Carrollian fluids and holographic applications

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Highlights

Fluid/gravity correspondence

*Branch of AdS/CFT: Relationship Einstein's and **relativistic** Euler's*

[Bhattacharyya, Haack, Hubeny, Loganayagam, Minwalla, Rangamani, Yarom, . . . '07]

$$\text{Einstein spacetime } \mathscr{E} \leftrightarrow \text{relativistic fluid on } \mathscr{M} = \partial\mathscr{E}$$

*Historically: **non-relativistic incompressible-fluid** equations emerge from perturbations of the black-hole horizon [Damour '79]*

- *Membrane paradigm [Price, Thorne '86; Oz et al '09]*
- *Attempts to set up a "holographic" correspondence involving Navier–Stokes equations possibly with asymptotically flat spacetime [Strominger et al '10; Caldarelli, Taylor, Skenderis et al '13; Klemm et al '14]*

Scrutinizing the flat-spacetime holography:

1. *Which surface $\mathscr{S}$ would replace the AdS conf. bry. $\mathscr{M}$ and what is its geometry?*
2. *Which are the degrees of freedom hosted by $\mathscr{S}$, what is their dynamics, how are the observables packaged?*

Wide dispersion – mixing sometimes AdS

- *Minkowski, surface at finite r..., relativistic, Rindler ...*
- *incompressible classical fluids, relativistic fluids, Brown–York tensor, conserved energy–momentum tensor...*

Our aim: unravel a clear pattern

Scattered facts

- *Ricci-flat limit is related to some non-relativistic contraction of Poincaré algebra* [Barnich et al. '10; Bagchi et al. '10–12; Duval et al. '14; Jensen, Karch '15]
- *Null infinity plays a privileged role for hosting the degrees of freedom* [He, Kapec, Mitra, Pasterski, Raclariu, Shao, Strominger '16–'17]

Method: setting $k \to 0$ inside the derivative expansion ($\Lambda = -3k^2$)

- $\mathscr{S}$ *is the 2-dim spatial surface at* $\mathscr{I}^+$ *and* $\mathscr{M} \to \mathscr{S} \times \mathbb{R}$ *is a Carrollian geometry*
- *the degrees of freedom are a conformal Carrollian fluid whose observables obey conformal Carrollian fluid dynamics*

Ricci-flat spacetime $\leftrightarrow$ conformal Carrollian fluid on $\mathscr{S}$

Highlights

Carrollian covariance in d spatial dimensions

Geometry on $\mathscr{S}$: $d\ell^2 = a_{ij}(t,\mathbf{x})dx^i dx^j \quad \Omega(t,\mathbf{x}) \quad \boldsymbol{b} = b_i(t,\mathbf{x})dx^i$

- Carrollian diffs.: $t' = t'(t,\mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$
- Jacobian: $J(t,\mathbf{x}) = \frac{\partial t'}{\partial t} \quad j_i(t,\mathbf{x}) = \frac{\partial t'}{\partial x^i} \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}$
- transfs.: $a'_{ij} = a_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j} \quad \Omega' = \frac{\Omega}{J} \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}_{\ \ k}$

Do not confuse with the Carrollian group of invariance present when $a_{ij} = \delta_{ij}$, $\Omega = 1$, $b_i =$ constant (here realized in tangent space)

$$\begin{cases} t' = t + B_i x^i + t_0, \\ x'^k = R^k_i x^i + x^k_0 \end{cases}$$

[Cartan, Bekaert, Bergshoeff, Duval, Gibbons, Gomis, Hartong, Horvathy, Longhi, Morand, Obers]

Relativistic uplift

Why?

Starting point for studying the $c \to 0$ limit of any system

Relativistic uplift: $d+1$-dim Randers–Papapetrou form

- $\mathrm{d}s^2 = -c^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$
- reproduces the wanted transformation under $x^{0\prime} = x^{0\prime}(x^0, \mathrm{x})$ $\mathrm{x}' = \mathrm{x}'(\mathrm{x})$ (here $x^0 = ct$) – Carrollian diffeomorphisms

$$J^{\mu}_{\nu}(t, \mathrm{x}) = \frac{\partial x^{\mu\prime}}{\partial x^{\nu}} = \begin{pmatrix} J(t, \mathrm{x}) & c j_j(t, \mathrm{x}) \\ 0 & J^i_j(\mathrm{x}) \end{pmatrix}$$

