Model independent extraction of the proton magnetic radius from electron scattering

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[Zachary Epstein, GP, Joydeep Roy, to appear]

Outline

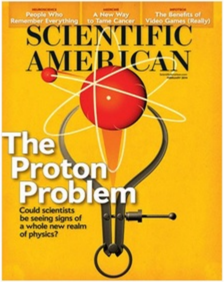
- Introduction
- Reminder: Model independent extraction of the electric radius
- Model independent extraction of the magnetic radius
- Conclusions and outlook

Introduction

Motivation

- Discrepancy between the proton electric radius as extracted from regular and muonic hydrogen

- Almost 4 years after first measurement puzzle is still not resolved

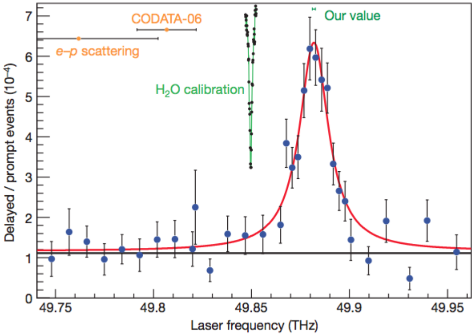


(Cover story of February 2014 Scientific American)

Motivation

- Great outreach opportunity!
  Problem easily communicated to a more general audience

- Example: Detroit high school students using data



[R. Pohl *et al.*, "The size of the proton," Nature **466**, 213 (2010)]
and the approximate formula, $f = 50.59\ \text{THz} - r^2 \dfrac{\text{THz}}{\text{fm}^2}$
to determine $r = 0.84$ fm

Form Factors

- Matrix element of EM current between nucleon states give rise to two form factors ($q = p_f - p_i$)

$$\langle N(p_f)|\sum_q e_q\,\bar{q}\gamma^\mu q|N(p_i)\rangle = \bar{u}(p_f)\left[\gamma^\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}}{2m}F_2(q^2)q^\nu\right]u(p_i)$$

- Sachs electric and magnetic form factors

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2) \qquad G_M(q^2) = F_1(q^2) + F_2(q^2)$$

$$G_E^p(0) = 1 \qquad G_M^p(0) = \mu_p \approx 2.793$$

- The slope of G_E^p and G_M^p

$$\langle r^2\rangle_E^p = 6\frac{dG_E^p(q^2)}{dq^2}\bigg|_{q^2=0}, \qquad \langle r^2\rangle_M^p = \frac{6}{G_M^p(0)}\frac{dG_M^p(q^2)}{dq^2}\bigg|_{q^2=0}$$

determines the charge radius $r_E^p \equiv \sqrt{\langle r^2\rangle_E^p}$

and magnetic radius $r_M^p \equiv \sqrt{\langle r^2\rangle_M^p}$

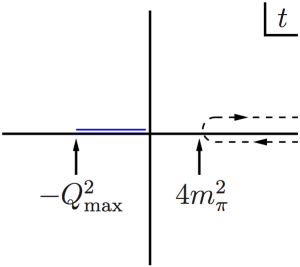
Form Factors: What we don't know

- The form factors are non-perturbative objects.
- **Nobody** knows the *exact* functional form of G_E^p and G_M^p
- They don't have to have a dipole/polynomial/spline or any other functional form
- Including such models can bias your extraction of r_E^p and r_M^p

Form Factors: What we do know

- Analytic properties of $G_E^p(t)$ and $G_M^p(t)$ are known
- They are analytic outside a cut $t \in [4m_\pi^2, \infty]$

  [Federbush, Goldberger, Treiman, Phys. Rev. **112**, 642 (1958)]
- $e - p$ scattering data is in $t < 0$ region



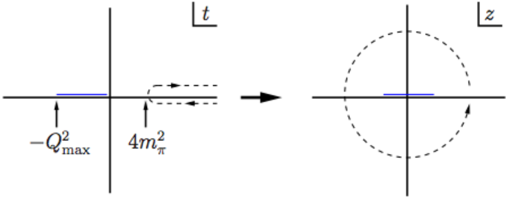
- If your form factor doesn't have this analytic structure it's **wrong!** (e.g. singularity at $4m_\pi^2$: why should the Taylor series converge?)

