D=4 EXTENDED GALILEAN SUSY AND N=4 GALILEAN SUPERPARTICLES WITH CENTRAL CHARGES

1. From general D=4 N-extended Poincaré to d=3 N-extended Galilean superalgebra
2. Example: D=4 N=4 Galilean SUSY with 12+1 central charges
3. N=4 d=3 Galilean superparticle models: action, phase space formulation and first quantization
4. Final remarks

based on

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D = 4 extended Galilei superalgebras with central charges

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abstract

We perform a nonrelativistic contraction of N-extended Poincaré superalgebra with internal symmetry U(N) and general set of N(N − 1) real central charges. We show that for even N = 2k and particular choice of the dependence of Z_{ij} on light velocity c one gets the N-extended Galilei superalgebra with unchanged number of central charges and compact internal symmetry algebra U(k; H) = USp(2k). The Hamiltonian positivity condition is modified only by one central charge. If we put all the central charges equal to zero one gets the 2k-extended Galilei superalgebra as the subalgebra of recently introduced extended Galilei conformal superalgebra (de Azcárraga, Lukierski (2009) [1] and Sakaguchi [2]).

1. Introduction

contraction $c \to \infty$ [7] if we perform the following c-dependent rescaling (P_i, M_i remain unchanged)

From $\mathcal{N}=4$ Galilean superparticle to three-dimensional non-relativistic $\mathcal{N}=4$ superfields

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Abstract

We consider the general $\mathcal{N}=4$, $d=3$ Galilean superalgebra with arbitrary central charges and study its dynamical realizations. Using the nonlinear realization techniques, we introduce a class of actions for $\mathcal{N}=4$ three-dimensional non-relativistic superparticle, such that they are linear in the central charge Maurer-Cartan one-forms. As a prerequisite to the quantization, we analyze the phase space constraints structure of our model for various choices of the central charges. The first class constraints generate gauge transformations, involving fermionic κ-gauge transformations. The quantization of the model gives rise to the collection of free $\mathcal{N}=4$, $d=3$ Galilean superfields, which can be further employed, e.g., for description of three-dimensional non-relativistic $\mathcal{N}=4$ supersymmetric theories.

1. From D=4 N-extended Poincaré to d=3 N-extended Galilean SUSY

N=extended Poincaré superalgebra with central charges generators

$$P_\mu, M_{\mu\nu}, T_A{}^B, A, Z^{AB}, \bar{Z}^{AB}, Q^A_\alpha, \bar{Q}^A_{\dot\alpha}:$$

$P_\mu, M_{\mu\nu}$ ↑ Poincaré; $T_A{}^B, A$ ↖↗ U(N); $Z^{AB}, \bar{Z}^{AB}$ ↖↗ central charges; $Q^A_\alpha, \bar{Q}^A_{\dot\alpha}$ supercharges

$$\left\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\right\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta^A_B$$

$$\left\{Q^A_\alpha, Q^B_\beta\right\} = 2\varepsilon_{\alpha\beta} Z^{AB} = 2\varepsilon_{\alpha\beta}(X^{AB} + iY^{AB})$$

$$\left\{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\right\} = 2\varepsilon_{\dot\alpha\dot\beta} \bar{Z}_{AB} = 2\varepsilon_{\dot\alpha\dot\beta}(X^{AB} - iY^{AB})$$

$\frac{N(N-1)}{2}$ complex $\leftrightarrow$ $N(N-1)$ real central charges.

Covariance relations:

$$\left[M_{\mu\nu}, Q_\alpha{}^A\right] = -\frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^A_\beta \qquad \left[P_\mu, Q^A_\alpha\right] = 0$$

$$\left[T^A_B, Q_\alpha{}^A\right] = \delta^C_B Q^A_\alpha - \frac{1}{N}\delta^A_B Q^C_\alpha \qquad \left[A, Q^A_\alpha\right] = \alpha Q^A_\alpha$$

$$(\alpha = 1 \rightarrow A = U(1))$$

D=4 Poincaré superalgebra → $d=3$ Galilean superalgebra (N=2k even)

i) Bosonic space-time sector

$$M_{\mu\nu} = (M_i, N_i) \qquad P_\mu = (P_i, P_0) \quad \xrightarrow[\;N_i = cB_i,\quad P_0 = m_0 c + \frac{H}{c}\;]{c\to\infty}$$

d=3 Galilean algebra

$$[M_i, B_j] = \varepsilon_{ijk} B_k$$
$$[H, B_i] = -P_i$$
$$[P_i, B_j] = -M\,\delta_{ij}$$
$$M = m_0 \text{ – Bargmann central charge}$$

ii) Internal sector $U(N) = SU(N) \oplus A$ (R-symmetries)

One splits generators of $SU(N)$ into symmetric Riemannian pair

$$h = T^{+A}_{\;\;B} \in USp(4) \qquad k = T^{-A}_{\;\;B} \in \frac{SU(N)}{USp(4)}$$

$$T^{\pm A}_{\;\;B} = \mp \Omega^{AC} T^{\pm D}_{\;\;C} \Omega_{DB} \longrightarrow \begin{array}{l} \Omega^{AB} = -\Omega^{BA} \\ \Omega^{AC}\Omega_{CB} = \delta^A_{\;B} \end{array} \Rightarrow \text{symplectic metric}$$