More on Carrollian geometries

- *isometries*
- *time and space connections, covariant derivatives, curvatures*
- *time and space Weyl connection, Weyl curvature*

Example: Carrollian space derivative $\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t$

- transfs.: $\hat{\partial}'_i = J^{-1}{}^j_i \hat{\partial}_j$
- connection: $\hat{\gamma}^i_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right)$
- covariant metric-compatible derivative: $\hat{\boldsymbol{\nabla}} = \hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\gamma}}$

Similarly: Weyl-covariant metric-compatible derivatives $\hat{\mathscr{D}}_i$, $\hat{\mathscr{D}}_t$
built on $\varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right)$ and $\theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}$

Highlights

Relativistic fluids

Obey $\nabla_\mu T^{\mu\nu} = 0$ *with*

$$T^{\mu\nu} = (\varepsilon + p)\frac{u^\mu u^\nu}{c^2} + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{c^2} + \frac{u^\nu q^\mu}{c^2}$$

- $\|\mathrm{u}\|^2 = -c^2,\ u^0 = \gamma c,\ u^i = \gamma v^i$
- $\varepsilon,\ p$: energy density and pressure
- $\tau^{\mu\nu},\ q^\mu$: viscous stress tensor and heat current – transverse

$$u^\mu q_\mu = 0 \quad u^\mu \tau_{\mu\nu} = 0$$

q^i and τ^{ij} carry all information on heat exchange and friction processes

Carrollian limit: Carrollian fluid

Relativistic fluid on Randers–Papapetrou at $c \to 0$: kinematics

v^i must vanish faster than c:

$$v^i = c^2 \Omega \beta^i + \mathrm{O}\left(c^4\right)$$

avoids blow-ups without trivializing

- kinematic variable $\beta^i = -\frac{\Omega u_i}{c u_0} - b_i = {}^{v_i}/_{c^2\Omega\left(1-\frac{v^j b_j}{\Omega}\right)}$
- $u^0 = \gamma c = {}^c/_\Omega + \mathrm{O}\left(c^3\right) \quad u^i = \gamma v^i = c^2 \beta^i + \mathrm{O}\left(c^4\right)$
- $u_0 = -c\Omega + \mathrm{O}\left(c^3\right) \quad u_i = c^2\left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right)$

[worth comparing with de Boer, Hartong, Obers, Sybesma, Vandoren ’17]

Limit inside the fluid data (microscopic justification yet to come)

- ε, p
- $q^i \to Q^i + c^2 \pi^i$ and $\tau^{ij} \to -\frac{1}{c^2}\Sigma^{ij} - \Xi^{ij}$

Limit inside the relativistic-fluid equations

$$\begin{cases} 0 = \frac{c}{\Omega}\nabla_\mu T^\mu{}_0 = \frac{1}{c^2}\mathcal{F} + \mathcal{E} + \mathrm{O}\left(c^2\right) \\ 0 = \nabla_\mu T^{\mu i} = \frac{1}{c^2}\mathcal{H}^i + \mathcal{G}^i + \mathrm{O}\left(c^2\right) \end{cases}$$

→ *Carrollian equations*

- scalar equations: $\mathcal{E} = 0 \quad \mathcal{F} = 0$
- vector equations: $\mathcal{G}^j = 0 \quad \mathcal{H}^i = 0$

→ *Covariant under Carrollian diffs.*

- $\mathcal{E}' = \mathcal{E} \quad \mathcal{F}' = \mathcal{F}$
- $\mathcal{G}'^i = J^i_j \mathcal{G}^j \quad \mathcal{H}'^i = J^i_j \mathcal{H}^j$

Carrollian hydrodynamics with $\boldsymbol{\beta} = 0$

Scalar equations

- $\mathcal{E} = -\frac{1}{\Omega}\partial_t \varepsilon - (\varepsilon + p)\theta - \hat{\nabla}_i Q^i - 2\varphi_i Q^i + \Xi^{ij}\xi_{ij} + \frac{1}{2}\Xi^i{}_i\theta = 0$
- $\mathcal{F} = \Sigma^{ij}\xi_{ij} + \frac{1}{2}\Sigma^i{}_i\theta = 0$

