Model-Independent Fits to ep Scattering Cross-Section Data

Proton Radius Puzzle Workshop, Mainz, Germany

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Outline

Determination of $\langle r_P \rangle$ Using ep Scattering

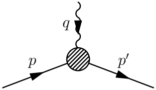
- The Mott cross-section for scattering of a relativistic electron off a recoiling proton is

$$\left(\frac{d\sigma}{d\Omega}\right)_M = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \cos^2 \frac{\theta}{2} \frac{E'}{E} .$$

- The Rosenbluth formula generalizes the above,

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \left(\frac{d\sigma}{d\Omega}\right)_M \frac{1}{1+\tau}\left[G_E^2 + \frac{\tau}{\epsilon} G_M^2\right],\ \tau = \frac{-q^2}{4M^2},\ \epsilon = \frac{1}{1 + 2(1+\tau)\tan^2 \frac{\theta}{2}} .$$

- The Sachs form factors $G_E(q^2), G_M(q^2)$ account for the finite size of the proton. In terms of the standard Dirac (F_1) and Pauli (F_2) form factors,



$$= \Gamma^\mu(q^2) = \underbrace{\frac{G_E + \tau G_M}{1+\tau}}_{F_1(q^2)} \gamma^\mu + \frac{i}{2M} \sigma^{\mu\nu} q_\nu \underbrace{\frac{G_M - G_E}{1+\tau}}_{F_2(q^2)} .$$

- The radii are defined by

$$\langle r^2 \rangle = \frac{6}{G(0)} \frac{\partial G}{\partial q^2}\bigg|_{q^2=0} , \quad G_E^p(0) = 1, G_M^p(0) = \mu_p .$$

Radiative Corrections

- The experimentally measured cross sections include radiative corrections. The form factors that we use to fit these data should be defined in a way to account for these corrections.
- A consistent definition of the form factors is required to compare extracted radii.
- In the following, we'll compare existing conventions in two analyses.

The Current State of Radiative Corrections

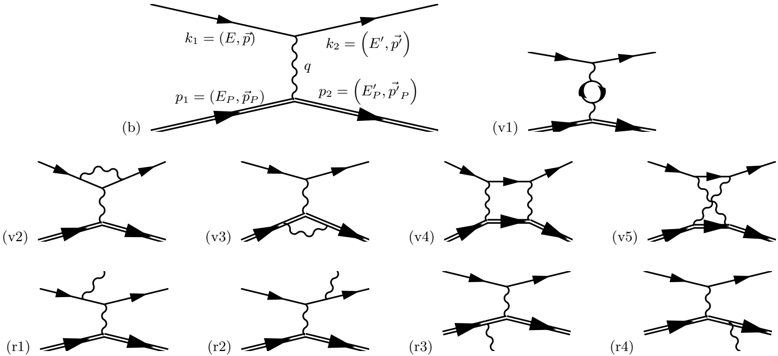


Fig. 3 from Bernauer et al.,1307.6227: (L-R), top-down

- Born, vac. pol.
- e vertex, p vertex, TPE box & cross
- Inelastic cross section, e and p bremsstrahlung

The Current State of Radiative Corrections (cont.)

$$\mathcal{M}_0 = -\frac{4\pi\alpha}{q^2}\bar{u}(k')\gamma^\mu u(k)\cdot\bar{u}(p')\Gamma_\mu^{(p)}(q)u(p)\,,\quad \mathcal{M}_1 = \sum_{i=1}^{5}\mathcal{M}_{\mathrm{v}i} + \sum_{j=1}^{4}\mathcal{M}_{\mathrm{r}j}$$

- There are two comprehensive calculations of the diagrams in the previous slide by Tsai (1961), and by Maximon & Tjon (2000) [MaTj], both treating the proton as a propagating Dirac particle.
- The MaTj calculation claims improvement since it applies the soft photon approximation differently – this essentially amounts to expressing integrals as Passarino-Veltman 4-point functions instead of 3-point functions.
- Two compilations of ep scattering cross sections: "world" (provided by J. Arrington), using Tsai, and Mainz (MAMI), using MaTj.

