Finite density particle condensation and scattering data

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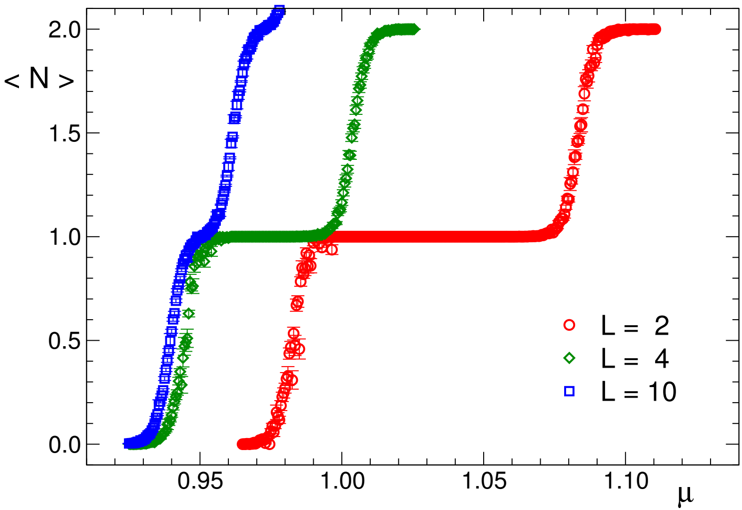
C. Gattringer, M. Giuliani and O. Orasch, Phys. Rev. Lett. 120, 241601 (2018)

M. Giuliani, O. Orasch, C. Gattringer, EPJ Web of Confs. 175, 07007 (2018)

Condensation at finite density

Condensation thresholds

Expectation value $\langle N \rangle$ of the particle number as a function of the chemical potential μ at very low temperature (charged scalar field):



- At critical values $\mu_n(L)$ one observes jumps from $\langle N \rangle = n{-}1$ to $\langle N \rangle = n$.
- The condensation thresholds $\mu_n(L)$ depend on the spatial extent L.

Connection of condensation thresholds and n-particle energies

- Grand canonical partition sum and grand potential:

$$Z \;=\; \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu \hat{N})} \;=\; e^{-\beta\,\Omega(\mu)}$$

- Low T: In each particle sector Z is governed by the minimal grand potential $\Omega(\mu)$

$$\Omega(\mu) \quad \xrightarrow{T\to 0} \quad \begin{cases} \Omega_{min}^{N=0} \;=\; 0\,, & \mu \;\in\; [0, \mu_1] \\ \Omega_{min}^{N=1} \;=\; m \,-\, 1\mu\,, & \mu \;\in\; [\mu_1, \mu_2] \\ \Omega_{min}^{N=2} \;=\; W_2 - 2\mu\,, & \mu \;\in\; [\mu_2, \mu_3] \\ \Omega_{min}^{N=3} \;=\; W_3 - 3\mu\,, & \mu \;\in\; [\mu_3, \mu_4] \\ \ldots & \end{cases}$$

- m: physical mass, W_2: minimal 2-particle energy, W_3: minimal 3-particle energy ...
- Use continuity of $\Omega(\mu)$ to relate the critical μ_n to m and the W_n.

F. Bruckmann, C. Gattringer, T. Kloiber, T. Sulejmanpasic, PRL 115, 231601 (2015)

Connection of condensation thresholds and n-particle energies

- Relations between the critical $\mu_n(L)$ and the minimal multi-particle energies:

$$m(L) = \mu_1(L) \, , \quad W_2(L) = \mu_1(L) + \mu_2(L) \, , \quad \ldots \quad W_n(L) = \sum_{k=1}^{n} \mu_k(L) \ldots$$

- The multi-particle energies are governed by low energy parameters.

- In particular their finite volume dependence can be related to scattering data.

  (K. Huang & C.N. Yang, M. Lüscher + half of the audience)

- We thus expect that one can describe the thresholds $\mu_n(L)$ with scattering data.

- Problem: Simulations at finite chemical potential μ typically have a sign problem.

