Torsional Newton-Cartan gravity and strong gravitational fields

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ongoing work

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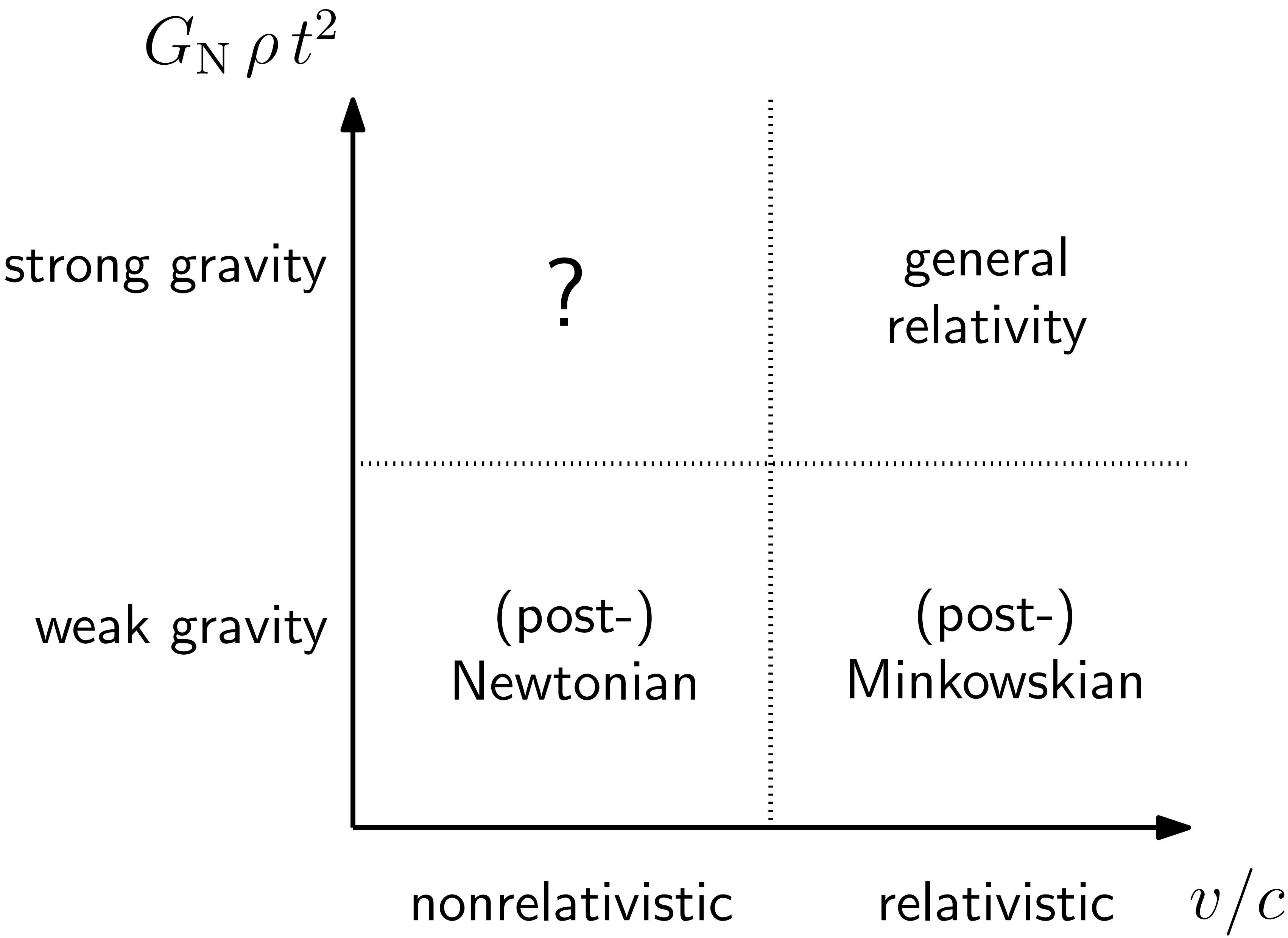
Gravity in various regimes