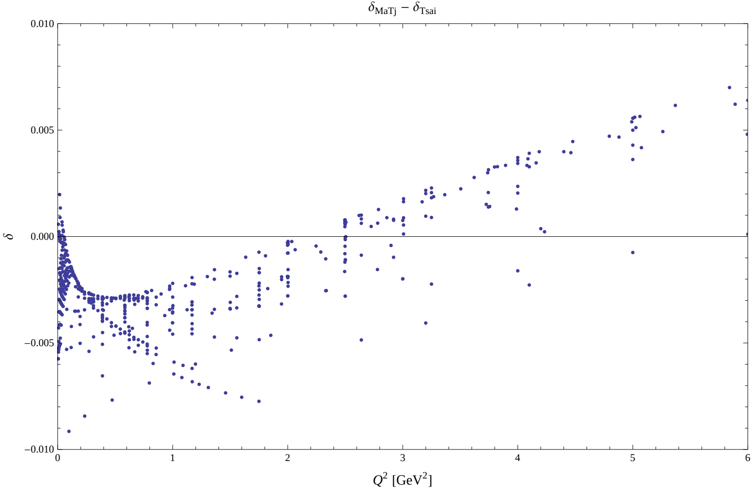
$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{exp}} = \left(\frac{d\sigma}{d\Omega}\right)_0 (1+\delta), \qquad \delta = \frac{2\Re(\mathcal{M}_0^\dagger \mathcal{M}_1)}{|\mathcal{M}_0|^2}\ \text{(world)}$$

RC type	World	Mainz
vac pol	hadronic* + all leptons	no hadronic + all leptons
TPE	Blunden	IR part (MaTj) + Feshbach

*According to Walker PRD 49, 5671 (1994).

A Scatter Plot for Model Dependence

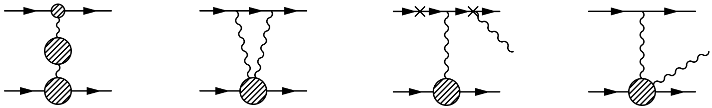
Points with larger $|\Delta\delta|$ tend to be at large scattering angle. Below is a scatter plot for the difference in the size of the radiative correction to the cross section between the two calculations.



Outline

Form Factor Definitions

- Before performing final fits, we want to make sure we have as complete of an understanding of the radiative corrections as possible, and to apply a consistent prescription in the analyses.
- We should think of a point particle scattering off a composite particle.



- Use on-shell renormalization, so that the form factors suffer infrared divergences.
- The single-photon exchange amplitude should read

$$\mathcal{M}_0 = -\frac{4\pi\alpha}{q^2}\frac{1}{1-\hat{\Pi}(q^2)}\bar{u}(k')\Gamma^{\mu(e)}(-q)u(k)\cdot\bar{u}(p')\Gamma_{\mu}^{(p)}(q)u(p)\,,$$

where $\hat{\Pi}(q^2)$ is the contribution from the vacuum polarization.

Form Factor Definitions (cont.)

- We should define reduced form factors that are finite for nonzero $q^2 \to 0$ in the $\lambda \to 0$ limit.

$$
\begin{aligned}
F_i^{(e)}(q^2,\lambda) &\equiv \tilde{F}_i^{(e)}(q^2)\Phi^{(e)}(q^2,\lambda)\,, \quad F_1^{(e)}(0) = 1\,, F_1^{(e)}(0) \sim \frac{\alpha}{2\pi}\,, \\
\frac{F_i^{(p)}(q^2,\lambda)}{1-\hat{\Pi}_{\rm had}(q^2)} &\equiv \tilde{F}_i^{(p)}(q^2)\Phi^{(p)}(q^2,\lambda)\,, \quad F_1^{(p)}(0) = 1\,, F_1^{(p)}(0) = a_p \sim 1.793\,, \\
\Phi^{(i)}(q^2,\lambda) &= 1 - \frac{\alpha}{2\pi}\left[\phi(q^2,m_i,\lambda) - \phi(0,m_i,\lambda)\right]\,, \\
\phi(q^2,m,\lambda) &= K(p_1,p_3) \text{ of Tsai.}
\end{aligned}
\tag{1}
$$

