

Kinematical (Higher-Spin) Chern-Simons Theories

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Applied Newton-Cartan Geometry
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Collaborators

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1612.02277, 1710.11105 , 1710.11110, work in progress

Outline

Holography & Symmetry

Kinematical Algebras

Chern–Simons Theories

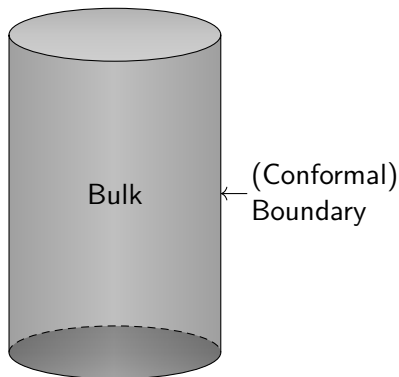
Metric Lie Algebras

Kinematical Chern–Simons Theories

Summary & Outlook

Holographic Principle

- ▶ (Non-gravitational) Boundary theory ($D - 1$) could be sufficient to describe a (gravitational) bulk theory (D)
['t Hooft '93, Susskind '95]
- ▶ Realized:
AdS/CFT correspondence
[Maldacena '98]



How General is Holography?

- ▶ Holographic principle independent of AdS/CFT
- ▶ Insights of AdS/CFT correspondence beyond AdS and CFT?
- ▶ Field theories not always relativistic and conformal (condensed matter systems)
- ▶ AdS one possibility of many geometries
- ▶ Symmetry groups of space and time classified → kinematical algebras [Bacry, Lévy Leblond '67]

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What are Kinematical Algebras?

- ▶ Kinematical Algebras: Take generators of ($D = 3 + 1$)
 - ▶ Time and space translations H, P_i
 - ▶ Spatial rotations J_i
 - ▶ Inertial transformations G_i
- ▶ Assumptions:
 - ▶ Space is isotropic

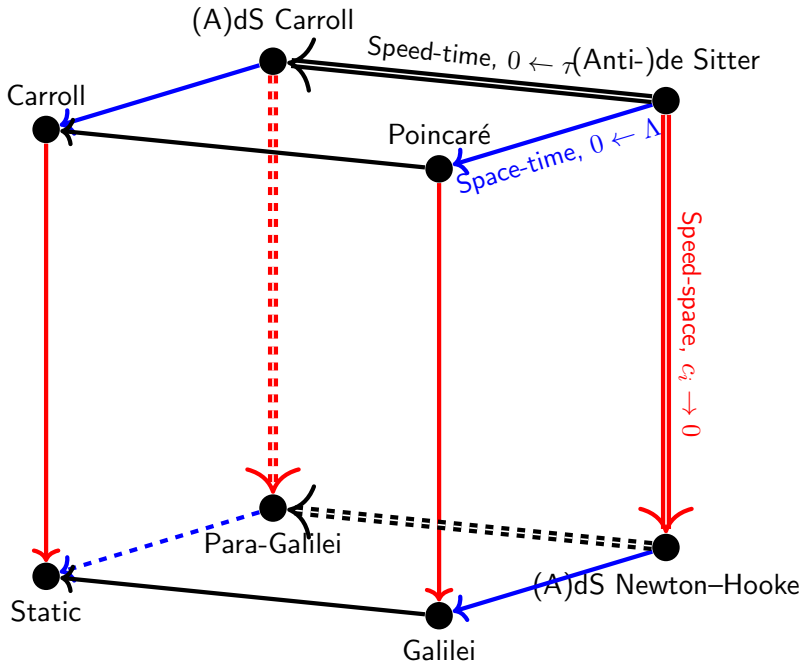
$$[J_i, H] = 0$$

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

$$[J_i, P_j] = \epsilon_{ijk} P_k$$

$$[J_i, G_j] = \epsilon_{ijk} G_k$$

- ▶ Inertial transformations are noncompact
 - ▶ Parity and time-reversal are automorphisms
- ▶ Check for consistent Lie algebras
- ▶ This can be generalized in various ways (José)



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Three spacetime dimensions

- ▶ Technically simpler \rightarrow (maybe) useful for conceptual questions
- ▶ Good balance between complexity and tractability
 - ▶ AdS_3 no gravitational waves [Staruszkiewicz '63] but black holes [Bañados et al '92]
- ▶ Boundary theory possibility $\text{CFT}_2 \rightarrow$ high degree of analytical control
- ▶ Higher spin theories have an (easy) action