Vector equations

- $\mathcal{G}_j = \hat{\partial}_j p + (\varepsilon + p)\varphi_j + \frac{1}{\Omega}\partial_t \pi_j + \pi_j\theta + 2Q^i\varpi_{ij} - \hat{\nabla}_i \Xi^i{}_j - \varphi_i \Xi^i{}_j = 0$
- $\mathcal{H}^i = \frac{a^{ij}}{\Omega}\partial_t Q_j + Q^i\theta - \hat{\nabla}_j \Sigma^{ji} - \varphi_j \Sigma^{ji} = 0$

Remarks

- *more involved for* $\beta^i \neq 0$
- *more elegant for conformal fluids*

φ_i, θ, ξ_{ij}, ϖ_{ij}: kinematic observables

Relativistic origin: acceleration, expansion, shear, vorticity

- $\frac{a_i}{c^2} = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right) + \mathrm{O}\left(c^2\right) = \varphi_i + \mathrm{O}\left(c^2\right)$
- $\Theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a} + \mathrm{O}\left(c^2\right) = \theta + \mathrm{O}\left(c^2\right)$
- $\sigma_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij} - \frac{1}{d}a_{ij}\partial_t \ln\sqrt{a}\right) + \mathrm{O}\left(c^2\right) = \xi_{ij} + \mathrm{O}\left(c^2\right)$
- $\frac{\omega_{ij}}{c^2} = \partial_{[i}b_{j]} + \frac{1}{\Omega}b_{[i}\partial_{j]}\Omega + \frac{1}{\Omega}b_{[i}\partial_t b_{j]} + \mathrm{O}\left(c^2\right) = \varpi_{ij} + \mathrm{O}\left(c^2\right)$

Remarks

- all Carrollian-covariant
- purely geometric
- more terms if $\beta^i \neq 0$

Highlights

The bulk reconstruction

Given the "initial data"

- boundary metric ds^2
- conserved energy–momentum tensor T

two options exist to get perturbatively the asymptotically AdS bulk:

1. Fefferman–Graham expansion: hard to resum
2. Derivative expansion: designed for fluid/gravity correspondence
   - Eddington–Finkelstein gauge in the bulk: close and sometimes identical to the BMS gauge – the radial coordinate is null
   - extra piece of bry. data: time-like hydrodynamic congruence u

The derivative expansion [Bhattacharyya et al '07]

- Guideline: **Weyl covariance** – the bulk metric must be invariant under boundary conformal transformations
- Tool: Weyl connection $A = \frac{1}{k^2}\left(a - \frac{\Theta}{2}u\right)$ and Weyl covariant derivative $\mathscr{D} = \nabla + wA$ (a is the acceleration and $\Theta = \nabla \cdot u$)
- Output: $ds^2_{\text{bulk}} =$ complicated expression based on the boundary data & their derivatives – order by order – including asymptotically locally AdS spacetimes [Leigh *et al* '10; Caldarelli *et al* '12; Mukhopadhyay *et al* '13; Gath *et al* '15]

The resummation in 4 dimensions [Caldarelli et al '12; Mukhopadhyay et al '13; Gath et al '15]

Assuming u shear-free a resummation is performed ($\Lambda = -3k^2$):

$$\mathrm{d}s^2_{\text{res. Einstein}} = 2\frac{\mathrm{u}}{k^2}(\mathrm{d}r + r\mathrm{A}) + r^2\mathrm{d}s^2 + \frac{\mathrm{S}}{k^4} + \frac{\mathrm{u}^2}{k^4\rho^2}\left(8\pi G\varepsilon r + c\gamma\right)$$

- boundary metric $\mathrm{d}s^2 = -k^2\left(\Omega\mathrm{d}t - b_i\mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$
- $\mathrm{u} = -k^2\left(\Omega\mathrm{d}t - b_i\mathrm{d}x^i\right) \Leftrightarrow \mathrm{u} = \frac{1}{\Omega}\partial_t$ with $\|\mathrm{u}\|^2 = -k^2$
- $\varepsilon = 2p$: conformal-fluid energy density
- $S = -2\mathrm{u}\nabla_\nu\omega^\nu{}_\mu\mathrm{d}x^\mu - \omega_\mu{}^\lambda\omega_{\lambda\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - \frac{1}{2}\mathrm{u}^2\mathscr{R}$
- $\mathscr{R} = R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu,$
- $\rho^2 = r^2 + \frac{1}{2k^4}\omega_{\alpha\beta}\omega^{\alpha\beta} = r^2 + \gamma^2$
- c: Cotton (3rd-order derivative of the metric) $\nabla^\lambda C_{\lambda\mu} = 0$