z expansion

- Even with the right analytic structure you have to be careful
  e.g. modeling $\text{Im}\,G(t)$ as poles+continuum form
  how to estimate model dependence?
- A better approach: z expansion
  We can map the domain of analyticity onto the unit circle

$$z(t, t_{\text{cut}}, t_0) = \frac{\sqrt{t_{\text{cut}} - t} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} - t} + \sqrt{t_{\text{cut}} - t_0}}$$

where $t_{\text{cut}} = 4m_\pi^2$, $z(t_0, t_{\text{cut}}, t_0) = 0$



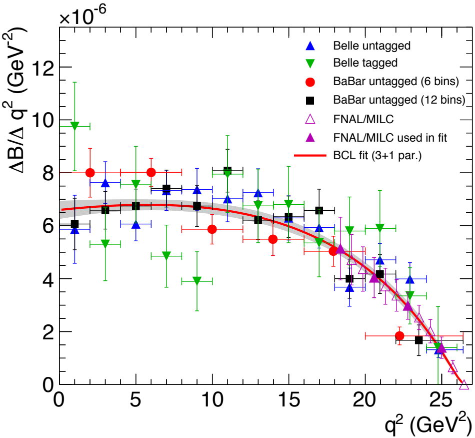
- Expand $G^p_{E,M}$ in a Taylor series in z: $G^p_{E,M}(q^2) = \sum_{k=0}^{\infty} a_k\, z(q^2)^k$

z expansion

- **Standard** tool in analyzing **meson** transition form factors
  - Bourrely et al., NPB **189**, 157 (1981)
  - Boyd et al., PRL **74**, 4603 (1995)
  - Boyd et al., NPB **461**, 493 (1996)
  - Lellouch et al., NPB **479**, 353 (1996)
  - Caprini et al., NPB **530**, 153 (1998)
  - Arnesen et al., PRL **95**, 071802 (2005)
  - Becher et al., PLB **633**, 61 (2006)
  - Hill, PRD **74**, 096006 (2006)
  - Bourrely et al., PRD **79**, 013008 (2009)
  - Bharucha et al., JHEP **1009**, 090 (2010)
  - ...

z expansion

- For meson form factors, z expansion **is** the method
- E.g. $|V_{ub}|$ from exclusive $B \to \pi \ell \bar{\nu}$



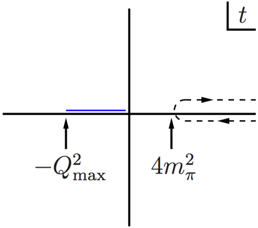
[Heavy Flavor Averaging Group, arXiv:1207.1158]

z expansion

- Recall $G_{E,M}^p(q^2) = \sum_{k=0}^{\infty} a_k \, z(q^2)^k$
  For a fit independent of k we need to bound a_k

- For meson form factor such as $B \to \pi$ unitarity implies a bound on $\sum_{k=0}^{\infty} a_k^2$

- For the nucleon form factors only the region above the two-nucleon threshold is constrained by unitarity



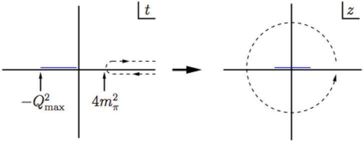
We can still establish a bound on a_k

Reminder: Model independent extraction of the electric radius

[Hill, GP PRD **82** 113005 (2010)]

Analytic structure and a_k

$$z(t, t_{\rm cut}, t_0) = \frac{\sqrt{t_{\rm cut} - t} - \sqrt{t_{\rm cut} - t_0}}{\sqrt{t_{\rm cut} - t} + \sqrt{t_{\rm cut} - t_0}}$$