Chern–Simons Action

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{M_3} \langle dA \wedge A + \frac{1}{3}[A, A] \wedge A \rangle \quad A = A^a{}_{\mu} T_a \otimes dx^{\mu}$$

Chern–Simons Action — Gauge Algebra

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{M_3} \langle dA \wedge A + \frac{1}{3} [A, A] \wedge A \rangle \quad A = A^a{}_{\mu} \mathbf{T}_a \otimes dx^{\mu}$$

- ▶ Gauge algebra \mathfrak{g} with $[\mathbf{T}_a, \mathbf{T}_b] = f_{ab}{}^c \mathbf{T}_c$ describes the field content

- ▶ Anti-de Sitter $\mathfrak{AdS} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$

$$I_{\text{CS}} \propto \int \sqrt{|g|} \left(R + \frac{2}{\ell^2} \right) d^3x$$

where

$$g_{\mu\nu} = \frac{1}{2} \langle A_{\mu} A_{\nu} \rangle$$

and $\mathfrak{sl}(2, \mathbb{R})$

$$[\mathbf{L}_n, \mathbf{L}_m] = (n - m) \mathbf{L}_{n+m} \quad (n, m = \pm 1, 0)$$

- ▶ De Sitter $\mathfrak{dS} = \mathfrak{sl}(2, \mathbb{C})$ and Poincaré \mathfrak{poi}

Chern–Simons Action — Gauge Algebra

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{M_3} \langle dA \wedge A + \frac{1}{3}[A, A] \wedge A \rangle \quad A = A^a{}_{\mu} \mathbf{T}_a \otimes dx^{\mu}$$

- ▶ Spin-3 on AdS $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$
- ▶ Interpretation: Spin-3 field coupled to gravity (hard in higher dimensions)

$$g_{\mu\nu} = \frac{1}{2} \langle A_{\mu} A_{\nu} \rangle \quad \varphi_{\lambda\mu\nu} = \frac{1}{3!} \langle A_{(\lambda} A_{\mu} A_{\nu)} \rangle$$

- ▶ $\mathfrak{sl}(3, \mathbb{R})$

$$[\mathbf{L}_n, \mathbf{L}_m] = (n - m) \mathbf{L}_{n+m}$$

$$[\mathbf{L}_n, \mathbf{W}_m] = (2n - m) \mathbf{W}_{n+m}$$

$$[\mathbf{W}_n, \mathbf{W}_m] = -(n - m)(2n^2 + 2m^2 - nm - 8) \mathbf{L}_{n+m}$$

Chern–Simons Action — Gauge Algebra

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{M_3} \langle dA \wedge A + \frac{1}{3}[A, A] \wedge A \rangle \quad A = A^a{}_{\mu} \mathbf{T}_a \otimes dx^{\mu}$$

- ▶ Spin-3 gravity in metric formulation [Campoleoni et al. '13, ...]

$$\begin{aligned} I_{\text{CS}} \propto \int d^3x \sqrt{|g|} & \left[\left(R + \frac{2}{\ell^2} \right) + \varphi^{\mu\nu\rho} \left(\mathcal{F}_{\mu\nu\rho} - \frac{3}{2} g_{(\mu\nu} \mathcal{F}_{\rho)} \right) \right. \\ & - \frac{3}{2} R \varphi_{\mu\nu\rho} \varphi^{\mu\nu\rho} + \frac{9}{4} R_{\rho\sigma} (2\varphi^{\rho}{}_{\mu\nu} \varphi^{\sigma\mu\nu} - \varphi^{\rho} \varphi^{\sigma}) \\ & \left. - \frac{1}{\ell^2} (6\varphi_{\mu\nu\rho} \varphi^{\mu\nu\rho} - 9\varphi_{\mu} \varphi^{\mu}) \right] + \mathcal{O}(\varphi^4) \end{aligned}$$

- ▶ Nontrivial generalization of GR (new geometric understanding needed)
- ▶ Toy model for Vasiliev theory

Chern–Simons Action — Invariant Metric

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{M_3} \langle dA \wedge A + \frac{1}{3}[A, A] \wedge A \rangle \quad A = A^a{}_{\mu} \mathbf{T}_a \otimes dx^{\mu}$$

► Invariant metric is bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$

1. Symmetry:

$$\langle X, Y \rangle = \langle Y, X \rangle \quad \text{for all } X, Y \in \mathfrak{g}$$

