathbf{P}]=\mathbf{P}$	$[Z,H]=2H$		
	19_γ	9_γ	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\gamma\mathbf{B}$	$[H,\mathbf{P}]=\mathbf{P}$	$[H,Z]=(1+\gamma)Z$		$-1<\gamma<1$
	20_γ	9_γ	$[\mathbf{B},\mathbf{P}]=H$	$[Z,\mathbf{B}]=\gamma\mathbf{B}$	$[Z,\mathbf{P}]=\mathbf{P}$	$[Z,H]=(1+\gamma)H$		$-1<\gamma<1$
	21_γ	9_γ	$[\mathbf{B},\mathbf{P}]=-Z$	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=\gamma\mathbf{P}$	$[H,Z]=(1+\gamma)Z$		$-1<\gamma<1$
	22_γ	9_γ	$[\mathbf{B},\mathbf{P}]=-H$	$[Z,\mathbf{B}]=\mathbf{B}$	$[Z,\mathbf{P}]=\gamma\mathbf{P}$	$[Z,H]=(1+\gamma)H$		$-1<\gamma<1$
	23	10	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=\mathbf{B}+\mathbf{P}$	$[H,Z]=2Z$		
NR $\rightarrow$	24	10	$[\mathbf{B},\mathbf{P}]=-Z$	$[H,\mathbf{B}]=\mathbf{B}+\mathbf{P}$	$[H,\mathbf{P}]=\mathbf{P}$	$[H,Z]=2Z$		
	25	10	$[\mathbf{B},\mathbf{P}]=H$	$[Z,\mathbf{B}]=\mathbf{B}$	$[Z,\mathbf{P}]=\mathbf{B}+\mathbf{P}$	$[Z,H]=2H$		
NR $\rightarrow$	26_θ	11_θ	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\theta\mathbf{B}+\mathbf{P}$	$[H,\mathbf{P}]=-\mathbf{B}+\theta\mathbf{P}$	$[H,Z]=2\theta Z$		$\theta>0$
	27_ε	12_ε	$[\mathbf{B},\mathbf{P}]=Z$	$[H,\mathbf{B}]=\mathbf{B}$	$[Z,\mathbf{P}]=\varepsilon\mathbf{B}$	$[H,Z]=Z$	$[\mathbf{P},\mathbf{P}]=\varepsilon\mathbf{J}$	$\varepsilon=\pm1$
	28_ε	12_ε	$[\mathbf{B},\mathbf{P}]=H$	$[H,\mathbf{B}]=\varepsilon\mathbf{P}$	$[Z,\mathbf{P}]=\mathbf{P}$	$[Z,H]=H$	$[\mathbf{B},\mathbf{B}]=\varepsilon\mathbf{J}$	$\varepsilon=\pm1$
	29_ε	12_ε	$[\mathbf{B},\mathbf{P}]=-H$	$[Z,\mathbf{B}]=\mathbf{B}$	$[H,\mathbf{P}]=\varepsilon\mathbf{B}$	$[Z,H]=H$	$[\mathbf{P},\mathbf{P}]=\varepsilon\mathbf{J}$	$\varepsilon=\pm1$