Contraction after rescaling

$$T^{+A}_{\;\;B} = \mathbb{T}^{+A}_{\;\;B} \qquad T^{-A}_{\;\;B} = c\,\mathbb{T}^{-A}_{\;\;B} \qquad A = c\,\mathbb{A}$$

$$[h', h] \subset h \quad [h, k] \subset k \quad [k, k] \subset h \quad \xrightarrow{c\to\infty} \quad \begin{array}{l} [h, h] \subset h \\ [h, \mathbb{k}] \subset \mathbb{k} \\ [\mathbb{k}, \mathbb{k}] = 0 \end{array}$$

One gets inhomogeneous $USp(N)$ as maximal Galilean R-symmetry

iii) Fermionic sector and central charges

We introduce new symplectic–covariant basis of supercharges

$$Q_{\alpha}^{\pm A} = \frac{1}{\sqrt{2}}\left(Q_{\alpha}^{A} \pm \varepsilon_{\alpha\beta}\Omega^{AB}\bar{Q}_{\dot{\beta}B}\right) \qquad \bar{Q}^{\pm}_{\dot{\alpha}A} = \frac{1}{\sqrt{2}}\left(\bar{Q}_{\dot{\alpha}A} \mp \varepsilon_{\dot{\alpha}\dot{\beta}}\Omega_{AB}Q_{\beta}^{B}\right)$$

Such supercharges satisfy symplectic-Majorana condition

$$(Q_{\alpha}^{\pm A})^{+} \equiv \bar{Q}^{\pm}_{\dot{\alpha}A} = \mp\varepsilon_{\dot{\alpha}\dot{\beta}}\Omega_{AB}Q_{\beta}^{\pm B}$$

Due to this constraint it is enough to consider only holomorphic sector described by $Q_{\alpha}^{\pm A}$ (or antiholomorphic $\bar{Q}_{\dot{\alpha}}^{\pm A}$)

N-extended D=4 Poincaré superalgebra in holomorphic basis

$$\begin{aligned}\left\{Q_{\alpha}^{\pm A}, Q_{\beta}^{\pm B}\right\} &= \pm 2\Omega^{AB}\varepsilon_{\alpha\beta}P_0 + \varepsilon_{\alpha\beta}(Z^{AB} - \Omega^{AC}\bar{Z}_{CD}\Omega^{DB}) \\ &= \pm 2\Omega^{AB}\varepsilon_{\alpha\beta}P_0 + 2\varepsilon_{\alpha\beta}(X_{-}^{AB} + iY_{+}^{AB})\end{aligned}$$

$$\begin{aligned}\left\{Q_{\alpha}^{+A}, Q_{\beta}^{-B}\right\} &= 2\Omega^{AB}(\sigma_i P_i)_{\alpha\beta} + \varepsilon_{\alpha\beta}(Z^{AB} + \Omega^{AC}\bar{Z}_{CD}\Omega^{DB}) \\ &= 2\Omega^{AB}(\sigma_i P_i)_{\alpha\beta} + 2\varepsilon_{\alpha\beta}(X_{+}^{AB} + iY_{-}^{AB})\end{aligned}$$

where

$$\Omega^{AC}X_{\pm}^{CD}\Omega^{DB} = \pm X_{\pm}^{AB} \qquad\qquad \Omega^{AC}Y_{\pm}^{CD}\Omega^{DB} = \pm Y_{\pm}^{AB}$$

Rescalings of supercharges:

$$Q^{+A}_{\alpha} = c^{-\frac{1}{2}}\mathbb{Q}^{A}_{\alpha} \qquad\qquad Q^{-A}_{\alpha} = c^{\frac{1}{2}}\mathbb{S}^{A}_{\alpha}$$

Rescalings of central charges $Z^{AB} = X^{AB} + iY^{AB}$ $\left(X^{AB} = \tilde{X}^{AB} + \Omega^{AB}X\right)$

$$X^{AB}_{-} = -m_0 c\Omega^{AB} + \frac{1}{c}\mathbb{X}^{AB}_{-} \qquad Y^{AB}_{+} = \frac{1}{c}\mathbb{Y}^{AB}_{+}$$

$$X^{AB}_{+} = \mathbb{X}^{AB}_{+} \qquad Y^{AB}_{-} = \mathbb{Y}^{AB}_{-} \qquad \leftarrow \quad \text{not rescaled!}$$

where $\Omega^{AC}X^{CD}_{\pm}\Omega^{DB} = \pm X^{AB}_{\pm}$ $\quad \Omega^{AC}Y^{CD}_{\pm}\Omega^{DB} = \pm Y^{AB}_{\pm}$