Overview

- I will revisit the covariant large c expansion of GR

  ⟶ Relax an assumption on the relativistic Levi-Civita connection

- outcome

  ⟶ TTNC gravity

- physical interpretation

  ⟶ non-relativistic, strong gravity regime

  extends Newtonian physics to include strong gravitational time dilation

  ⟶ static sector of GR is captured exactly, expansion around static solutions

Expanding the geometry

- Large c expansion

$$g_{\mu\nu}(c) = \sum_{i=-1}^{\infty} \overset{(2i)}{g}_{\mu\nu} c^{-2i} \qquad g^{\mu\nu}(c) = \sum_{i=0}^{\infty} \overset{(2i)}{g}{}^{\mu\nu} c^{-2i}$$

Ansatz: $\overset{(-2)}{g}_{\mu\nu} = -\tau_\mu \tau_\nu$ (natural via $x^0 = ct$)

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- Constraint

$$g_{\mu\nu}(c) g^{\nu\rho}(c) = \delta_\mu^\rho$$

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Ansatz: $\overset{(-2)}{g}_{\mu\nu} = -\tau_\mu \tau_\nu$ (natural via $x^0 = ct$)

- Constraint

$$g_{\mu\nu}(c) g^{\nu\rho}(c) = \delta^\rho_\mu$$

- At first order

$$\overset{(0)}{g}{}^{\mu\nu} = h^{\mu\nu} \qquad \text{with} \qquad h^{\mu\nu}\tau_\nu = 0$$

Expanding the geometry

LO

$$\overset{(-2)}{g}_{\mu\nu} = -\tau_\mu \tau_\nu$$

$$\overset{(0)}{g}{}^{\mu\nu} = h^{\mu\nu}$$

NLO

$$\overset{(0)}{g}_{\mu\nu} = -2\hat{\Phi}\tau_\mu \tau_\nu + \hat{h}_{\mu\nu}$$

$$\overset{(2)}{g}{}^{\mu\nu} = -\hat{\tau}^\mu \hat{\tau}^\nu + \hat{\beta}^{\mu\nu}$$

NNLO

$$\overset{(2)}{g}_{\mu\nu} = \tau_\mu \hat{B}_\nu + \tau_\nu \hat{B}_\mu - \hat{h}_{\mu\rho}\hat{h}_{\nu\sigma}\hat{\beta}^{\rho\sigma}$$

$$\overset{(4)}{g}{}^{\mu\nu} = 2\hat{\Phi}\hat{\tau}^\mu \hat{\tau}^\nu + 2\hat{\tau}^{(\mu} h^{\nu)\rho}\hat{B}_\rho + \hat{\gamma}^{\mu\nu}$$

Expanding metric compatibility

- Unique Levi-Civita connection

$$\nabla_\mu g_{\nu\rho} = 0 \qquad \nabla_\mu g^{\nu\rho} = 0$$

- At NLO

$$\begin{aligned}
\overset{(0)}{\nabla}_\mu h^{\nu\lambda} &= -\overset{(-2)}{\Gamma}{}^{\nu}_{\mu\rho}\overset{(2)}{g}{}^{\rho\lambda} - \overset{(-2)}{\Gamma}{}^{\lambda}_{\mu\rho}\overset{(2)}{g}{}^{\rho\nu} \\
\overset{(0)}{\nabla}_\mu (\tau_\nu \tau_\lambda) &= -\overset{(-2)}{\Gamma}{}^{\rho}_{\mu\nu}\overset{(0)}{g}_{\rho\lambda} - \overset{(-2)}{\Gamma}{}^{\rho}_{\mu\lambda}\overset{(0)}{g}_{\rho\nu}
\end{aligned}$$

Expanding metric compatibility

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\end{aligned}$$

- Can be massaged into

$$\overset{(\text{nc})}{\nabla}_\mu h^{\nu\lambda} = 0 \qquad \text{and} \qquad \overset{(\text{nc})}{\nabla}_\mu \tau_\nu = 0$$