- The slopes of the reduced form factors ($\tilde{F}_i$ related to $\tilde{G}_i$ as before) are used to define the radii.

$$
\begin{aligned}
m_p^2 \tilde{F}_1^{(p)\prime}(0) &= m_p^2 F_1^{(p)\prime}(0) - \frac{\alpha}{3\pi}\left(\log\frac{m_p}{\lambda} + \frac{1}{4}\right) + \hat{\Pi}'_{\rm had}(0) \\
&\equiv \frac{1}{6}m_p^2 r_E^2 - \frac{a_p}{4}\,, \\
m_p^2 \tilde{F}_2^{(p)\prime}(0) &= m_p^2 \tilde{F}_2^{(p)\prime}(0) + \left[-\frac{\alpha}{3\pi}\left(\log\frac{m_p}{\lambda} + \frac{1}{4}\right) + \hat{\Pi}'_{\rm had}(0)\right]a_p \\
&\equiv \frac{1}{6}\left[(1+a_p)m_p^2 r_M^2 - m_p^2 r_E^2\right] + \frac{a_p}{4}\,.
\end{aligned}
\tag{2}
$$

What Remains?

- We know how to compute results for $\tilde{F}_i^{(e)}$ and the leptonic contributions to the vacuum polarization in perturbation theory.
- For soft bremsstrahlung, the result previously given is exact, although the evaluation of the resulting integrals differed between the two calculations.
- For TPE, one can define a conventional separation of the IR divergent and finite parts of the amplitude. In the Tsai and MaTj calculations, the IR separation used corresponded to how the soft photon approximation was used. Is there a way to calculate the TPE contribution in a model-independent way?
- What about the hadronic vacuum polarization?

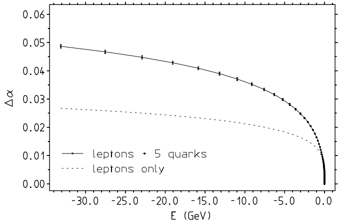
Hadronic Vacuum Polarization $\hat{\Pi}_h(q^2)$

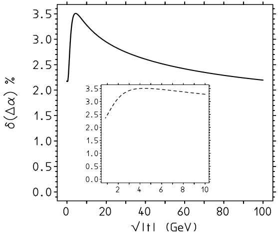
- This is included in the spectroscopic extractions for the radius.
- Walker (1994) uses a fit and writes the correction to the cross-section

$$\delta^h_{\text{vac}} = -2\Big[-1.513\times10^{-3} - 2.822\times10^{-3}\log\Big(1+1.218\frac{Q^2}{\text{GeV}}\Big)\Big];$$

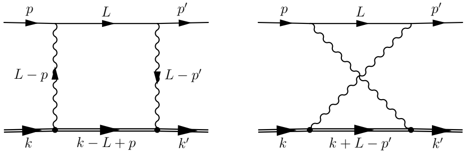
however, the expression for the lepton contribution in the paper is incorrect.

- Use the optical theorem and $e^+e^- \to \text{hadrons}$ data. Jegerlehner (1996), Friar et al. (1998)
- For our purposes, it is enough to extract $\hat{\Pi}'_h(0) \sim -9.31(20)\times10^{-3}\,\text{GeV}^{-2}$, which translates to a 0.1% or 10^{-3} fm difference.





TPE in NRQED



- At low energies, NRQED provides a systematic and model-independent approach to computing corrections and to relating observables at different energies.
- Leading $\mathcal{O}(\alpha)$ correction to $ep \to ep$, $m_e \ll E \ll M$: Hill et al., 2012

$$\mathcal{M}_{2\gamma} = \frac{4\pi^2\alpha}{Q^2}\bar{u}(p)\gamma^0 u(p)\left\{\frac{\pi}{2}\frac{Q}{2E+Q} + i\left(\frac{Q^2}{(2E)^2-Q^2}\log\frac{Q}{2E} - 2\log\frac{\lambda}{Q}\right)\right.$$
$$\left. + \mathcal{O}[\alpha^2, \lambda/E, m_e/E, E/M]\right\}, \quad (3)$$

- In the limit of $M \to \infty$, we recover the Feshbach correction,

$$\delta_F = \alpha\pi\frac{\sin(\theta/2)(1-\sin(\theta/2))}{\cos^2(\theta/2)},$$

which was the form of the Coulomb distortion correction computed by Rosenfelder and applied by the Mainz collaboration. Note that this correction is always positive.