- Analytic structure implies:
  Information about $\mathrm{Im}\, G_E^p(t + i0) \Rightarrow$ information about a_k
- $G(t) = \sum_{k=0}^{\infty} a_k\, z(t)^k$, z^k are orthogonal over $|z| = 1$

$$\begin{aligned} a_0 &= G(t_0) \\ a_k &= \frac{2}{\pi} \int_{t_{\rm cut}}^{\infty} \frac{dt}{t - t_0} \sqrt{\frac{t_{\rm cut} - t_0}{t - t_{\rm cut}}}\, \mathrm{Im}\, G(t) \sin[k\theta(t)]\,, \quad k \geq 1 \\ \sum_k a_k^2 &= \frac{1}{\pi} \int_{t_{\rm cut}}^{\infty} \frac{dt}{t - t_0} \sqrt{\frac{t_{\rm cut} - t_0}{t - t_{\rm cut}}} |G|^2 \end{aligned}$$

- How to constrain $\mathrm{Im}\, G(t)$?

Size of a_k: vector dominance ansatz

- The isovector and isoscalar form factors are

$$G_E^{(0)} = G_E^p + G_E^n\,, \quad G_E^{(1)} = G_E^p - G_E^n$$

- Assume vector dominance ansatz [Höhler NPB **95**, 210 (1975)]

$$F_i^{(I=0)} \sim \frac{\alpha_i m_\omega^2}{m_\omega^2 - t - i\Gamma_\omega m_\omega}\,, \quad F_i^{(I=1)} \sim \frac{\beta_i m_\rho^2}{m_\rho^2 - t - i\Gamma_\rho m_\rho}\,,$$

α_i and β_i are fixed by $F_i^I(0)$

- For $G(t) \sim 1/(t - m_V^2)$, $\mathrm{Im}\, G(t + i0) = -i\pi\delta(t - m_V^2)$
$\Rightarrow |a_k/a_0| \leq 2\sqrt{(t_\mathrm{cut} - t_0)/(m_v^2 - t_\mathrm{cut})}$
Taking $t_0 = 0$: $|a_k| < 1.3$ for $G_E^{(0)}$,$|a_k| < 0.78$ for $G_E^{(1)}$

- Conclusion: $|a_k| \leq 10$ is a very conservative estimate for this ansatz

Size of a_k: $\pi\pi$ continuum

- $\pi\,\pi$ is the lightest state that can contribute to $\mathrm{Im}\,G_E^{(1)}$

$$\mathrm{Im}\,G_E^{(1)}(t) = \frac{2}{m_N\sqrt{t}}\left(t/4 - m_\pi^2\right)^{\frac{3}{2}} F_\pi(t)^* f_+^1(t)$$

$F_\pi(t)$ pion form factor, $f_+^1(t)$ is a partial amplitude for $\pi\pi \to N\bar{N}$ [Federbush et al. Phys. Rev. **112**, 642 (1958), Frazer et al. Phys. Rev. **117**, 1609 (1960), Belushkin et al. PRC **75**, 035202 (2007)]

- Since they share the same phase up to $t < 16m_\pi^2$, we can use $|F_\pi|$ (For determining bound on a_k we assume phase equality through ρ peak)
- Using $|F_\pi(t)|$ data from
  - NA7 experiment [Amendolia et al. PLB **138**, 454 (1984)]
  - SND experiment [Achasov et al. arXiv:hep-ex/0506076]
- Using $f_+^1(t)$ tables from [G. Höhler, Pion-nucleon scattering, Springer-Verlag, Berlin, 1983]
- For $t_0 = 0$: $a_0 \approx 2.1$ $a_1 \approx -1.4$, $a_2 \approx -1.6$, $a_3 \approx -0.9$, $a_4 \approx 0.2$ Using $|\sin(k\theta)| \leq 1$ in the integral gives $|a_k| \lesssim 2.0$ for $k \geq 1$.