Expanding metric compatibility

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- Can be massaged into

$$\overset{\text{(nc)}}{\nabla}_\mu h^{\nu\lambda} = 0 \qquad \text{and} \qquad \overset{\text{(nc)}}{\nabla}_\mu \tau_\nu = 0$$

- Where

$$\begin{aligned} \overset{\text{(nc)}}{\Gamma}{}^\lambda_{\mu\nu} \;=\; & \frac{1}{2} h^{\lambda\rho} \Big(\partial_\mu \hat{h}_{\rho\nu} + \partial_\nu \hat{h}_{\mu\rho} - \partial_\rho \hat{h}_{\mu\nu} \\ & + 2\partial_\rho \hat{\Phi} \tau_\mu \tau_\nu - 4\hat{\Phi}(\tau_\mu \partial_{[\nu}\tau_{\rho]} + \tau_\nu \partial_{[\mu}\tau_{\rho]}) \Big) \\ & + \hat{\tau}^\lambda \partial_\mu \tau_\nu \end{aligned}$$

Expanding the Einstein equations

- Setup

$$R_{\mu\nu} = 8\pi G_{\mathrm{N}}\,\mathcal{T}_{\mu\nu}\,, \qquad \mathcal{T}_{\mu\nu} = c^{-4}\left(T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} g^{\rho\sigma} T_{\rho\sigma}\right)$$

Given the expansion of the metric

$$R_{\mu\nu} = \sum_{i=-2}^{\infty} \overset{(2i)}{R}_{\mu\nu} c^{-2i}$$

Consistency

$$\mathcal{T}_{\mu\nu} = \sum_{i=-2}^{\infty} \overset{(2i)}{\mathcal{T}}_{\mu\nu} c^{-2i}$$

In full generality $\overset{(-4)}{\mathcal{T}}_{\mu\nu} \neq 0$ we will restrict to the case

$$\overset{(-4)}{\mathcal{T}}_{\mu\nu} = 0 \qquad\qquad \text{(assumption)}$$

Expanding the Einstein equations

- Result: TTNC gravity

LO

$$h^{\kappa\lambda}h^{\rho\sigma}\partial_{[\kappa}\tau_{\rho]}\partial_{[\lambda}\tau_{\sigma]} = 0$$

↳ twistless torsion: $\partial_{[\mu}\tau_{\nu]} = \tau_{[\mu}\hat{a}_{\mu]}$

NLO

$$-\tau_\mu\tau_\nu h^{\lambda\rho}D_\lambda\hat{a}_\rho = 8\pi G_{\rm N}\overset{\text{(-2)}}{\mathcal{T}}_{\mu\nu}$$

NNLO

$$\overset{\text{(nc)}}{R}_{\mu\nu} = -\hat{h}_\mu{}^\rho\hat{h}_\nu{}^\sigma D_\rho\hat{a}_\sigma - \overline{\mathcal{K}}_{\mu\rho}\tau_\nu h^{\rho\sigma}\hat{a}_\sigma$$
$$-\tau_\mu\tau_\nu\left(h^{\rho\sigma}\hat{a}_\rho\partial_\sigma(\hat{\Phi}+\frac{1}{2}\hat{\beta}) - D_\rho(\hat{\beta}^{\rho\sigma}\hat{a}_\sigma)\right)$$
$$+8\pi G_{\rm N}\overset{\text{(0)}}{\mathcal{T}}_{\mu\nu}$$

Picking a time

$$\tau_\mu = \psi(t,x)\delta^t_\mu \qquad \psi = e^{\frac{\lambda}{2}}$$

$$\begin{aligned} ds^2 = & -c^2\, e^\lambda dt^2 + e^{-\lambda} h_{ij} dx^i dx^j - 2e^\lambda dt(\Phi dt - C_i dx^i) \\ & + c^{-2}\left((e^{-\lambda}\beta_{ij} - e^\lambda C_i C_j) dx^i dx^j + B_i dt dx^i + \Sigma dt^2\right) \\ & + \mathcal{O}(c^{-4}) \end{aligned}$$