Contraction limits in holomorphic basis: d=3 Galilean superalgebra

$$\begin{aligned}
\left\{\mathbb{Q}^{A}_{\alpha}, \mathbb{Q}^{B}_{\beta}\right\} &= \lim_{c\to\infty}\left[c\cdot\{2\Omega^{AB}\varepsilon_{\alpha\beta}(P_0 + X)\} + 2\varepsilon_{\alpha\beta}(\tilde{X}^{AB}_{-} + iY^{AB}_{+})\right] \\
&= 2\Omega^{AB}\varepsilon_{\alpha\beta}(\mathbb{H} + \mathbb{X}) + 2\varepsilon_{\alpha\beta}(\tilde{\mathbb{X}}^{AB}_{-} + i\mathbb{Y}^{AB}_{+}) \\
\left\{\mathbb{Q}^{A}_{\alpha}, \mathbb{S}^{B}_{\beta}\right\} &= 2\Omega^{AB}(\sigma_i P_i)_{\alpha\beta} + 2\varepsilon_{\alpha\beta}(\mathbb{X}^{AB}_{+} + i\mathbb{Y}^{AB}_{-}) \\
\left\{\mathbb{S}^{A}_{\alpha}, \mathbb{S}^{B}_{\beta}\right\} &= \lim_{c\to\infty}\left[\frac{1}{c}2\Omega^{AB}\varepsilon_{\alpha\beta}(-P_0 + X)\right] + 2\varepsilon_{\alpha\beta}(\tilde{X}^{AB}_{-} + iY^{AB}_{+}) = 4m_0\varepsilon_{\alpha\beta}\Omega^{AB}
\end{aligned}$$

Rescalings: $\tilde{X}^{AB}_{-} = \frac{1}{c}\tilde{\mathbb{X}}^{AB}_{-}$ $\quad Y^{AB}_{+} = \frac{1}{c}\mathbb{Y}^{AB}_{+}$; $\quad$ not rescaled $\quad X^{AB}_{+} = \mathbb{X}^{AB}_{+}$ $\quad Y^{AB}_{+} = \mathbb{Y}^{AB}_{+}$

If $c \to \infty$ one obtains identical in form Galilean symplectic Majorana condition (A,B=1,2...N; N=2k)

$$\bar{\mathbb{Q}}_{\dot{\alpha}A} = -\varepsilon_{\dot{\alpha}\dot{\beta}}\Omega_{AB}\mathbb{Q}^B_\beta \qquad \bar{\mathbb{S}}_{\dot{\alpha}A} = \varepsilon_{\dot{\alpha}\dot{\beta}}\Omega_{AB}\mathbb{S}^B_\beta$$

Resolving Galilean symplectic Majorana condition by choosing as independent supercharges

$$(\mathbb{Q}^A_\alpha, \mathbb{S}^A_\alpha) \to (\mathbb{Q}^i_\alpha, \bar{\mathbb{Q}}^i_{\dot{\alpha}i}), (\mathbb{S}^i_\alpha, \bar{\mathbb{S}}^i_{\dot{\alpha}i}) \qquad i = 1, 2, \ldots \tfrac{N}{2} = k \qquad \left(\text{for } \Omega = \begin{pmatrix} 0 & -\mathbb{1}_k \\ \mathbb{1}_k & 0 \end{pmatrix}\right)$$

one passes from holomorphic superalgebra to Hermitean superalgebra

$$\left\{\mathbb{Q}^i_\alpha, \bar{\mathbb{Q}}^j_\beta\right\} = 2\delta^{ij}(\mathbb{H} + \mathbb{X}) + 2\varepsilon_{\alpha\beta}(\tilde{\mathbb{X}}^{ik+j}_- + i\mathbb{Y}^{ik+j}_+)$$

$$\left\{\mathbb{Q}^i_\alpha, \mathbb{Q}^j_\beta\right\} = 2\varepsilon_{\alpha\beta}(\tilde{\mathbb{X}}^{ij}_- + i\mathbb{Y}^{ij}_+)$$

$$\left\{\mathbb{Q}^i_\alpha, \bar{\mathbb{S}}^j_\beta\right\} = 2\delta^{ij}(\sigma_i P_i)_{\alpha\beta} + 2\varepsilon_{\alpha\beta}(\mathbb{X}^{ik+j}_+ + i\mathbb{Y}^{ik+j}_-)$$

$$\left\{\mathbb{Q}^i_\alpha, \mathbb{S}^j_\beta\right\} = 2\varepsilon_{\alpha\beta}(\mathbb{X}^{ij}_+ + i\mathbb{Y}^{ij}_-)$$

$$\left\{\mathbb{S}^i_\alpha, \bar{\mathbb{S}}^j_{\dot{\beta}}\right\} = 4m_0\delta_{\alpha\dot{\beta}}\delta^{ij} \qquad \left\{\mathbb{S}^i_\alpha, \mathbb{S}^j_\beta\right\} = 0$$

Proper basis for describing N-extended Galilean SUSY of QM!