Size of a_k: $t > 4m_N^2$ region

- For the region $t > 4m_N^2$ we can use $e^+e^- \to N\bar{N}$ data, e.g.
- $p - \bar{p}$: BES collaboration [Ablikim et al. arXiv:hep-ex/0506059]
- $n - \bar{n}$: FENICE experiment [Antonelli et al. NPB **517**, 3 (1998)]

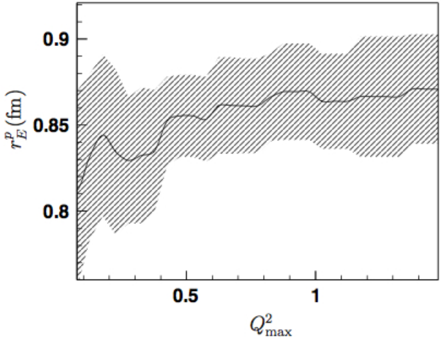
- We find a very small contribution from this region
- $|\delta a_k| \lesssim 0.006 + 0.002$ for the proton
- $|\delta a_k| \lesssim 0.013 + 0.025$ for the neutron

Size of a_k: Summary

- In all of the above $|a_k| \leq 10$ appears very conservative
- In practice we find max $|a_k| \sim 2$
- Final results are presented for both $|a_k| \leq 5$ and $|a_k| \leq 10$

Results: proton data

- Use tables from [Arrington et al. PRC **76**, 035205 (2007)]
  We fit with $k_{\rm max} = 10,\ t_0 = 0,\ |a_k| \le 10$



- Beyond $Q^2 \gtrsim {\rm few} \times 0.1\,{\rm GeV}^2$ the impact of additional data is minimal

$$\text{For } Q^2_{\rm max} = 0.5\,{\rm GeV}^2: \quad r_E^p = 0.870 \pm 0.023 \pm 0.012\,{\rm fm}$$

- Notice: If using only proton data, r_E^p is only 1σ away from μH!

Results: Proton and Neutron data

- Including neutron data $\Rightarrow$ fit $G_E^{(0)}$ and $G_E^{(1)}$ separately
  For isoscalar $t_{\rm cut} = 9m_\pi^2 \Rightarrow$ smaller value of $|z|_{\rm max}$
- Using

- G_E^p up to $Q^2_{\rm max} = 0.5\,{\rm GeV}^2$

- 20 data points for G_E^n

- Neutron charge radius from [PDG 2010]

$$\langle r^2 \rangle_E^n = -0.1161(22)\,{\rm fm}^2\,.$$

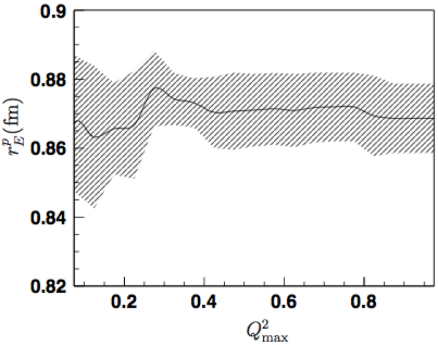
- We get

$$r_E^p = 0.880^{+0.017}_{-0.020} \pm 0.007\,{\rm fm}$$

Results: Proton, Neutron and $\pi\,\pi$ data

- $\pi\,\pi$ data allows us to set $t_{\rm cut} = 16 m_\pi^2$ for $G_E^{(1)}$

$$G_E^{(1)}(t) = G_{\rm cut}(t) + \sum_k a_k z^k(t, t_{\rm cut} = 16 m_\pi^2, t_0)$$



- We get: $r_E^p = 0.871 \pm 0.009 \pm 0.002 \pm 0.002\,\text{fm}$

  Last error 30% normalization for $f_1^+(t)$

Results: Summary

- Proton: $Q^2 < 0.5\,\mathrm{GeV}^2$

$$r_E^p = 0.870 \pm 0.023 \pm 0.012\,\mathrm{fm}$$

- Proton and neutron data

$$r_E^p = 0.880^{+0.017}_{-0.020} \pm 0.007\,\mathrm{fm}$$

- Proton, neutron and $\pi\,\pi$ data

$$r_E^p = 0.871 \pm 0.009 \pm 0.002 \pm 0.002\,\mathrm{fm}$$

Model independent extraction of the proton magnetic radius from electron scattering

[Zachary Epstein, GP, Joydeep Roy, to appear]