2. Non-degeneracy:

$$\text{If } \langle X, Y \rangle = 0 \quad \text{for all } Y \in \mathfrak{g} \quad \text{then } X = 0$$

3. Invariance:

$$\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}$$

► Metric Lie algebras

Chern–Simons Action — Invariant Metric

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{M_3} \langle dA \wedge A + \frac{1}{3}[A, A] \wedge A \rangle \quad A = A^a{}_{\mu} \mathbf{T}_a \otimes dx^{\mu}$$

- ▶ Example: Trace for (semi)simple Lie algebras

$$\mathbf{L}_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathbf{L}_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{L}_{+1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\langle \mathbf{L}_a \mathbf{L}_b \rangle = \text{Tr}(\mathbf{L}_a \mathbf{L}_b) = \left(\begin{array}{c|ccc} & \mathbf{L}_{-1} & \mathbf{L}_0 & \mathbf{L}_{+1} \\ \hline \mathbf{L}_{-1} & 0 & 0 & -1 \\ \mathbf{L}_0 & 0 & \frac{1}{2} & 0 \\ \mathbf{L}_{+1} & -1 & 0 & 0 \end{array} \right)$$

- ▶ Trace proportional to Killing form \rightarrow Non-semisimple degenerate \rightarrow Are there Lie algebras beyond semisimple?

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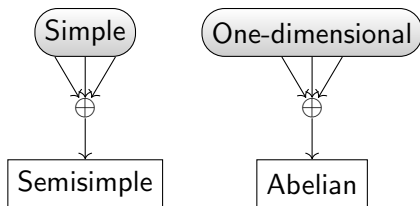
Kinematical Chern–Simons Theories

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Lie algebras with Invariant Metric

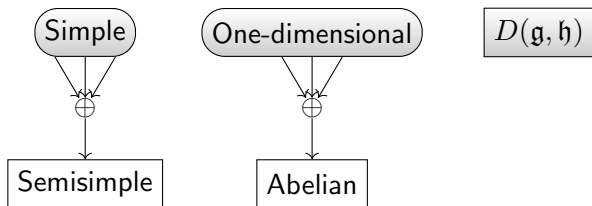
[Medina, Revoy '85; Figueroa-O'Farrill,

Stanciu '95]



Lie algebras with Invariant Metric [Medina, Revoy '85; Figueroa-O'Farrill,

Stanciu '95]



Double Extension $D(\mathfrak{g}, \mathfrak{h})$ [Medina, Revoy '85]

$D(\mathfrak{g}, \mathfrak{h})$ defined on the vector space $\mathfrak{g} \dot{+} \mathfrak{h} \dot{+} \mathfrak{h}^*$ by

$$[G_i, G_j] = f_{ij}^k G_k$$

Double Extension $D(\mathfrak{g}, \mathfrak{h})$ [Medina, Revoy '85]

$D(\mathfrak{g}, \mathfrak{h})$ defined on the vector space $\mathfrak{g} \dot{+} \mathfrak{h} \dot{+} \mathfrak{h}^*$ by

$$[\mathbf{G}_i, \mathbf{G}_j] = f_{ij}^k \mathbf{G}_k$$

$$[\mathbf{H}_\alpha, \mathbf{G}_i] = f_{\alpha i}^j \mathbf{G}_j$$

$$[\mathbf{H}_\alpha, \mathbf{H}_\beta] = f_{\alpha\beta}^\gamma \mathbf{H}_\gamma$$

Double Extension $D(\mathfrak{g}, \mathfrak{h})$ [Medina, Revoy '85]

$D(\mathfrak{g}, \mathfrak{h})$ defined on the vector space $\mathfrak{g} \dot{+} \mathfrak{h} \dot{+} \mathfrak{h}^*$ by

$$[\mathbf{G}_i, \mathbf{G}_j] = f_{ij}{}^k \mathbf{G}_k + f_{\alpha i}{}^k \Omega_{kj}^{\mathfrak{g}} \mathbf{H}^\alpha$$

$$[\mathbf{H}_\alpha, \mathbf{G}_i] = f_{\alpha i}{}^j \mathbf{G}_j$$

$$[\mathbf{H}_\alpha, \mathbf{H}_\beta] = f_{\alpha\beta}{}^\gamma \mathbf{H}_\gamma$$

$$[\mathbf{H}_\alpha, \mathbf{H}^\beta] = -f_{\alpha\gamma}{}^\beta \mathbf{H}^\gamma$$

$$[\mathbf{H}^\alpha, \mathbf{G}_j] = 0$$

$$[\mathbf{H}^\alpha, \mathbf{H}^\beta] = 0$$

Double Extension $D(\mathfrak{g}, \mathfrak{h})$ [Medina, Revoy '85]