Remarks:

1. One central charge we distinguish, namely

$$Z^{AB} = Z\Omega^{AB} \quad Z = X + iY = -m_0 c + \frac{\mathbb{X}}{c} + i\mathbb{Y} - \text{unique for N=2}$$

We denote by $\tilde{X}^{AB}$ the central charges X^{AB} without X.
The special role of central charge Z is to compensate the term $m_0 c$

2. If only $Z \neq 0$ (one complex central charge), the R-symmetry $SU(N)$ is reduced to $USp(N) \equiv U(\frac{N}{2}; H)$

3. For $N = 2\mathbb{k}$ one can introduce k complex quasi-triangular central charges $Z_1 \ldots Z_k$

$$Z^{AB} = \left(\begin{array}{cc|c|c} 0 & Z_1 & 0 & 0 \\ -Z_1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & \ddots & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & \begin{matrix} 0 & Z_1 \\ -Z_1 & 0 \end{matrix} \end{array}\right) \Rightarrow \begin{array}{c} \text{one gets R-symmetry:} \\ \underbrace{USp(2) \otimes \ldots \otimes USp(2)}_{\text{k times}} \end{array}$$

For $N \geq 4$ also important off-diagonal central charges.

2. Example: d=3 N=4 Galilean SUSY with 12+1 central charges.

a) d=3 N= 1 Galilean superalgebra (Puzalowski 1978)

Hermitean basis

$$\left\{\mathbb{S}_\alpha, \bar{\mathbb{S}}^{\dot\beta}\right\} = 4m_0\delta_\alpha{}^{\dot\beta} \qquad [\mathbb{B}_i, \mathbb{S}_\alpha] = 0$$

$$[J_i, \mathbb{S}_\alpha] = (\sigma_i)_\alpha{}^\beta \mathbb{S}_\beta \qquad \left[J, \bar{\mathbb{S}}^{\dot\alpha}\right] = -\bar{\mathbb{S}}^{\dot\beta}(\sigma_i)_{\dot\beta}{}^{\dot\alpha}$$

b) d=3 N=2 Galilean superalgebra (Bergman, Thorn 1995 etc.)

Hermitean basis

$$\left\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}^{\beta}\right\} = 2\delta_\alpha{}^{\dot\beta}(\mathbb{H} + \mathbb{X})$$

$$\left\{\mathbb{Q}_\alpha, \bar{\mathbb{S}}^{\dot\beta}\right\} = 2(\sigma_i P_i)_\alpha{}^{\dot\beta} P_i + 2i\mathbb{Y}\delta_\alpha{}^{\dot\beta}$$

$$\left\{\mathbb{S}_\alpha, \bar{\mathbb{S}}^{\dot\beta}\right\} = 4m_0\delta_\alpha{}^{\dot\beta}$$

$$[B_i, \mathbb{Q}_\alpha] = (\sigma_i)_\alpha{}^\beta \mathbb{S}_\beta \qquad [B_i, \mathbb{S}_\alpha] = 0$$

Hermitean basis $(\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_\alpha, \mathbb{S}_\alpha, \bar{\mathbb{S}}_{\dot\beta})$ is related with holomorphic one $(\mathbb{Q}^A_\alpha, \mathbb{S}^A_\alpha)$ (A=1,2) by the explicite relations

$$\mathbb{Q}_\alpha = \mathbb{Q}^1_\alpha \quad \bar{\mathbb{Q}}_{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\mathbb{Q}^2_\beta \quad \mathbb{S}_\alpha = \mathbb{S}^1_\alpha \quad \bar{\mathbb{S}}_{\dot\alpha} = \varepsilon_{\alpha\dot\beta}\mathbb{S}^2_\alpha$$

c) d=3 N= 4 Galilean SUSY

The relativistic N=4 central charges matrix, which is denoted as representation of $USp(2)\otimes USp(2) = SU(2)\otimes SU(2) = O(4)$ internal symmetries

$$Z^{AB} = \left(\begin{array}{c|c} \mathbb{Z}_1\varepsilon_{ab} & \mathbb{Z}_{a\tilde{b}} \\ \hline -\mathbb{Z}_{a\tilde{b}} & \mathbb{Z}_2\varepsilon_{\tilde{a}\tilde{b}} \end{array}\right) \qquad \begin{array}{l} A = (a,\tilde{a}) \\ B = (b,\tilde{b}) \end{array}$$

where $a = (1,2)$ and $\tilde{a} = (\tilde{1},\tilde{2})$ describe two $USp(2)\simeq SU(2)$ spinorial indices. The supercharges are

$$\mathbb{Q}^A_\alpha = (\mathbb{Q}^a_\alpha, \mathbb{Q}^{\tilde{a}}_\alpha) \quad \longrightarrow \quad \begin{array}{ll} \mathbb{Q}_\alpha = \mathbb{Q}^1_\alpha & \bar{\mathbb{Q}}_\alpha = -\varepsilon_{\alpha\beta}\mathbb{Q}^{\beta 2} \\ \tilde{\mathbb{Q}}_\alpha = \mathbb{Q}^2_\alpha & \bar{\tilde{\mathbb{Q}}}_\alpha = -\varepsilon_{\alpha\beta}\mathbb{Q}^{\beta\tilde{2}} \end{array}$$