The proton magnetic radius problem

- The proton *magnetic* radius

$$\langle r^2 \rangle_M^p = \frac{6}{G_M^p(0)} \frac{dG_M^p(q^2)}{dq^2}\bigg|_{q^2=0}$$

- PDG 2012:

- Recent high precision data from A1 experiment at Mainz
  $r_M^p = 0.777 \pm 0.017$ fm [Bernauer et al. PRL **105**, 242001 (2010)]
  Older data sets
- $r_M^p = 0.876 \pm 0.019$ fm [Borisyuk NPA **843**, 59 (2010)]
- $r_M^p = 0.854 \pm 0.005$ fm [Belushkin et al. PRC **75**, 035202 (2007)]
  Are we facing a magnetic radius puzzle too?

- We need a model independent extraction of r_M^p!

Model independent extraction of r_M^p

- Analysis follows [Hill, GP PRD **82** 113005 (2010)]
- Fitting reported magnetic form factor data in the literature
- Check the effect of proton; proton and neutron; and proton, neutron, and $\pi\pi$ data on r_M^p

Bound on $|a_k|$: General

- Recall $G_M^p(0) = \mu_p \approx 2.793$, $G_M^n(0) = \mu_n \approx -1.913$.

- Analyzing p and n data, separate G_M^p and G_M^n to isospin channels

$$G_M^{(0)} = G_M^p + G_M^n \qquad G_M^{(1)} = G_M^p - G_M^n$$
$$G_M^{(0)}(0) = \mu_p + \mu_n \approx 0.88 \qquad G_M^{(1)}(0) = \mu_p - \mu_n \approx 4.7$$
$$\Rightarrow I = 0, \quad a_0 = 0.88 \qquad \Rightarrow I = 1, \quad a_0 = 4.7$$

- We use

- Vector dominance ansatz
- $\pi\pi$ continuum
- $e^+e^- \to N\bar{N}$

Bound on $|a_k|$: vector dominance ansatz

- Vector dominance ansatz

$$\mathrm{Im}\, G(t+i0) = \frac{\mathcal{N} m_V^3 \Gamma_V}{(t-m_V^2)^2 + \Gamma_V^2 m_V^2}\,\theta(t-t_{\rm cut})$$

Using dispersion relation can find $G(t+i0)$ analytically
[Bhattacharya, Hill, GP PRD **84**, 073006 (2011)]

$$\left|\frac{a_k}{a_0}\right| \le \frac{2|\mathcal{N}|}{|G_M(t_0)|}\mathrm{Im}\left(\frac{-m_V^2}{t - m_V^2 + i\Gamma_V m_V + \sqrt{(t_{\rm cut}-t_0)(t-m_V^2+i\Gamma_V m_V)}}\right)$$

- We find

- $I=0$ (ω exchange) $|a_k/a_0| \le 1.3 \Rightarrow |a_k| \le 1.1$
- $I=1$ (ρ exchange) $|a_k/a_0| \le 1.1 \Rightarrow |a_k| \le 5.1$

Bound on $|a_k|$: $\pi\pi$ continuum

- $\pi\,\pi$ is the lightest state that can contribute to $\mathrm{Im}\, G_M^{(1)}$

$$\mathrm{Im}\; G_M^{(1)}(t) = \sqrt{\frac{2}{t}}\left(t/4 - m_\pi^2\right)^{\frac{3}{2}} F_\pi(t)^* f_-^1(t)$$

  $F_\pi(t)$ pion form factor, $f_-^1(t)$ is a partial amplitude for $\pi\pi \to N\bar{N}$ [Federbush et al. Phys. Rev. **112**, 642 (1958), Frazer et al. Phys. Rev. **117**, 1609 (1960), Belushkin et al. arXiv:hep-ph/0608337]
- Since they share the same phase up to $t < 16 m_\pi^2$, we can use $|F_\pi|$ (For determining bound on a_k we assume phase equality through ρ peak)
- Using $|F_\pi(t)|$ data from
  - NA7 experiment [Amendolia et al. PLB **138**, 454 (1984)]
  - SND experiment [Achasov et al. arXiv:hep-ex/0506076]
- Using $f_-^1(t)$ tables from [G. Höhler, Pion-nucleon scattering, Springer-Verlag, Berlin, 1983]
- For $t_0 = 0$: $a_0 \approx 7.9$, $a_1 \approx -5.5$, $a_2 \approx -6.0$, $a_3 \approx -2.9$, $a_4 \approx 1.2$ Using $|\sin(k\theta)| \leq 1$ in the integral gives $|a_k| \lesssim 7.2$ for $k \geq 1$.