TPE in NRQED (cont.)

- The NRQED Lagrangian, up to $\mathcal{O}(1/M^2)$ in the heavy fermion ψ, is:

$$\mathcal{L} = \psi^\dagger \left\{ iD_t + c_2 \frac{\boldsymbol{D}^2}{2M} c_F g \frac{\boldsymbol{\sigma} \cdot \boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial} \cdot \boldsymbol{E}]}{8M^2} + ic_S g \frac{\boldsymbol{\sigma} \cdot (\boldsymbol{D} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{D})}{8M^2} \right\} \psi \,.$$

- If we include 4-fermion operators, with ℓ a relativistic lepton,

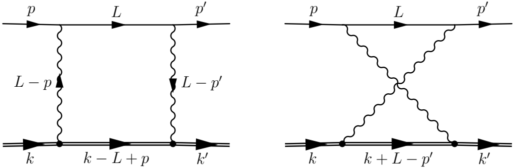
$$\mathcal{L}_{4\text{f}} = \frac{b_1}{M^2} \psi^\dagger \psi \, \bar{\ell} \gamma^0 \ell + \frac{b_2}{M^2} \psi^\dagger \sigma^i \psi \, \bar{\ell} \gamma^i \gamma_5 \ell \,.$$

- Power corrections to $\mathcal{M}_{2\gamma}$ will involve :

$$1/M : c_F \,,$$
$$1/M^2 : c_D, b_1, b_2 \,.$$

- One can also use NRQED to compute corrections to S-state energies in bound states in eH. Hill et al. (2012)

TPE with Form Factor Insertions



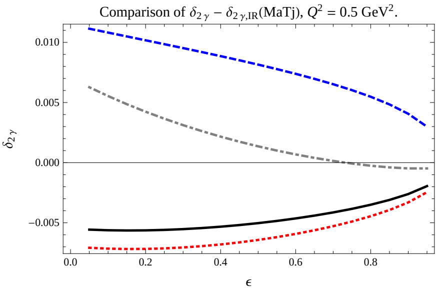
- There has been some effort at modelling the non-IR part of the TPE by the "Sticking in Form Factors" (SIFF) procedure. Blunden (2003, 2005)
- Treat the proton as a propagating Dirac particle and insert Γ^μ at each of the vertices, using simple form factor ansätze for F_1, F_2.
- We investigated the model dependence of this calculation:

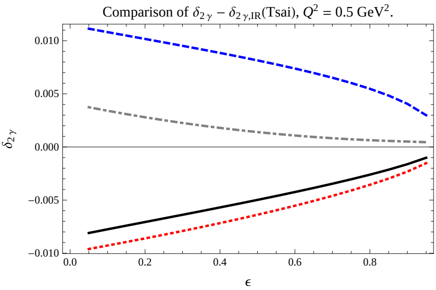
$$F_1 = F_2/(\mu_p - 1) = (1 - q^2/\Lambda^2)^{-1}\,, \qquad \text{monopole, } \Lambda^2 = 0.71\ \mathrm{GeV}^2\,,$$

$$F_1 = F_2/(\mu_p - 1) = (1 - q^2/\Lambda^2)^{-2}\,, \qquad \text{dipole, } \Lambda^2 = 0.71\ \mathrm{GeV}^2\,,$$

$$F_i = \sum_{j=1}^{3} \frac{a_{ij}}{b_{ij} - q^2}\,, \quad \sum_{j=1}^{3} \frac{a_{ij}}{b_{ij}} = F_i(0)\,, \qquad \text{Blunden sum of monopoles (2005).}$$

TPE with Form Factor Insertions (cont.)