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_\mu u_\nu}{k} + \frac{ck}{2}g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}$$

Resummability conditions (on top of the absence of shear)

- transverse duality (with $\eta_{\mu\nu} = -\frac{u^\rho}{k}\eta_{\rho\mu\nu}$)

$$q_\mu = \frac{1}{8\pi G}\eta^{\nu}{}_{\mu}c_\nu \quad \tau_{\mu\nu} = -\frac{1}{8\pi G k^2}\eta^{\rho}{}_{\mu}c_{\rho\nu}$$

  origin: "electric–magnetic gravitational duality"

- energy–momentum conservation $\nabla^\lambda T_{\lambda\mu} = 0$ for the boundary data ε, a_{ij}, Ω and b_i

*Output: algebraically special Einstein spacetimes – Goldberg–Sachs generalizations – asymptotically **locally** AdS*

- Kerr–Taub–NUT (perfect fluids)
- Robinson–Trautman
- Plebański–Demiański
- ...

Highlights

Foreword

Non-relativistic geometries and relativistic uplifts

From relativistic hydrodynamics to Carrollian fluids

Relativistic fluid/AdS gravity

Carrollian fluid/Ricci-flat gravity

Summary

The Ricci-flat limit $\Lambda = -3k^2 \to 0$

In the boundary data: $k \equiv$ *speed of light*

$$\mathrm{d}s^2 = -k^2 \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j \quad \|u\|^2 = -k^2$$

$\underset{k\to 0}{\longrightarrow}$ Carrollian limit: $\mathscr{M} \to \mathscr{S} \times \mathbb{R}$ reached at $\mathscr{I}^+$

In the bulk metric: the limit is well-defined and Weyl-covariant

$$\begin{aligned} \lim_{k\to 0} \mathrm{d}s^2_{\text{res. Einstein}} &= \mathrm{d}s^2_{\text{res. flat}} \\ &= -2\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)\left(\mathrm{d}r + r\boldsymbol{\alpha} + \tfrac{r\theta\Omega}{2}\mathrm{d}t\right) + r^2 \mathrm{d}\ell^2 \\ &\quad + \boldsymbol{s} + \frac{\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)^2}{\rho^2}\left(8\pi G \varepsilon r + c * \varpi\right) \end{aligned}$$

- $\mathrm{d}\ell^2$, Ω, $\boldsymbol{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{s}$, θ, $*\varpi$: Carrollian-geometric data
- $\varepsilon = 2p$: conformal Carrollian-fluid energy density

The boundary fluid $\underset{k\to 0}{\longrightarrow}$ *conformal Carrollian fluid*

- data: ε, Q_i, π_i, Σ_{ij}, Ξ_{ij}
- dynamics: conformal Carrollian

richer than a divergence-free 2-dim "energy–momentum" tensor

The boundary Cotton tensor $\underset{k\to 0}{\longrightarrow}$ *conformal Carrollian 3rd-derivative*

c, χ_i, ψ_i, X_{ij}, Ψ_{ij} obeying similar equations

The Carrollian-boundary resummability conditions:

$$Q_i = \frac{1}{8\pi G}\eta^j{}_i\chi_j \quad \Sigma_{ij} = \frac{1}{8\pi G}\eta^l{}_i X_{lj} \quad \Xi_{ij} = \frac{1}{8\pi G}\eta^l{}_i\Psi_{lj}$$

*Output : algebraically special Ricci-flat spactimes – Goldberg–Sachs – asymptotically **locally** flat*

Example

Boundary data: $d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta}$, $\Omega = 1$, $b_i = 0$

- $\xi_{ij} = 0$, $\omega_{ij} = 0$, $\varphi_i = 0$, $\theta = -2\partial_t \ln P$
- "Cotton": $c = 0$, $\boldsymbol{\psi} = 0$, $\boldsymbol{\chi} = \frac{\mathrm{i}}{2}\left(\partial_\zeta K \mathrm{d}\zeta - \partial_{\bar{\zeta}} K \mathrm{d}\bar{\zeta}\right)$, $\boldsymbol{\Psi} = 0$, $\boldsymbol{X} = \frac{\mathrm{i}}{P^2}\left(\partial_\zeta \left(P^2 \partial_t \partial_\zeta \ln P\right) \mathrm{d}\zeta^2 - \partial_{\bar{\zeta}} \left(P^2 \partial_t \partial_{\bar{\zeta}} \ln P\right) \mathrm{d}\bar{\zeta}^2\right)$