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$$[\mathbf{H}_\alpha, \mathbf{G}_i] = f_{\alpha i}{}^j \mathbf{G}_j$$

$$[\mathbf{H}_\alpha, \mathbf{H}_\beta] = f_{\alpha\beta}{}^\gamma \mathbf{H}_\gamma$$

$$[\mathbf{H}_\alpha, \mathbf{H}^\beta] = -f_{\alpha\gamma}{}^\beta \mathbf{H}^\gamma$$

$$[\mathbf{H}^\alpha, \mathbf{G}_j] = 0$$

$$[\mathbf{H}^\alpha, \mathbf{H}^\beta] = 0$$

It has the invariant metric

$$\Omega_{ab}^{\mathfrak{d}} = \begin{matrix} & \mathbf{G}_j & \mathbf{H}_\beta & \mathbf{H}^\beta \\ \mathbf{G}_i & \left(\begin{array}{ccc} \Omega_{ij}^{\mathfrak{g}} & 0 & 0 \\ 0 & h_{\alpha\beta} & \delta_\alpha^\beta \\ 0 & \delta_\beta^\alpha & 0 \end{array} \right) \\ \mathbf{H}_\alpha & & & \\ \mathbf{H}^\alpha & & & \end{matrix}$$

Double Extension $D(\mathfrak{g}, \mathfrak{h})$ [Medina, Revoy '85]

$D(\mathfrak{g}, \mathfrak{h})$ defined on the vector space $\mathfrak{g} \dot{+} \mathfrak{h} \dot{+} \mathfrak{h}^*$ by

$$[\mathbf{G}_i, \mathbf{G}_j] = f_{ij}{}^k \mathbf{G}_k + f_{\alpha i}{}^k \Omega_{kj}^{\mathfrak{g}} \mathbf{H}^\alpha$$

$$[\mathbf{H}_\alpha, \mathbf{G}_i] = f_{\alpha i}{}^j \mathbf{G}_j$$

$$[\mathbf{H}_\alpha, \mathbf{H}_\beta] = f_{\alpha\beta}{}^\gamma \mathbf{H}_\gamma$$

$$[\mathbf{H}_\alpha, \mathbf{H}^\beta] = -f_{\alpha\gamma}{}^\beta \mathbf{H}^\gamma$$

$$[\mathbf{H}^\alpha, \mathbf{G}_j] = 0$$

$$[\mathbf{H}^\alpha, \mathbf{H}^\beta] = 0$$

$D(0, \mathfrak{so}(2, 1))$

$$[\mathbf{J}_A, \mathbf{J}_B] = \epsilon_{AB}{}^C \mathbf{J}_C \quad [\mathbf{J}_A, \mathbf{P}^B] = -\epsilon_{AC}{}^B \mathbf{P}^C \quad [\mathbf{P}^A, \mathbf{P}^B] = 0$$

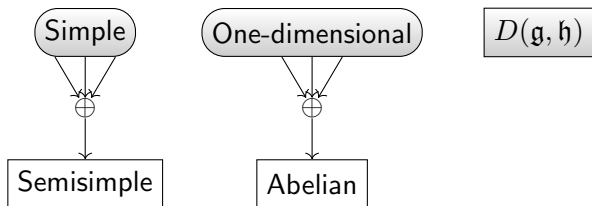
and the invariant metric

$$\langle \mathbf{J}_A, \mathbf{J}_B \rangle = \eta_{AB} \quad \langle \mathbf{J}_A, \mathbf{P}^B \rangle = \delta_A{}^B \quad \langle \mathbf{P}^A, \mathbf{P}^B \rangle = 0.$$