holomorphic basis — Hermitean basis

i) N=4 NR superalgebra in holomorphic basis

$$\left\{Q^a_\alpha, Q^b_\beta\right\} = 2\varepsilon^{ab}\varepsilon_{\alpha\beta}(\mathbb{H}+\mathbb{X}_1)$$
$$\left\{Q^{\tilde{a}}_\alpha, Q^{\tilde{b}}_\beta\right\} = 2\varepsilon^{ab}\varepsilon_{\alpha\beta}(\mathbb{H}+\mathbb{X}_2) \quad \longleftarrow \quad \text{quasidiagonal central charges}$$

$$\left\{Q^a_\alpha, Q^{\tilde{b}}_\beta\right\} = 2\varepsilon_{\alpha\beta}W^{a\tilde{b}} \qquad W^{a\tilde{b}} = \mathbb{X}^{a\tilde{b}}_- + i\mathbb{Y}^{a\tilde{b}}_+ \quad \longleftarrow \quad \text{off-diagonal central charges}$$

$$\left\{Q^a_\alpha, S^b_\beta\right\} = 2\varepsilon^{ab}((\sigma_i)_{\alpha\beta}P_i + i\varepsilon_{\alpha\beta}\mathbb{Y}_1)$$

quasi-diagonal

$$\left\{Q^{\tilde a}_\alpha, S^{\tilde b}_\beta\right\} = 2\varepsilon^{\tilde a\tilde b}((\sigma_i)_{\alpha\beta}P_i + i\varepsilon_{\alpha\beta}\mathbb{Y}_2)$$

off-diagonal

$$\left\{Q^a_\alpha, S^{\tilde b}_\beta\right\} = \left\{\mathbb{S}^a_\alpha, Q^{\tilde b}_\beta\right\} = 2i\varepsilon_{\alpha\beta}V^{a\tilde b} \qquad V^{a\tilde b} = \mathbb{X}^{a\tilde b}_+ + i\mathbb{Y}^{a\tilde b}_-$$

$$\left\{S^a_\alpha, S^b_\beta\right\} = -4m_0\varepsilon^{ab}\varepsilon_{\alpha\beta} \qquad \left\{S^{\tilde a}_\alpha, S^{\tilde b}_\beta\right\} = -4m_0\varepsilon^{\tilde a b}\varepsilon_{\alpha\beta}$$

$$\left\{S^a_\alpha, S^{\tilde b}_\beta\right\} = 0$$

ii) Hermitean basis: $(Q_\alpha, \bar Q_\alpha, \tilde Q_\alpha, \bar{\tilde Q}_\alpha)$

$$\begin{pmatrix} Q_\alpha \equiv Q^1_\alpha & \bar Q_\alpha = -\varepsilon_{\alpha\beta}Q^2_\alpha \\ \tilde Q_\alpha = Q^{\tilde 1}_\alpha & \bar{\tilde Q}_\alpha = -\varepsilon_{\alpha\beta}\tilde Q^{\tilde 2}_\alpha \end{pmatrix}$$

$$\left\{Q_\alpha, \bar Q_{\dot\beta}\right\} = 2\delta_{\alpha\beta}(\mathbb{H} + \mathbb{X}_1)$$

$$\left\{\tilde Q_\alpha, \bar{\tilde Q}_{\dot\beta}\right\} = 2\delta_{\alpha\beta}(\mathbb{H} + \mathbb{X}_2)$$

$$\left\{Q_\alpha, \bar{\tilde Q}_\beta\right\} = -2\delta_{\alpha\beta}W^{1\tilde 2} \qquad \left\{\tilde Q_\alpha, \bar Q_\alpha\right\} = 2\delta_{\alpha\dot\beta}W^{2\tilde 1}$$

$$\left\{Q_\alpha, \tilde Q_\beta\right\} = 2\varepsilon_{\alpha\beta}W^{1\tilde 1} \qquad \left\{\bar Q_\alpha, \bar{\tilde Q}_{\bar\beta}\right\} = 2\varepsilon_{\dot\alpha\dot\beta}W^{2\tilde 2}$$

etc.

General features of d=3 N-extended Galilean SUSY:

i) We derived NR superalgebra which is d=3 NR counterpart of Haag–Łopuszański–Sohnius classification of D=4 Poincaré superalgebras

ii) The central charges $Z^{AB} = X^{AB} + iY^{AB}$ after contraction enter as

$$\{Q,Q\} \simeq \mathbb{H} + \frac{N(N-1)}{2} \quad \text{real central charges} \quad W^{AB} = \mathbb{X}_-^{AB} + i\mathbb{Y}_+^{AB}$$

$$\{Q,S\} \simeq \sigma_i P_i + \frac{N(N-1)}{2} \quad \text{real central charges} \quad V^{AB} = X_+^{AB} + iY_-^{AB}$$

$$\{S,S\} \simeq m_0 \leftarrow \quad \text{additional Bargmann central charge}$$