$(K = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta \ln P)$

Fluid data: ε, $\boldsymbol{\pi} = 0$, $\boldsymbol{Q} = \frac{1}{8\pi G} * \chi$, $\boldsymbol{\Sigma} = \frac{1}{8\pi G} * \boldsymbol{X}$, $\boldsymbol{\Xi} = 0$

- momentum equation: $\partial_i \varepsilon = 0 \Rightarrow \varepsilon(t) = {}^{M(t)}/_{4\pi G}$
- energy equation: $\Delta\Delta \ln P + 12 M \partial_t \ln P - 4\partial_t M = 0$

Resummation: Ricci-flat Robinson–Trautman

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Summary

About Carrollian fluids on a_{ij}, Ω, b_i

- obtained from Randers-Papapetrou relativistic fluids
- described in terms of ε, p Q_i, π_i, Σ_{ij}, Ξ_{ij} (and β_i)
- obey Carrollian-covariant (possibly conformal) set of equations

About flat holography (4 spacetime dimensions – shearless situation)

1. Which surface $\mathscr{S}$ would replace the AdS conf. bry. $\mathscr{M}$ and what is its geometry? Spatial surface at $\mathscr{I}^+$ equipped with Carrollian geometry
2. Which are the degrees of freedom hosted by $\mathscr{S}$, what is their dynamics, how are the observables packaged? Carrollian conformal fluid with ε, Q_i, π_i, Σ_{ij}, Ξ_{ij} obeying Carrollian fluid equations

Fundamental question: microscopic behaviour at at $c \to 0$

- *Boltzmann equation*
- *QFT*
- *...*

Highlights

Galilean covariance

Galilean fluids

More on Carrollian fluids and Carrollian geometry

Curvature and Cotton

Galilean covariance in d spatial dimensions

Geometry: $d\ell^2 = a_{ij}(t, \mathbf{x})dx^i dx^j, \quad \Omega = \Omega(t), \quad \mathbf{w} = w^i(t, \mathbf{x})\partial_i$

- Galilean diffs.: $t' = t'(t), \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$
- Jacobian: $J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial t}, \quad J^i_j(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}$
- transfs.: $a'_{ij} = a_{kl} J^{-1}{}^k_i J^{-1}{}^l_j, \ \Omega' = \frac{\Omega}{J}, \ w'^k = \frac{1}{J}\left(J^k_i w^i + j^k\right)$
- absolute Newtonian time (invariant): $\Omega(t)\mathrm{d}t$

[Cartan, Bekaert, Bergshoeff, Duval, Gibbons, Gomis, Hartong, Horvathy, Longhi, Morand, Obers]

Do not confuse with the Galilean group of invariance present when $a_{ij} = \delta_{ij}, \Omega = 1, w^i = constant$

$$\begin{cases} t' = t + t_0, \\ x'^k = R^k_i x^i + V^k t + x^k_0 \end{cases}$$

Simple realization and relativistic uplift

Particle: $x^i = x^i(t), \quad v^i = {}^{dx^i}/_{dt}$, $\mathbf{v} = v^i \partial_i$

- transfs.: $v'^k = \frac{1}{J}\left(J^k_i v^i + j^k\right)$, $\quad \frac{\mathbf{v}-\mathbf{w}}{\Omega}$ d-dim vector
- free dynamics: $\mathcal{L}(\mathbf{v}, \mathbf{x}, t) = \frac{1}{2\Omega^2} a_{ij}\left(v^i - w^i\right)\left(v^j - w^j\right)$

Relativistic uplift: $d+1$*-dim Zermelo form*

- $\mathrm{d}s^2 = -\Omega^2 c^2 \mathrm{d}t^2 + a_{ij}\left(\mathrm{d}x^i - w^i \mathrm{d}t\right)\left(\mathrm{d}x^j - w^j \mathrm{d}t\right)$
- form-invariant under $J^\mu_\nu(x) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} = \begin{pmatrix} J(t) & 0 \\ \frac{j^i(t,\mathbf{x})}{c} & J^i_j(t,\mathbf{x}) \end{pmatrix}$
- relevant limit: $c \to \infty$