Lie algebras with Invariant Metric

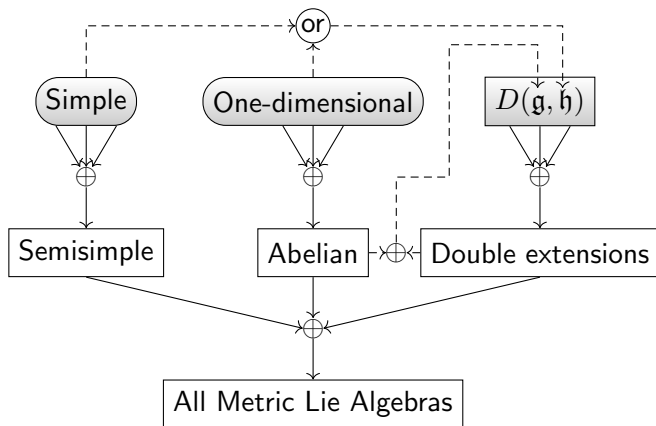
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Lie algebras with Invariant Metric [Medina, Revoy '85; Figueroa-O'Farrill,

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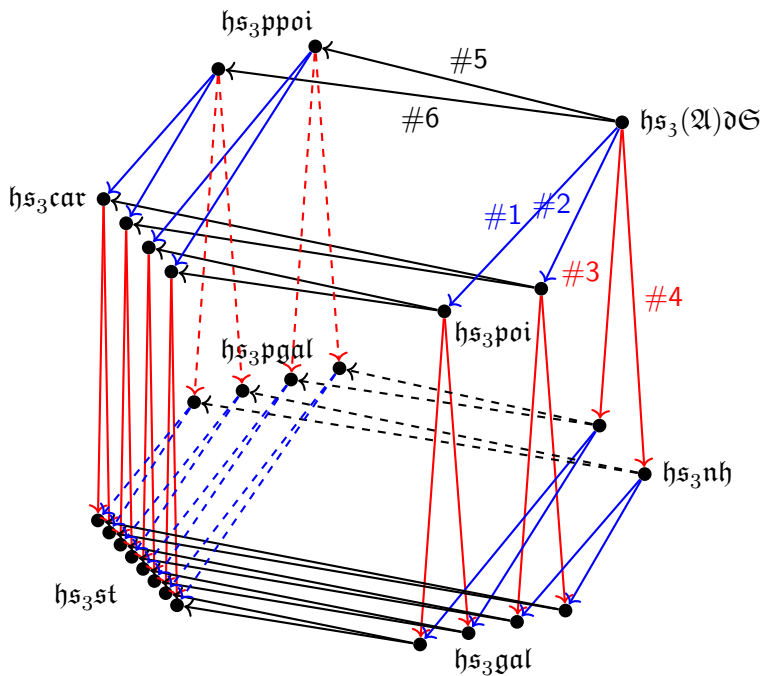
- ▶ So Lie algebras are sometimes metric, sometimes not → which of the kinematical Lie algebras are?
- ▶ Are they connected via contractions/approximations?
 - ▶ In general this restricts the class of possible invariant metrics,
 - ▶ In this case there is no loss of generality.
- ▶ Do the contractions also “work” on the level of the action?
 - ▶ Generically not

Kinematical Chern–Simons Theories

- ▶ What is the actual problem:
 - ▶ Start with (A)dS (rigid)
 - ▶ Contractions of Levy-Leblond, Bacry on the Lie algebra level
 - ▶ Does the invariant metric stay nondegenerate?
 - ▶ This means the action is still well defined
- ▶ “Contraction” not only on the Lie algebra level also on the invariant metric
 - ▶ Well defined contraction of Lie algebra and invariant metric \rightarrow well defined contraction of the action \rightarrow equation of motion !
 - ▶ Algebraic problem
- ▶ Constructive

Kinematical Higher Spin Theories

- ▶ Are there any?
- ▶ Use IW contractions:
 - ▶ Start with higher spin AdS, $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$
 - ▶ Restricted to spin-2 part \rightarrow contraction should be that of kinematical algebras
 - ▶ Resulting contraction not all commutators of the spin-3 part are vanishing.
 - ▶ Under this conditions we classified the contractions



Non-AdS Higher Spin Gravity?

- ▶ Non-AdS Higher Spin Algebras ✓
- ▶ CS action would be nice

$$I_{\text{CS}} = \frac{k}{4\pi} \int_{M_3} \langle dA \wedge A + \frac{1}{3}[A, A] \wedge A \rangle$$

- ▶ Do Lie algebras have an invariant metric?
 - ▶ HS AdS = $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R}) \rightarrow$ semi-simple ✓
 - ▶ HS Poincaré \rightarrow Double extension ✓
 - ▶ HS Poincaré \rightarrow HS Carroll ✓
 - ▶ Invariant metric preserving contraction (natural generalization of IW contraction to double extensions)
 - ▶ HS Galilei ✗

HS Galilei: What can we do?