- L/R: TPE IR subtraction scheme of MaTj/Tsai.
- Blue dashed: monopole,
- Grey dot-dashed: Feshbach,
- Black solid: Blunden,
- Red dotted: dipole.

Outline

Fitting Procedure for Global Elastic ep Scattering Dataset

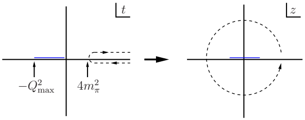
- Experiments since the 1960's have used the Rosenbluth separation: varying E and θ to keep q^2 (momentum transfer) fixed, and looking at the resulting fit as a function of ϵ.
- To extract the radii values, earlier experiments used functional forms for G_E, G_M that have pathological behaviour. Hill & Paz (2010)

	$k_{\max} = 1$	2	3	4	5
polynomial	836^{+8}_{-9}	867^{+23}_{-24}	866^{+52}_{-56}	959^{+85}_{-93}	1122^{+122}_{-137}
	$\chi^2 = 34.49$	32.51	32.51	31.10	28.99
continued fraction	882^{+10}_{-10}	869^{+26}_{-25}	–	–	–
	$\chi^2 = 32.81$	32.51			
z expansion (no bound)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+64}_{-69}	1022^{+102}_{-114}	1193^{+152}_{-174}
	$\chi^2 = 36.14$	32.52	32.48	30.35	28.92
z expansion ($\lvert a_k \rvert \le 10$)	918^{+9}_{-9}	868^{+28}_{-29}	879^{+38}_{-59}	880^{+39}_{-61}	880^{+39}_{-62}
	$\chi^2 = 36.14$	32.52	32.48	32.46	32.45

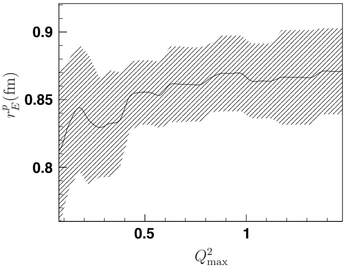
Values in 10^{-18} m, $G_{\text{CF}}(q^2) = \dfrac{1}{1+a_1 \dfrac{q^2/t_{\text{cut}}}{1+a_2 \dfrac{q^2/t_{\text{cut}}}{1+\ldots}}}$.

Functional Forms of G_E, G_M

- Hill & Paz use the analyticity of the form factor to give a model-independent constraint on its functional form. This technique is widely used in the meson community. The form factor is a truncated series in the variable



$$z(t; t_{\text{cut}}, t_0) = \frac{\sqrt{t_{\text{cut}} - t} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} - t} + \sqrt{t_{\text{cut}} - t_0}}, \; t = q^2 \; .$$



Hill & Paz, fit to extracted FF from world data.
Just proton: $r_P^E = 0.870 \pm 0.023 \pm 0.012$ fm
from sample point at $Q^2 = 0.5\ \text{GeV}^2$ with the bounds $|a_k|_{\max} = 5, 10,$ (max. variation for the bound of 10).
See talk by Gil Paz for the magnetic result.

Fitting Procedure

- Use the form factors expanded in z

$$G_E(z) = \sum_{k=1}^{k_{\max}} a_k z^k \,, \qquad G_M(z) = \sum_{k=1}^{k_{\max}} b_k z^k \,,$$

 and fit the world cross-section data compiled by Arrington (procedure outlined in PRC 68, 034325, updated from 2007 fit), extracting the charge and magnetic radii.
- Use MINUIT to minimize the chi-square function

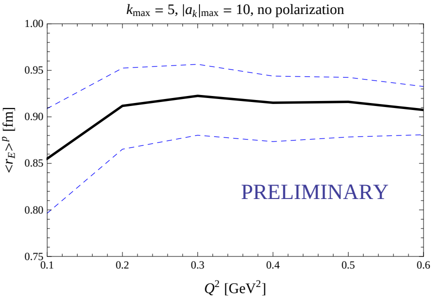
$$\chi^2 = \chi^2_\sigma + \chi^2_{\rm pol} + \chi^2_{\rm norm} \,,$$