Relevant fields:

$\lambda\,, \quad h_{ij}\,, \quad C_i\,, \quad \Phi$ (or β_{ij})

TTNC in partially gaugefixed form

Fields: $\lambda\,,\quad h_{ij}\,,\quad C_i\,,\quad \Phi$ (or β_{ij})

$$
\begin{aligned}
\frac{1}{2}e^{2\lambda}\nabla_i\partial^i\lambda &= 0 \\
R_{ij} &= \frac{1}{2}\partial_i\lambda\partial_j\lambda \\
\frac{1}{2}\nabla^j\left(e^{2\lambda}K_{ij}\right) &= -h^{jk}\nabla_{[j}\dot{h}_{i]k} - \frac{1}{2}(h^{jk}\dot{h}_{jk} - \dot{\lambda})\partial_i\lambda - \partial_i\dot{\lambda} + \dot{h}_{ij}\partial^j\lambda \\
-e^{\frac{3}{2}\lambda}\nabla^i(e^{\frac{1}{2}\lambda}G_i) &= -\frac{1}{4}e^{4\lambda}K_{ij}K^{ij} + \frac{1}{2}h^{ij}\ddot{h}_{ij} - \frac{1}{4}h^{ik}h^{jl}\dot{h}_{ij}\dot{h}_{kl} \\
&\quad -\frac{3}{2}(\ddot{\lambda} + \dot{\lambda}(\frac{1}{2}h^{ij}\dot{h}_{ij} - \dot{\lambda})) + e^{\frac{5}{2}\lambda}C^i\partial_i\partial_t e^{-\frac{1}{2}\lambda} \\
&\quad -\frac{1}{2}e^{2\lambda}\left(\frac{1}{2}h^{jk}\dot{h}_{jk}C^i\partial_i\lambda - C^i\dot{h}_{ij}\partial^j\lambda\right) \\
&\quad +\frac{1}{2}e^{2\lambda}\nabla_i(\bar{\beta}^{ij}\partial_j\lambda)
\end{aligned}
$$

NC in partially gaugefixed form

Fields: $\lambda = 0\,, \quad h_{ij}\,, \quad C_i\,, \quad \Phi$

$$
\begin{aligned}
0 &= 0 \\
R_{ij} &= 0 \\
\frac{1}{2}\nabla^j K_{ij} &= -h^{jk}\nabla_{[j}\dot{h}_{i]k} - \frac{1}{2}h^{jk}\dot{h}_{jk} \\
-\nabla^i G_i &= -\frac{1}{4}K_{ij}K^{ij} + \frac{1}{2}h^{ij}\ddot{h}_{ij} - \frac{1}{4}h^{ik}h^{jl}\dot{h}_{ij}\dot{h}_{kl}
\end{aligned}
$$

$$K = dC \qquad G = d\Phi + \dot{C}$$

TTNC in partially gaugefixed form

Fields: $\lambda\,,\quad h_{ij}\,,\quad C_i$

$$
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\end{aligned}
$$

TTNC in partially gaugefixed form

Fields: $\lambda\,, \quad h_{ij}\,, \quad C_i$

$$\frac{1}{2}e^{2\lambda}\nabla_i\partial^i\lambda = 0$$

$$R_{ij} = \frac{1}{2}\partial_i\lambda\partial_j\lambda$$

← Static Einstein eqns!

$$\frac{1}{2}\nabla^j\left(e^{2\lambda}K_{ij}\right) = -h^{jk}\nabla_{[j}\dot{h}_{i]k} - \frac{1}{2}(h^{jk}\dot{h}_{jk} - \dot{\lambda})\partial_i\lambda - \partial_i\dot{\lambda} + \dot{h}_{ij}\partial^j\lambda$$