- ▶ Central extensions, but can be interpreted as double extensions
- ▶ Galilei double extension = Extended Bargmann algebra
- ▶ Naturally generalize spin-2 to higher spin Extended Bargmann algebras
- ▶ Not possible with only central extensions
- ▶ Theories analyzed on linear level [Bergshoeff '16]

Extended Bargmann 1 (Part I)

$$[J, G_a] = \epsilon_{am} G_m$$

$$[J, P_a] = \epsilon_{am} P_m$$

$$[G_a, H] = -\epsilon_{am} P_m$$

$$[G_a, G_b] = \epsilon_{ab} H^*$$

$$[P_a, G_b] = \epsilon_{ab} J^*$$

$$[J, J_a] = \epsilon_{am} J_m$$

$$[J, G_{ab}] = -\epsilon_{m(a} G_{b)m}$$

$$[J, H_a] = \epsilon_{am} H_m$$

$$[J, P_{ab}] = -\epsilon_{m(a} P_{b)m}$$

$$[G_a, J_b] = -(\epsilon_{am} G_{bm} + \epsilon_{ab} G_{mm})$$

$$[G_a, H_b] = -(\epsilon_{am} P_{bm} + \epsilon_{ab} P_{mm})$$

$$[H, J_a] = \epsilon_{am} H_m$$

$$[H, G_{ab}] = -\epsilon_{m(a} P_{b)m}$$

$$[P_a, J_b] = -(\epsilon_{am} P_{bm} + \epsilon_{ab} P_{mm})$$

Extended Bargmann 1 (Part II)

$$[J_a, J_b] = \epsilon_{ab} J$$

$$[J_a, G_{bc}] = \delta_{a(b} \epsilon_{c)m} G_m$$

$$[J_a, P_{bc}] = \delta_{a(b} \epsilon_{c)m} P_m$$

$$[G_{ab}, G_{cd}] = \epsilon_{(a(c} \delta_{d)b)} H^*$$

$$[P_a, G_{bc}] = \epsilon_{a(b} J_c^*$$

$$[G_a, P_{bc}] = \epsilon_{a(b} J_c^*$$

$$[J, J_a^*] = \epsilon_{am} J_m^*$$

$$[J_a, J^*] = -\epsilon_{am} J_m^*$$

$$[J_a, J_b^*] = \epsilon_{ab} J^*$$

$$[H_a, H^*] = -\epsilon_{am} J_m^*$$

$$[J_a, H_b] = \epsilon_{ab} H$$

$$[G_{ab}, H_c] = -\delta_{c(a} \epsilon_{b)m} P_m$$

$$[P_{ab}, G_{cd}] = \epsilon_{(a(c} \delta_{d)b)} J^*$$

$$[G_a, G_{bc}] = \epsilon_{a(b} H_c^*$$

$$[J, H_a^*] = \epsilon_{am} H_m^*$$

$$[H, H_a^*] = \epsilon_{am} J_m^*$$

$$[J_a, H^*] = -\epsilon_{am} H_m^*$$

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- ▶ Kinematical algebras + invariant metrics in $2 + 1$ dimension is rich (see also José)
- ▶ For Bacry and Levy-Leblond algebras + central extensions:
Always well defined action + limit
- ▶ (Partially) generalizes to higher spin theories
- ▶ Contracted Lie algebras with invariant metric generically double extensions
- ▶ Spin-2 Extended Bargmann related to Hořava-Lifshitz gravity
[Lei et al. '15] → Related to Spin-3 Hořava Gravity (?)

Outlook

- ▶ “Do everything that has been done for AdS_3 ”
 - ▶ A lot of tools available
 - ▶ Nonrelativistic (entanglement) entropy (Daniel, Alejandra)?
 - ▶ Holography: Maybe stick to negative cosmological constant (?)
- ▶ Solution space (Coset constructions)
- ▶ Boundary conditions (done for Carroll Spin-2)
- ▶ Boundary theories
 - ▶ Interesting math (Drinfeld Sokolov reduction for nonsemisimple Lie algebras)
- ▶ In general interesting math (non semisimple Lie algebras, but restricted to kinematical)
- ▶ Fronsdal-like equations for higher spins
- ▶ Lie superalgebras
 - ▶ Double superextensions
- ▶ Double extensions and deformations/contractions

Thank you very much for your attention.