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Non-relativistic Galilean fluid

Relativistic fluid on Zermelo at $c \to \infty$: Galilean fluid

$$u^0 = \frac{c}{\Omega} + \mathrm{O}\left(1/c\right) \quad u^i = \frac{v^i}{\Omega} + \mathrm{O}\left(1/c^2\right)$$

- v^i
- $e, \ p, \ \varrho$
- $q_i \to Q_i$ and $\tau_{ij} \to -\Sigma_{ij}$

Galilean-covariant equations on $a_{ij}(t, \mathrm{x}), \ \Omega(t), \ w^i(t, \mathrm{x})$

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Simple realization and relativistic uplift

Extended object: $t = t(\mathbf{x})$, $\beta_i = \Omega \partial_i t - b_i$, $\boldsymbol{\beta} = \beta_i dx^i$

- transfs.: $\beta'_k = \beta_i J^{-1i}{}_k$ (d-dim form)
- free dynamics: $\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}) = \frac{1}{2} a^{ij} \left(\Omega \partial_i t - b_i\right) \left(\Omega \partial_j t - b_j\right)$

Relativistic uplift: $d+1$*-dim Randers–Papapetrou form*

- relevant limit: $c \to 0$
- $\mathrm{d}s^2 = -c^2 \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij} \mathrm{d}x^i \mathrm{d}x^j$
- form-invariant under Carrollian diffeomorphisms ($x^0 = ct$)

$$J^\mu_\nu(t, \mathbf{x}) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} = \begin{pmatrix} J(t, \mathbf{x}) & c j_j(t, \mathbf{x}) \\ 0 & J^i_j(\mathbf{x}) \end{pmatrix}$$

The Carrollian limit

Inside the perfect-fluid energy–momentum tensor

$$\gamma = \frac{1 + c^2 \boldsymbol{\beta} \cdot \boldsymbol{b}}{\Omega \sqrt{1 - c^2 \boldsymbol{\beta}^2}} = \frac{1}{\Omega} \left(1 + \frac{c^2}{2} \boldsymbol{\beta} \cdot (\boldsymbol{\beta} + 2\boldsymbol{b}) + \mathrm{O}\left(c^4\right) \right)$$

$$\begin{cases} T_{\text{perf}}{}^{0}{}_{0} = -\varepsilon - c^2 (\varepsilon + p) \beta^k \left(b_k + \beta_k\right) + \mathrm{O}\left(c^4\right) \\ c\Omega T_{\text{perf}}{}^{0}{}_{i} = c^2 (\varepsilon + p) \left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right) \\ \frac{c}{\Omega} T_{\text{perf}}{}^{j}{}_{0} = -c^2 (\varepsilon + p) \beta^j + \mathrm{O}\left(c^4\right) \\ T_{\text{perf}}{}^{j}{}_{i} = p \delta^j_i + c^2 (\varepsilon + p) \beta^j \left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right) \end{cases}$$

Conformal Carrollian geometry

Weyl transformation on Carrollian geometry

$$a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2} \quad b_i \to \frac{b_i}{\mathcal{B}} \quad \Omega \to \frac{\Omega}{\mathcal{B}}$$

Spatial Weyl derivative for a weight-w vector V^l

$$\hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i$$

Temporal Weyl derivative for a weight-w vector V^l

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{2}\theta V^l + \xi^l{}_i V^i$$

Conformal Carrollian fluids $\boldsymbol{\beta} = 0$

From relativistic to Carrollian conformal properties

- $\varepsilon = dp$ and $\tau^{\mu}{}_{\mu} = 0 \underset{c\to 0}{\longrightarrow} \Xi^{i}{}_{i} = \Sigma^{i}{}_{i} = 0$
- $\varepsilon \to \mathcal{B}^{d+1}\varepsilon,\ \pi_i \to \mathcal{B}^{d}\pi_i,\ Q_i \to \mathcal{B}^{d}Q_i,$
  $\Xi_{ij} \to \mathcal{B}^{d-1}\Xi_{ij},\ \Sigma_{ij} \to \mathcal{B}^{d-1}\Sigma_{ij}$