We have k^2 central charges $\mathbb{X}_-^{AB}(Y_-^{AB})$ and $k(k-1)$ of $X_+^{AB}(Y_+^{AB})$

iii) Special feature of N=4:

We have 2 complex quasi-diagonal central charges Z_1, Z_2 and 4 complex off-diagonal ones described by complex fourvector $Z_A \quad (A = 1, 2, \ldots 4)$

$$X^{a\tilde{b}} = (\sigma_A^E)^{a\tilde{b}} Z_A \qquad \sigma_\mu^E = (\sigma_i, -iI_2) \qquad Z_A = n_A + iv_A$$

where σ_μ^E are D=4 Euclidean σ-matrices and n_A, v_A are two real O(4) internal fourvectors ⇒ new type of KK theory?

3. N=4 d=3 Galilean superparticle models: action, phase space formulation, first quantization

One performs the following steps:

i) Maurer-Cartan (MC) one-form $\omega(g_r)$ for the coset G

$$G = \frac{\mathcal{G}}{H}$$

$\mathcal{G}$ – $N=4$ $d=3$ Galilean supergroup with generators $\hat{g}_r$

$H - USp(4) \times O(3) \times A$ – stability group (standard choice)

One gets

$$G^{-1}dG = i\sum_r \omega(g_r)\cdot \hat{g}_r$$

$\omega(g_r)$ – linear representation of stability group H

$$\hat{g}_r = (\underbrace{H, B_i, P_i, M,}_{\text{Galilei algebra}/O(3)}\ \underbrace{T^{-A}_{\ \ B}}_{SU(4)/USp(4)}, X_1, X_2, \underbrace{Y_1, Y_2, X^{a\tilde{b}}}_{\text{12 central charges}}, Y^{a\tilde{b}}, \underbrace{Q^a_\alpha, Q^{\tilde{a}}_\alpha, S^a_\alpha, S^{\tilde{a}}_\alpha}_{\text{all supercharges}})$$

One gets the model of classical mechanics if all parameters of G are promoted to $d=1$ fields: $t \to t(\tau), k_i \to k_i(\tau), x_i \to x_i(\tau), s \to s(\tau), \ldots$

Important: one can in H-covariant way eliminate some of $d=1$ fields by imposing algebraic inverse Higgs constraints.

Simple example how to describe free Galilean particle with mass m_0:

$$G = \frac{\text{d=3 Galilei group}}{O(3)}$$

8 generators: H, P_i, B_i, M
8 group parameters: t, x_i, k_i, s

MC one-forms:

$$\omega(H) = dt \quad \omega(B_i) = dk_i \quad \omega(P_i) = dx_i + k_i dt$$

$$\omega(M) = ds + k_i \omega(P_i) - \tfrac{1}{2} k^2 dt$$

Inverse Higgs mechanism: $\omega(P_i) = 0 \ \rightarrow \ k_i = -\frac{dx_i}{dt} = -v_i$

Action: $\mathcal{S}_0 = -m_0 \int \omega(M) = -m_0 \int d\tau (\dot{s} - \tfrac{1}{2} k^2 \dot{t})$ $\qquad \dot{a} = \frac{da}{d\tau} \qquad k_i = \frac{dx_i}{d\tau} \cdot \frac{d\tau}{dt}$

$$\frac{dx_i}{dt} = \frac{dx_i}{d\tau}\frac{d\tau}{dt} = \frac{\dot{x}_i}{\dot{t}}$$

$$= \frac{m_0}{2} \int d\tau \left(\frac{\dot{x}_i}{\dot{t}}\right)^2 \cdot \dot{t} = \frac{m_0}{2} \int d\tau \frac{\dot{x}_i^2}{\dot{t}}$$

τ – evolution parameter

$$\int d\tau \dot{s} = 0$$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m_0 \frac{\dot{x}_i}{\dot{t}}$$

$$p_0 = \frac{\partial L}{\partial \dot{t}} = -\frac{m_0}{2}\frac{\dot{x}_i^2}{\dot{t}^2}$$

$\Rightarrow p_0 = -\frac{p_i^2}{2m_0}$ $\quad p_0$ – energy E

One gets NR energy-momentum dispersion relation for Schrödinger free particle!

Quantization: $p_i = i\frac{\partial}{\partial x_i}$ $\ _0 = -i\frac{\partial}{\partial t} \ \Rightarrow \ i\frac{\partial}{\partial t}\psi = -\frac{\Delta}{2m_0}\psi$ free Schrödinger eq.