Bound on $|a_k|$: $e^+e^- \to N\bar{N}$

- For the region $t > 4m_N^2$ we can use $e^+e^- \to N\bar{N}$ data, e.g.
  - $p - \bar{p}$: BES collaboration [Ablikim et al. arXiv:hep-ex/0506059]
  - $n - \bar{n}$: FENICE experiment [Antonelli et al. NPB **517**, 3 (1998)]

- We find a very small contribution from this region
  - $|\delta a_k| \lesssim 0.013 + 0.004$ for the proton
  - $|\delta a_k| \lesssim 0.011 + 0.047$ for the neutron

Bound on $|a_k|$: Summary

- Vector dominance ansatz:
  - $I = 0$ (ω exchange) $|a_k/a_0| \leq 1.3 \Rightarrow |a_k| \leq 1.1$
  - $I = 1$ (ρ exchange) $|a_k/a_0| \leq 1.1 \Rightarrow |a_k| \leq 5.1$
- Between $t = 4m_\pi^2$ and $t = 16m_\pi^2$ only $\pi\pi$ contributes

  $I = 1$: $|a_k| \leq 7.2$
- Above $t = 4m_N^2$ use $e^+e^- \to N\bar{N}$: negligible contribution to a_k
- Two options
  - Use $|a_k| \leq 10$ and $|a_k| \leq 15$ (default)
  - Use $|a_k/a_0| \leq 5$ and $|a_k/a_0| \leq 10$ (used as a check)

r_M^p from proton data (*Preliminary*)

- $G_M^p(q^2)$ values from $e - p$ scattering data
  [Arrington et al. PRC **76**, 035205 (2007)]

- Extracted values don't depend on number of parameters
  (results shown for for $k_{\max} = 8$)

- $Q^2 \leq 0.5$ GeV^2
- $|a_k| \leq 10$: $r_M^p = 0.91^{+0.03}_{-0.06}$ fm
- $|a_k| \leq 15$: $r_M^p = 0.92^{+0.04}_{-0.07}$ fm

- $Q^2 \leq 1.0$ GeV
- $|a_k| \leq 10$: $r_M^p = 0.90^{+0.03}_{-0.07}$ fm
- $|a_k| \leq 15$: $r_M^p = 0.91^{+0.04}_{-0.07}$ fm

r_M^p from proton and neutron data (*Preliminary*)

- $G_M^p(q^2)$ from [Arrington et al. PRC **76**, 035205 (2007)]
- $G_M^n(q^2)$ from [Lachniet et al. PRL **102** 192001 (2009); Anderson et al. PRC**75**, 034003 (2007); Kubon et al. PLB **524**, 26 (2002); Xu et al. PRL **85**, 2900 (2000); Anklin et al. PLB **428**, 248 (1998); Anklin et al. PLB **336**, 313 (1994); Gao et al. PRC **50**, 546 (1994); Lung et al. PRL **70**, 718 (1993)]

- Fit both $G_M^{(0)}$ and $G_M^{(1)}$

- $Q^2 \leq 0.5\ \text{GeV}^2$
  - $|a_k| \leq 10$: $r_M^p = 0.87^{+0.04}_{-0.05}$ fm
  - $|a_k| \leq 15$: $r_M^p = 0.87^{+0.05}_{-0.05}$ fm

- $Q^2 \leq 1.0$ GeV
  - $|a_k| \leq 10$: $r_M^p = 0.88^{+0.02}_{-0.05}$ fm
  - $|a_k| \leq 15$: $r_M^p = 0.88^{+0.04}_{-0.05}$ fm

r_M^p from proton and neutron and $\pi\pi$ data (*Preliminary*)