Scalar equations

- $\mathcal{E} = -\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon - \hat{\mathscr{D}}_i Q^i + \Xi^{ij}\xi_{ij} = 0$
- $\mathcal{F} = \Sigma^{ij}\xi_{ij} = 0$

Vector equations

- $\mathcal{G}_j = \frac{1}{d}\hat{\mathscr{D}}_j\varepsilon + \frac{1}{\Omega}\hat{\mathscr{D}}_t\pi_j + \pi_i\xi^{i}{}_{j} + 2Q^i\varpi_{ij} - \hat{\mathscr{D}}_i\Xi^{i}{}_{j} = 0$
- $\mathcal{H}_j = \frac{1}{\Omega}\hat{\mathscr{D}}_t Q_j + Q_i\xi^{i}{}_{j} - \hat{\mathscr{D}}_i\Sigma^{i}{}_{j} = 0$

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The Conformal Carrollian curvature tensors in 2 dim

The Ricci tensor (space)

$$\hat{\mathscr{R}}_{ij} = \hat{\mathscr{R}}^k{}_{ikj} = \hat{r}_{ij} + a_{ij}\hat{\nabla}_k\varphi^k = \hat{s}_{ij} + \hat{\mathscr{K}} a_{ij} + \hat{\mathscr{A}}\eta_{ij}$$

$$\hat{\mathscr{K}} = \frac{1}{2}a^{ij}\hat{\mathscr{R}}_{ij} = \hat{K} + \hat{\nabla}_k\varphi^k, \quad \hat{\mathscr{A}} = \frac{1}{2}\eta^{ij}\hat{\mathscr{R}}_{ij}$$

$\hat{r}_{ij}$ is the Ricci of $\hat{\nabla}_k$ and $2\hat{K} = \hat{r} = a^{ij}\hat{r}_{ij}$

The vector (time)

$$\hat{\mathscr{R}}_i = \frac{1}{\Omega}\partial_t\varphi_i - \frac{1}{2}\left(\hat{\partial}_i + \varphi_i\right)\theta$$

Reminder: the Cotton tensor

In 3 dim the Weyl tensor vanishes – conformal properties are captured by the Cotton tensor

$$C_{\mu\nu} = \eta_{\mu}{}^{\rho\sigma}\nabla_{\rho}\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right)$$

$(\eta_{\mu\nu\sigma} = \sqrt{-g}\epsilon_{\mu\nu\sigma})$

- symmetric and traceless
- conformally covariant of weight 1
- identically conserved: $\nabla_{\mu}C^{\mu\nu} = 0$

Decomposition wrt u

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_\mu u_\nu}{k} + \frac{ck}{2} g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}$$

At large k with $u = \frac{1}{\Omega}\partial_t$

- $c_i = \chi_i + k^2 \psi_i$
- $c_{ij} = X_{ij} + k^2 \Psi_{ij}$
- weight 2 and 1 respectively

The "Cotton" in two-dimensional Carrollian geometry

$$\begin{cases} c = \left(\hat{\mathscr{D}}_l \hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right) * \varpi \\ \chi_j = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{K}} + \frac{1}{2}\hat{\mathscr{D}}_j \hat{\mathscr{A}} - 2 * \varpi \hat{\mathscr{R}}_j \\ \psi_j = 3\eta^l{}_j \hat{\mathscr{D}}_l * \varpi^2 \\ X_{ij} = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{R}}_i + \frac{1}{2}\eta^l{}_i \hat{\mathscr{D}}_j \hat{\mathscr{R}}_l \\ \Psi_{ij} = \hat{\mathscr{D}}_i \hat{\mathscr{D}}_j * \varpi - \frac{1}{2} a_{ij} \hat{\mathscr{D}}_l \hat{\mathscr{D}}^l * \varpi - \eta_{ij} \frac{1}{\Omega} \hat{\mathscr{D}}_t * \varpi^2 \end{cases}$$

Conservation identities

$$\begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t c + \hat{\mathscr{D}}_i \chi^i = 0 \\ \frac{1}{2}\hat{\mathscr{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t \psi_j - \hat{\mathscr{D}}_i \Psi^i{}_j = 0 \\ \frac{1}{\Omega}\hat{\mathscr{D}}_t \chi_j - \hat{\mathscr{D}}_i X^i{}_j = 0 \end{cases}$$