The coset element G in our d=3 N=4 NR model can be written as

$$G = G_{(1)}\, G_{(2)}\, G_{(3)}\, G_{(4)}\, G_{(5)}\, G_{(6)} \equiv \hat{G}\, G_{(6)}\,,$$

where explicitly

$$G_{(1)} = \exp i\{t\mathbb{H} + x^i\mathbb{P}^i\}\,,$$
$$G_{(2)} = \exp i\{\xi^\alpha_a\mathbb{Q}^a_\alpha + \xi^\alpha_{\tilde{a}}\mathbb{Q}^{\tilde{a}}_\alpha\}\,,$$
$$G_{(3)} = \exp i\{\theta^\alpha_a\mathbb{S}^a_\alpha + \theta^\alpha_{\tilde{a}}\mathbb{S}^{\tilde{a}}_\alpha\}\,,$$
$$G_{(4)} = \exp i\{k^i\mathbb{B}^i\}\,,$$
$$G_{(5)} = \exp i\{s\mathbb{M} + h_1\mathbb{X}_1 + h_2\mathbb{X}_2 + h_{a\tilde{b}}\mathbb{X}^{a\tilde{b}} + f_1\mathbb{Y}_1 + f_2\mathbb{Y}_2 + f_{a\tilde{b}}\mathbb{Y}^{a\tilde{b}}\}\,,$$
$$G_{(6)} = \exp i\{u^b_a\mathbb{T}^{-a}{}_b + u^{\tilde{b}}_{\tilde{a}}\mathbb{T}^{-\tilde{a}}{}_{\tilde{b}} + u^{\tilde{b}}_a\mathbb{T}^{-a}{}_{\tilde{b}}\}.$$

The factors $G_{(1)}$, $G_{(4)}$ are parametrized by d=3 Galilei group parameters, $G_{(5)}$ by the central charge parameters dual to central charges, $G_{(6)}$ represents the abelian 5-dimensional coset IUSp(4)/USp(4) and $G_{(2)}$, $G_{(3)}$ collect parameters of the fermionic (odd) sector.

We can write the MC one-forms in the following way

$$\hat{G}^{-1}d\hat{G} := i\sum_K \hat{\omega}_{(K)}T_{(K)}\,,$$

where $T_{(K)}$ stand for all coset G generators, and $\hat{\omega}_{(K)}$ denote the

corresponding MC one-forms.

$$
\begin{aligned}
&\hat{\omega}^{\alpha}_{(Q)\,a} = d\xi^{\alpha}_{a}\,, \hat{\omega}^{\alpha}_{(Q)\,\tilde{a}} = d\xi^{\alpha}_{\tilde{a}}\,,\\
&\hat{\omega}^{\alpha}_{(S)\,a} = d\theta^{\alpha}_{a} + \tfrac{1}{2}\,k_i(\sigma_i)_{\beta}{}^{\alpha} d\xi^{\beta}_{a}\,,\\
&\hat{\omega}^{\alpha}_{(S)\,\tilde{a}} = d\theta^{\alpha}_{\tilde{a}} + \tfrac{1}{2}\,k_i(\sigma_i)_{\beta}{}^{\alpha} d\xi^{\beta}_{\tilde{a}},\\
&\hat{\omega}_{(H)} = dt + i\left(\xi^{\alpha}_{a} d\xi^{a}_{\alpha} + \xi^{\alpha}_{\tilde{a}} d\xi^{\tilde{a}}_{\alpha}\right),\\
&\hat{\omega}_{(B)\,i} = dk_i\,,\\
&\hat{\omega}_{(P)\,i} = \left[dx_i + 2i(\sigma_i)_{\alpha\beta}\left(\theta^{b\alpha} d\xi^{\beta}_{b} + \theta^{\tilde{b}\alpha} d\xi^{\beta}_{\tilde{b}}\right)\right] + k_i\,\hat{\omega}_{(H)}\,,\\
&\hat{\omega}_{(M)} = ds + k_i\,\hat{\omega}_{(P)\,i} - \tfrac{1}{2}\,k^2\hat{\omega}_{(H)} - 2i\left(\theta^{\alpha}_{a} d\theta^{a}_{\alpha} + \theta^{\alpha}_{\tilde{a}} d\theta^{\tilde{a}}_{\alpha}\right),\\
&\hat{\omega}_{(X)\,1} = dh_1 + i\xi^{\alpha}_{a} d\xi^{a}_{\alpha}\,,\\
&\hat{\omega}_{(X)\,2} = dh_2 + i\xi^{\alpha}_{\tilde{a}} d\xi^{\tilde{a}}_{\alpha}\,,\\
&\hat{\omega}_{(X)\,a\tilde{b}} = dh_{a\tilde{b}} + i\left(\xi^{\alpha}_{a} d\xi_{\alpha\tilde{b}} - \xi^{\alpha}_{\tilde{b}} d\xi_{\alpha a}\right),\\
&\hat{\omega}_{(Y)\,1} = df_1 + 2\theta^{\alpha a} d\xi_{a\alpha}\,,\\
&\hat{\omega}_{(Y)\,2} = df_2 + 2\theta^{\alpha\tilde{a}} d\xi_{\alpha\tilde{a}}\,,\\
&\hat{\omega}_{(Y)\,a\tilde{b}} = df_{a\tilde{b}} - 2\left(\theta^{\alpha}_{a} d\xi_{\alpha\tilde{b}} - \theta^{\alpha}_{\tilde{b}} d\xi_{\alpha a}\right),
\end{aligned}
$$

where $k^2 := k_i k_i$. Using as action $S_0 = \int \hat{\omega}(M)$ one gets d=3 N=4 NR superparticle model.