- $G_M^p(q^2)$ from [Arrington et al. PRC **76**, 035205 (2007)]
- $G_M^n(q^2)$ from [Lachniet et al. PRL **102** 192001 (2009); Anderson et al. PRC**75**, 034003 (2007); Kubon et al. PLB **524**, 26 (2002); Xu et al. PRL **85**, 2900 (2000); Anklin et al. PLB **428**, 248 (1998); Anklin et al. PLB **336**, 313 (1994); Gao et al. PRC **50**, 546 (1994); Lung et al. PRL **70**, 718 (1993)]
- $\mathrm{Im}\, G_M^{(1)}$ between $t = 4m_\pi^2$ and $t = 16m_\pi^2$ from $\pi\pi$ data [Höhler, Landolt-Börnstein database Vol. 9b1 (1983); Amendolia et al. PLB **138**, 454 (1984); Achasov et al. JETP **101**, 1053 (2005)]

- $Q^2 \leq 0.5$ GeV^2
  - $|a_k| \leq 10$: $r_M^p = 0.87^{+0.01}_{-0.02}$ fm
  - $|a_k| \leq 15$: $r_M^p = 0.87^{+0.01}_{-0.02}$ fm

- $Q^2 \leq 1.0$ GeV
  - $|a_k| \leq 10$: $r_M^p = 0.87^{+0.01}_{-0.01}$ fm
  - $|a_k| \leq 15$: $r_M^p = 0.88^{+0.01}_{-0.02}$ fm

r_M^p extraction: comments

- Our results
- do not depend on the number of parameters
- are very consistent over the range of Q^2
- barely change (less than 1 σ)

  using $|a_k| \leq 20$, or $|a_k/a_0| \leq 5$, or $|a_k/a_0| \leq 10$

- The reduction in the error bar by inclusion of $\pi\pi$ data arises from the increase in $t_{\rm cut}$ to $16m_\pi^2$ for $G_M^{(1)}$

Conclusions and outlook

Conclusions

- Proton electric radius problem not resolved yet
  Are we facing a magnetic radius puzzle too?

- *Preliminary* results from model independent extraction
- Proton data :$r_M^p = 0.91^{+0.03}_{-0.06} \pm 0.02$ fm
- Proton and neutron data: $r_M^p = 0.87^{+0.04}_{-0.05} \pm 0.01$ fm
- Proton, neutron and $\pi\,\pi$ data: $r_M^p = 0.87^{+0.01}_{-0.02}$ fm

- Consistent results, independent of $k_{\rm max}$ and cut on Q^2

- Error larger than the r_E^p extraction, but exhibits similar features

- Another successful test of the z expansion
  (See also Gabriel Lee's talk)

Outlook

- *Preliminary* results from model independent extraction
- Proton data :$r_M^p = 0.91^{+0.03}_{-0.06} \pm 0.02$ fm
- Proton and neutron data: $r_M^p = 0.87^{+0.04}_{-0.05} \pm 0.01$ fm
- Proton, neutron and $\pi\pi$ data: $r_M^p = 0.87^{+0.01}_{-0.02}$ fm

- PDG 2012:
- $r_M^p = 0.777 \pm 0.017$ fm [Bernauer et al. PRL **105**, 242001 (2010)]
- $r_M^p = 0.876 \pm 0.019$ fm [Borisyuk NPA **843**, 59 (2010)]
- $r_M^p = 0.854 \pm 0.005$ fm [Belushkin et al. PRC **75**, 035202 (2007)]

- Other non-PDG values:
- $r_M^p = 0.855 \pm 0.035$ fm [Sick Prog.Part.Nucl.Phys. **55**, 440 (2005)]
- $r_M^p = 0.86^{+0.02}_{-0.03}$ fm [Lorenz et al. EPJA **48**, 151 (2012)]
- $r_M^p = 0.78 \pm 0.08$ fm [Karshenboim arXiv:1405.6515]

- Future directions: analyze other data sets using the z expansion