Examples: Schwarschild

Standard Newtonian expansion

$$
\begin{aligned}
ds^2 &= -c^2\left(1-\frac{2m}{c^2\,r}\right)dt^2+\left(1-\frac{2m}{c^2\,r}\right)^{-1}dr^2+r^2d\Omega^2 \\
&= -c^2dt^2+\frac{2m}{r}dt^2+dr^2+r^2d\Omega^2+\mathcal{O}(c^{-2})
\end{aligned}
$$

$$\tau_0 = 1,\ \tau_i = 0 \qquad h_{ij} = \delta_{ij},\ h_{\mu 0} = 0 \qquad \Phi = \tfrac{m}{r},\ C_i = 0$$

- This solves NC eom
- $\Phi = \frac{m}{r}$ Newtonian gravity of point mass
- $d\tau = 0$ $\Rightarrow$ no torsion

Examples: Schwarschild

Extremely massive/strong field expansion: $M = \frac{m}{c^2}$ fixed

$$ds^2 = -c^2\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$\tau_\mu = \left(1-\frac{2M}{r}\right)^{1/2}\delta^t_\mu, \quad h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad C_\mu = 0$$

- This solves TTNC eom
- $\Phi = C_0 = 0$ vanishing Newtonian potential !
- $d\tau \neq 0 \Rightarrow$ non-vanishing torsion!
- curved spatial geometry!

Examples: Kerr

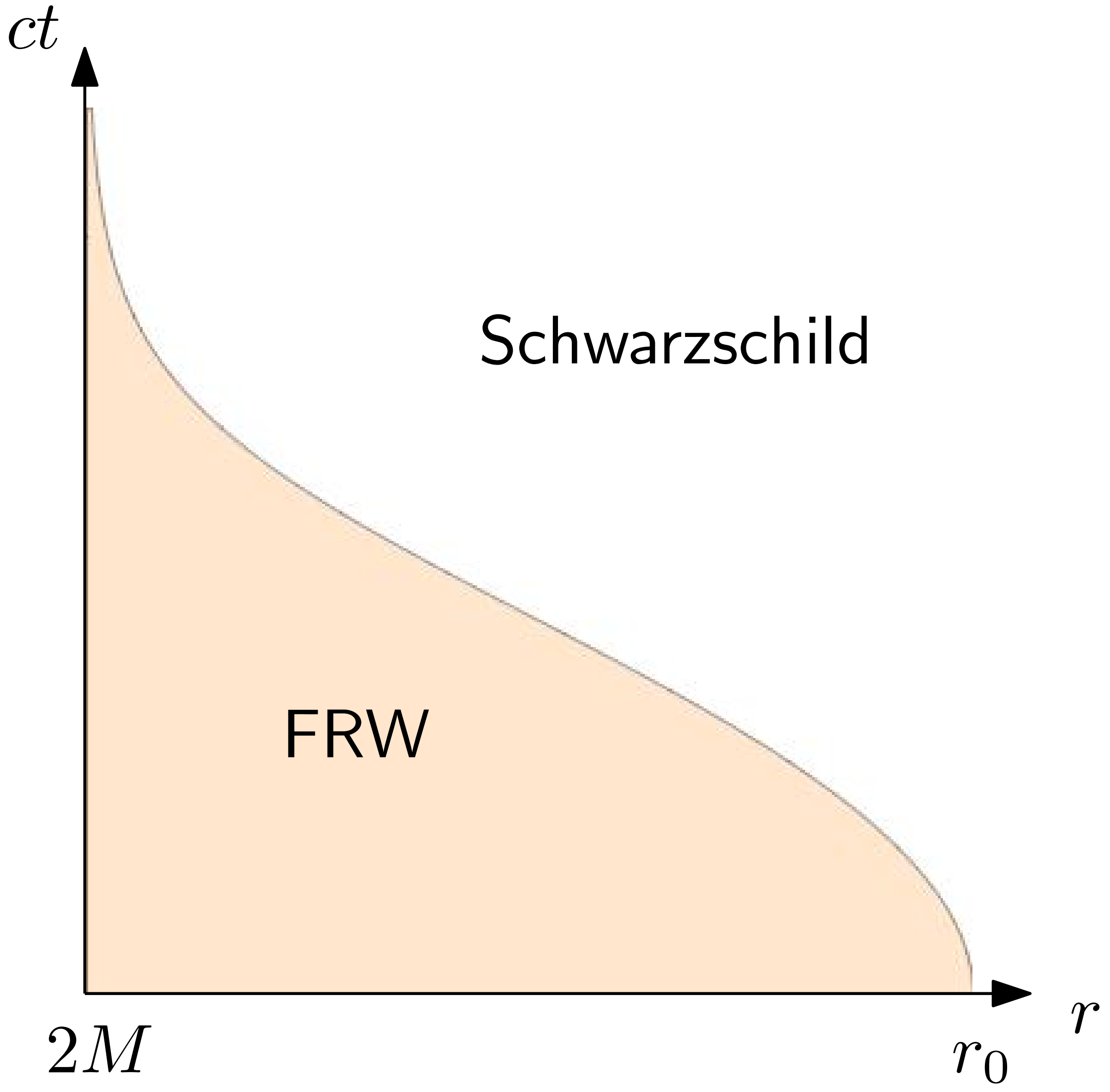
Strong field expansion: $M = m/c^2$ and $A = ac = J/M$ fixed