Our general model: **linear combination of MC forms** associated with all **12+1 central charges**

$$S = S_0 + S_1 + S_2 \qquad S_1 = \sum_{i=1}^{2} \int (m_i \omega(X_i) + \mu_i \omega(Y_i) \qquad \text{quasidiagonal}$$

$$S_2 = \sum_{a,\tilde{b}} \int (n^{a\tilde{b}} \omega(X^{a\tilde{b}}) + \nu^{a\tilde{b}} \omega(Y^{a\tilde{b}}) \qquad \text{off-diagonal}$$

Two kind of models:

i) $S = S_0 + S_1 \qquad m_1 = -\frac{(\mu_1)^2}{2m_0}$ and/or $m_2 = -\frac{(\mu_2)^2}{2m_0} \Rightarrow$

$\Rightarrow$ **necessary conditions** for getting first class constraints

Odd (fermionic/coordinates:

16 real Grassmann variables: $\theta_\alpha, \tilde{\theta}_\alpha, \xi_\alpha, \tilde{\xi}_\alpha, \bar{\theta}_\alpha, \bar{\tilde{\theta}}_{\dot{\alpha}}, \bar{\xi}_\alpha, \bar{\tilde{\xi}}_\alpha$

One gets: 8 first class constraints $\longrightarrow$ 8 wave equations for Ψ

8 second class constraints $\longrightarrow$ 4 wave equations for Ψ

(Gupta–Bleuler quantization!)

Superfield solution Ψ of all 12 wave equations depends on arbitrary **doubly chiral superfield** $\chi^{(2)}(\theta_\alpha, \tilde{\theta}_\alpha)$, which **determines initial values of Ψ at $\tau = 0$**

ii) $S = S_0 + S_2$ off-diagonal central charges

$$\left(\frac{v_A^2}{2m_0}\right)^2 = n_A^2 \quad \Leftarrow$$

required for 1-st class constraints n_A, v_A – two $O(4)$ fourvectors

One gets 16 constraints which follow from the definition of 16 odd (fermionic) momenta – one gets:

4 first class constraints $\Rightarrow$ 4 wave equations

12 second class constraints $\Rightarrow$ 6 wave equations

(Gupta-Bleuler quantization is used)

N=4 SUSY wave function Ψ:

$$\Psi(x_i, t; \underbrace{\theta_\alpha, \tilde{\theta}_\alpha, \zeta_\alpha, \tilde{\zeta}_\alpha, \bar{\theta}_\alpha, \bar{\tilde{\theta}}_\alpha, \bar{\zeta}_\alpha, \bar{\tilde{\zeta}}_\alpha}_{\text{16 odd variables}}) \xrightarrow[\text{10 wave equations}]{16-10=6} \text{solution depends arbitrarily on } \underbrace{\bar{\theta}_\alpha, \theta_\alpha, \zeta_\alpha}_{\text{6 odd variables}}$$

Form of the solution:

$$\Psi = \hat{R}(\tilde{\chi}, \bar{\theta}_\alpha, \bar{\tilde{\theta}}_\alpha, \bar{\zeta}_\alpha, \bar{\tilde{\zeta}}_\alpha; \frac{\partial}{\partial\theta_\alpha}, \frac{\partial}{\partial\tilde{\theta}_\alpha}, \frac{\partial}{\partial\tilde{\zeta}_\alpha})\ \chi^{(3)}(\theta_\alpha, \tilde{\theta}_\alpha, \bar{\tilde{\zeta}}_\alpha)$$

$\Uparrow$ algebraic dependence

$\chi^{(3)}$– arbitrary triply chiral superfield

All component fields $\psi_A(x_i, t)$ of the wave function Ψ satisfy free Schrödinger wave eq., due to the presence of S_0 in the action.

4. Final remarks

i) One can choose the actions as nonlinear functions of MC one-forms, e.g. $S_0 + S_2'$, where (by analogy with free relativistic particle)

$$S_2 = \int (k_1 \sqrt{\omega^{a\tilde{b}}(X)\omega_{a\tilde{b}}(X)} + k_2 \sqrt{\omega^{a\tilde{b}}(Y)\omega_{a\tilde{b}}(Y)})$$

with additional dynamic bosonic internal coordinates $h_{a\tilde{b}}$, $f_{a\tilde{b}}$, associated by duality with central charges $X^{a\tilde{b}}, Y^{a\tilde{b}}$. Model after quantization leads to particular SUSY KK theory and is under consideration.

ii) Other generalization is to introduce the couplings to EM, YM and gravitational (super)backgrounds. Because component fields have various spins with different couplings to such backgrounds, one expects spin-dependent modifications of the free Schrödinger eq.

iii) An important task is to formulate d=3 NR N=4 SUSY YM theory with nonvanishing central charges. It is interesting to obtain it as NR contraction limit $c \to \infty$ of relativistic D=4 N=4 SUSY YM field theory.

THANK YOU!