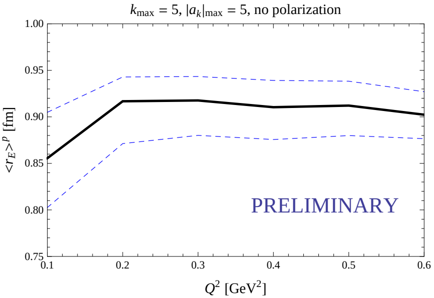
$$\chi^2_\sigma = \sum_{i=1}^{N_\sigma} \frac{\left(\sigma_i - \frac{\sigma_{i,\rm fit}}{\eta_{i,\rm fit}}\right)^2}{d\sigma_i^2} \,, \quad \chi^2_{\rm pol} = \sum_{i=1}^{N_{\rm pol}} \frac{(R_i - R_{i,\rm fit})^2}{dR_i^2} \,, \quad \chi^2_{\rm norm} = \sum_{i=1}^{N_{\rm exp}} \frac{(1-\eta_{i,\rm fit})^2}{d\eta_i^2} \,.$$

- 1σ values for the radii are determined by scanning the χ^2 curve in radius from the minimum until $\Delta\chi^2 = 1$ is reached.

Outline

Preliminary Results Excluding Polarization Data





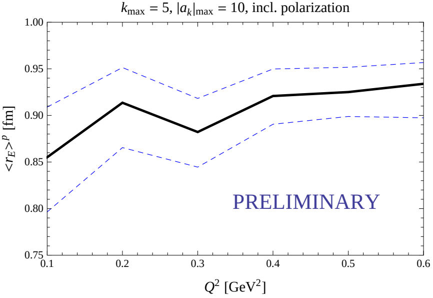
Following the same extraction as Hill & Paz, taking $Q^2_{\max} = 0.5\ \mathrm{GeV}^2$,

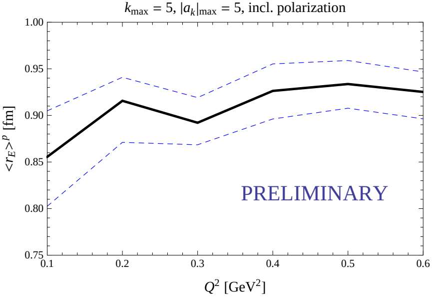
$$\langle r_E^2 \rangle^{1/2} = 0.912\ {}^{+0.026}_{-0.032} \pm 0.004\ \mathrm{fm}\,.$$

Compare with result of Hill & Paz,

$$\langle r_E^2 \rangle^{1/2} = 0.870 \pm 0.023 \pm 0.012\ \text{fm}\,.$$

Preliminary Results Including Polarization Data





Following the same extraction,

$$\langle r_E^2 \rangle^{1/2} = 0.934 \, {}^{+0.025}_{-0.026} \pm 0.009 \text{ fm} \,.$$

Compare with result of Zhan et al. (2011) with world data and polarization data below $Q^2 = 0.5 \text{ GeV}^2$:

$$\langle r_E^2 \rangle^{1/2} = 0.875 \pm 0.010 \text{ fm}$$

Further Investigation

- For the low Q^2 data, we did not find that a significant difference from applying different TPE schemes.
- We can choose different values for t_0, the spacelike q^2 point that is mapped to $z = 0$. Choosing $t_0 \neq 0$ yields a faster convergence, which can have some effect on the speed of the fitting, but a small impact on the result.
- We are working on results for $\langle r_M^2 \rangle^{1/2}$.
- We will include Mainz data in the fit.

Conclusion

- It is important, as a community, to agree on a set of radiative corrections to compare radius extractions from scattering and spectroscopic experiments.
- We have presented a framework for these radiative corrections that will be applied in our analyses of the world, polarization, and Mainz datasets.
- For low Q^2 and initial electron energies, we can compute the TPE corrections in a model-independent way with NRQED.
- The z-expansion, by construction, guarantees the proper analytic behaviour for the form factors, and yields a model-independent functional form for G_E, G_M.
- We presented preliminary results for radius extractions from the world cross-section data with uncertainties that are consistent with the Hill & Paz extraction that used the same functional form to fit extracted form factors. We find uncertainties that are two or three times those of previous analyses.