Coset Representation of Newton-Cartan Geometry

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Collaborators

Homogeneous Nonrelativistic Geometries as Coset Spaces, [arXiv:1712.03980] with

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Outline

Motivation

- Probes of geometry (geodesics, Killing vectors, etc.)
- Nonrelativistic gravity (e.g. HLG)
- Nonrelativistic holography
- Condensed matter
- Strings

Relativistic Example: Minkowski and (A)dS

$\mathfrak{g}$ = Poincaré, $\mathfrak{h}$ = Lorentz, $\mathfrak{m}$ = translations

$[M_{ab}, M_{cd}] = \eta_{ac} M_{bd} \pm \cdots$ and $[M_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a$

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Associated structure constants: ${f_{\mathfrak{h}\mathfrak{m}}}^{\mathfrak{m}} = {f_{[cd]a}}^{b} = \eta_{ca}\delta^b_d - \eta_{da}\delta^b_c$

Solution to ad-$\mathfrak{h}$ inv. symm. bil. form ${f_{\mathfrak{h}(a}}^{c}\Omega_{b)c} = 0$ is $\Omega_{ab} \propto \eta_{ab}$

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Metric on $\mathcal{M} = \mathcal{G}/\mathcal{H}$ is $ds^2 = \Omega_{ab} e^a e^b = \eta_{\mu\nu} dx^\mu dx^\nu$

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There is also Ω^{ab} such that $\Omega^{c(a} f_{\mathfrak{h}c}{}^{b)} = 0$, but Ω_{ab} and Ω^{ab} are inverses.

Bargmann

cf. 1407.7730 [Brauner, Endlich, Monin, Penco]

$\mathfrak{g} = \text{Barg}$, $\mathfrak{h} = \{J, G_i, N\}$, $\mathfrak{m} = \{H, P_i\}$ and $g = e^{Ht} e^{P_i x^i}$

$\Omega_{ab} e^a e^b = -dt^2$ (degenerate!)

But, contravariant version $\Omega^{c(a} f_{Ic}{}^{b)} = 0$ gives $\Omega^{ab} e_a e_b = \delta^{ij} \partial_i \partial_j$

$$\boxed{\tau = dt, \qquad h^{\mu\nu} \partial_\mu \partial_\nu = \delta^{ij} \partial_i \partial_j, \qquad m = 0.}$$

Loc. tan. space trans.: $g \to gh$ with $h \in \mathcal{H}$

From Relativistic to Non-relativistic

Two derivations of Bargmann coset from relativistic starting point:

- Inönü-Wigner contraction

$$\begin{array}{ccc} \text{Poinc} \times U(1) & \xrightarrow{c\to\infty} & \text{Barg} \\ \Big\downarrow \text{coset} & & \Big\downarrow \text{coset} \\ \Omega_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{l}{c^2} \end{pmatrix} & \xrightarrow{c\to\infty} & \Omega_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

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- Null-reduction ($N \in \mathfrak{m}$, with associated coordinate u)

$$ds^2 = 2\underbrace{dt}_{\tau}(du - \underbrace{dt}_{m}) + \underbrace{\delta_{ij}dx^i dx^j}_{h_{\mu\nu}dx^\mu dx^\nu}$$

Newton-Hooke

Add commutator $[H, P_i] = \Lambda G_i$

$$\boxed{\begin{array}{c} \tau = dt \\ h^{\mu\nu} = \delta^{ij}\delta^{\mu}_i\delta^{\nu}_j \\ m = -\frac{1}{2}\Lambda x^2 dt \end{array}} \implies \begin{array}{c} \lambda^i = -\sqrt{\Lambda}\, x^i \\ \sigma = \frac{1}{2}\sqrt{\Lambda}x^2 \\ t' = t \\ x'^i = e^{-\sqrt{\Lambda}t} x^i \end{array} \implies \boxed{\begin{array}{c} \tau' = dt' \\ h'^{\mu\nu} = \\ e^{-2\sqrt{\Lambda}t'}\delta^{ij}\delta^{\mu}_i\delta^{\nu}_j \\ m' = 0 \end{array}}$$

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Can also derive via contraction or null-reduction.

Schrödinger

cf. 0903.4245 [Schäfer-Nameki, Yamazaki, Yoshida]

$\mathfrak{h} = \{J, G_i, N + 2K\}$ and $\mathfrak{m} = \{N, H, P_i, D\}$

$$ds^2 = 2\frac{dt}{r^2}\left(du - \frac{\alpha}{2}\frac{dt}{r^2}\right) + \frac{dr^2 + dx^2}{r^2} \xrightarrow{\text{n-red}} \boxed{\begin{array}{c} \tau = \frac{dt}{r^2} \\ h^{\mu\nu} = r^2(\delta^\mu_r \delta^\nu_r + \delta^\mu_i \delta^\nu_i) \\ m = \frac{\alpha}{2}\frac{dt}{r^2} \end{array}}$$

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Can get this TTNC via $\mathfrak{h} = \{J, G_i, N, K\}$ and $\mathfrak{m} = \{H - \frac{\alpha}{2}N, P_i, D\}$.

TNC with $\mathbb{R} \times SO(3)$ isometries

$\mathfrak{g} = \underbrace{\mathfrak{so}(3)}_{J_a} \times \underbrace{\mathfrak{u}(1)}_{H} \times \underbrace{\mathfrak{u}(1)}_{N}$ and rescale $J \equiv J_3$ and $P_{i=1,2} = \frac{1}{R} J_i$

$\mathfrak{h} = \{J + aN + bH, N\}$ and $\mathfrak{m} = \{H, P_i\}$

$g = e^{Hv} e^{P_1 \alpha_1} e^{P_2 \alpha_2}$ and redefine $\alpha_1 = R\phi$ and $\alpha_2 = R\left(\eta - \frac{\pi}{2}\right)$.

Ω_{ab} is diagonal and, in general, non-degenerate.

<u>Choose</u> $\text{sign}\Omega_{ab} = (1, 0, 0)$ and $\text{sign}\Omega^{ab} = (0, 1, 1)$.

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$$\boxed{\begin{array}{c} \tau = dv + b \cos\eta \, d\phi \\ h_{\mu\nu} dx^\mu dx^\nu = \\ R^2(d\eta^2 + \sin^2\eta \, d\phi^2) \\ m = a \cos\eta \, d\phi \end{array}}$$

Outlook

Classification problems (algebras to coset spaces)

Quotient out by discrete subgroups (e.g., analog of BTZ)

Solutions of HLG or CS, cf. 1712.05794 [Hartong, Lei, Obers, Oling]

Nonrelativistic dual for Schrödinger field theories (or Lifshitz?)

Entanglement entropy and Ryu-Takayanagi.