$$
\begin{aligned}
\lambda &= \log\left(1 - \frac{2M}{r}\right) \\
h_{ij}dx^i dx^j &= dr^2 + \left(1 - \frac{2M}{r}\right) r^2 d\Omega^2 \\
C_i dx^i &= 2\frac{AM}{r}\sin^2\theta \left(1 - \frac{2M}{r}\right)^{-1} d\phi \\
\Phi &= \frac{A^2 M}{r^3}\left(1 - \frac{2M}{r}\right)^{-1} \cos^2\theta
\end{aligned}
$$

- Schwarschild + corrections
- Expansion valid in region $r \gg 2M + \frac{A^2}{2Mc^2} + \mathcal{O}(c^{-4})$

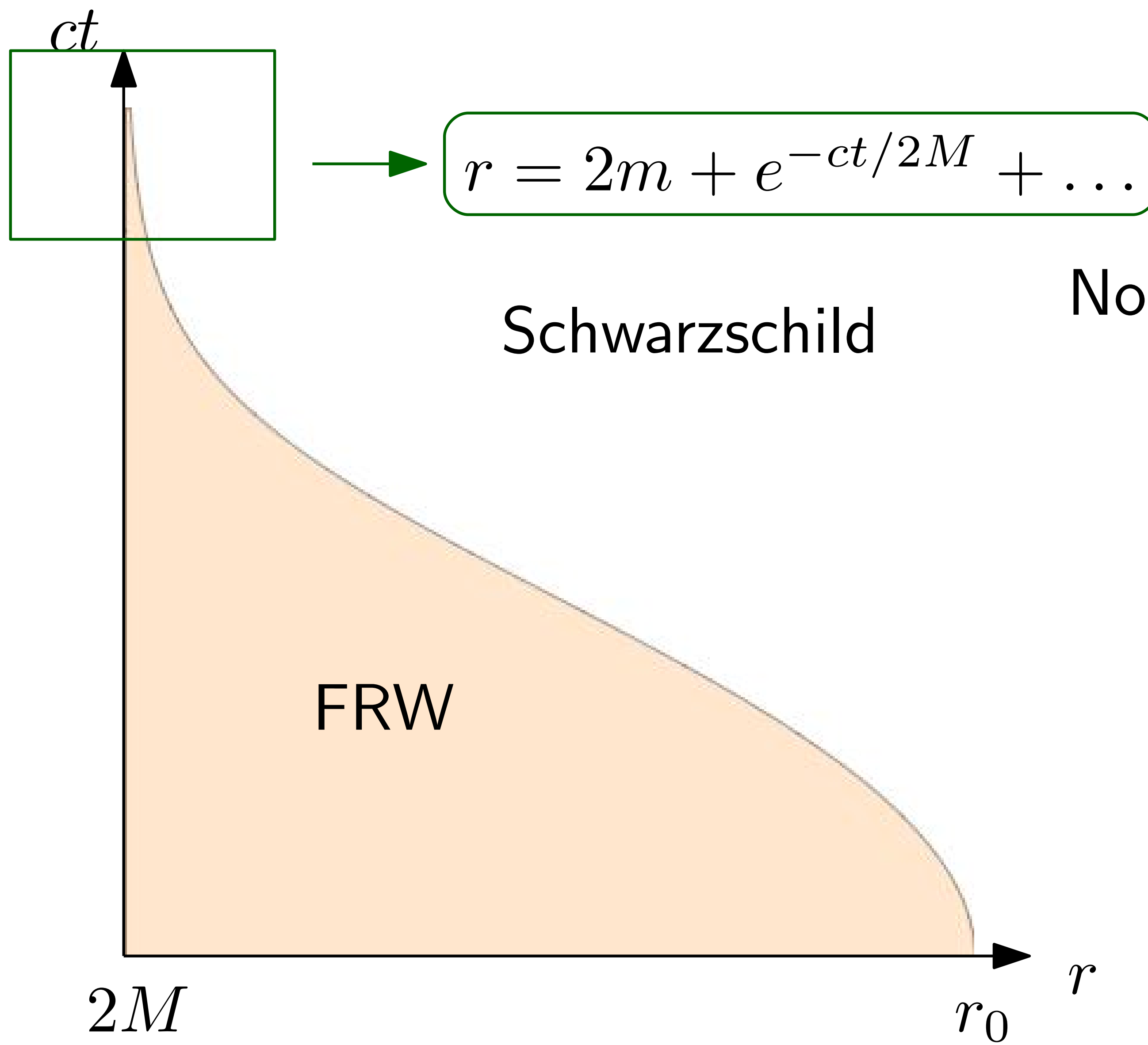
  Compare $\quad r_+ = 2M - \frac{A^2}{2Mc^2} + \mathcal{O}(c^{-4})$

Examples: Oppenheimer-Snyder



Examples: Oppenheimer-Snyder

Strong field expansion: $M = m/c^2$ and r_0 fixed



Non-analytic!!

Summary & outlook

Punch line:

- the large c expansion of GR can be extended beyond the Newtonian regime to include strong time dilation effects. The effective theory describing this is TTNC.

Open questions:

- how to go beyond stationary case?
- need to generalize expansion ansatz: odd powers, transseries?
- interesting applications?

Gaussian normal coordinates

There always exist coordinates such that (on a patch)

$$ds^2 \quad = \quad -c^2 d\sigma^2 + g_{ij} dx^i dx^j$$

$\Rightarrow$ doesn't that imply we can always remove the torsion?

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$$ds^2 \quad = \quad -c^2 d\sigma^2 + g_{ij} dx^i dx^j$$

$\Rightarrow$ doesn't that imply we can always remove the torsion?

The transformation to GN coordinates is not compatible with (standard) large c expansion

Schwarzschild (extremely massive)

$$\begin{aligned} ds^2 \quad &= \quad -c^2 d\sigma^2 + \frac{2M}{r(\sigma,\rho)} d\rho^2 + r(\sigma,\rho)^2 d\Omega^2 \\ &= \quad -c^2 d\sigma^2 + c^{4/3} \left(\frac{9}{2} M\right)^{2/3} \sigma^{4/3} d\Omega^2 + \mathcal{O}(\rfloor^{\infty/\ni}) \end{aligned}$$

$$r(\sigma,\rho) = \left(\tfrac{3}{2}\sqrt{2M}(c\sigma+\rho)\right)^{2/3}$$