Solving (some) sign problems with worldline techniques

Sign problem (complex action problem)

In some cases the action becomes complex at finite chemical potential μ

$$S[\phi] \; = \sum_{n \in \Lambda} \left[(m_b^2 + 8) \, |\phi_n|^2 \; + \lambda \, |\phi_n|^4 \; - \sum_{\nu=1}^{3} \Big[\phi_n^\star \phi_{n+\hat{\nu}} + \phi_{n+\hat{\nu}}^\star \phi_n \Big] - \Big[e^{-\mu} \phi_n^\star \phi_{n+\hat{4}} + e^{\mu} \phi_{n+\hat{4}}^\star \phi_n \Big] \right]$$

The Boltzmann factor is complex and does not have a probability interpretation:

$$\frac{1}{Z} \, e^{-S[\phi]} \;\in\; \mathbb{C}$$

No direct Monte Carlo simulation! "Sign problem"

Idea: Solve the sign problem by transforming the path integral to new variables.

Worldline representation for the charged ϕ^4 field

- Lattice action: $\quad (\, \phi_n \in \mathbb{C} \, , \quad M^2 = m_b^2 + 8 \,)$

$$S = \sum_n \Big[\, M^2 \, |\phi_n|^2 \; + \lambda \, |\phi_n|^4 \, \Big] - \sum_{n,\nu} \Big[e^{-\mu \, \delta_{\nu,4}} \, \phi_n^\star \, \phi_{n+\widehat{\nu}} \; + \; e^{\mu \, \delta_{\nu,4}} \, \phi_x \, \phi_{n+\widehat{\nu}}^\star \Big]$$

- Expand the nearest neighbor terms of e^{-S}:

$$\prod_{n,\nu} \exp \left(e^{-\mu \, \delta_{\nu 4}} \, \phi_n^\star \, \phi_{n+\widehat{\nu}} \right) \; = \; \prod_{n,\nu} \sum_{j_{n,\nu}=0}^{\infty} \frac{\left(e^{-\mu \, \delta_{\nu 4}} \right)^{j_{n,\nu}}}{j_{n,\nu} \, !} \, \left(\phi_n^\star \, \phi_{n+\widehat{\nu}} \right)^{j_{n,\nu}}$$

$$= \sum_{\{j\}} e^{-\mu \sum_n j_{n,4}} \prod_{n,\nu} \frac{1}{j_{n,\nu}!} \prod_n \phi_n^{\sum_\nu j_{n-\hat{\nu},\nu}} \phi_n^{\star \, \sum_\nu j_{n,\nu}}$$

- $j_{n,\nu}$ (and $\overline{j}_{n,\nu}$ for other NN-term) turn into the new worldline degrees of freedom.

Worldline representation - integrating out the fields

- Integral over $\phi_n \sim r\, e^{i\theta}$: ($F, \overline{F}$ are sums of $j_{\cdot,\nu}, \overline{j}_{\cdot,\nu}$ connected to n)

$$\int_{\mathbb{C}} \frac{d\,\phi_n}{2\pi}\; e^{-M^2|\phi_n|^2 - \lambda|\phi_n|^4}\; \phi_n^{\,F}\; \phi_n^{\star\,\overline{F}} \;=$$

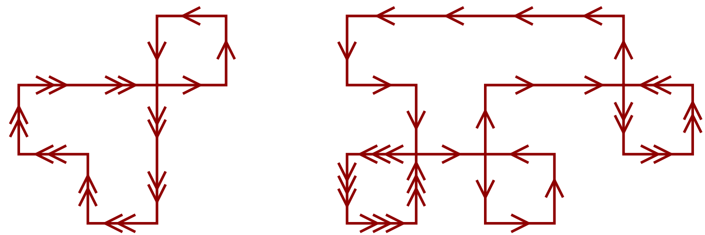
$$= \int_0^\infty dr\; r^{\,1+F+\overline{F}}\; e^{-M^2 r^2 - \lambda r^4} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\; e^{\,i\theta\,(F-\overline{F})} \;=\; \mathcal{R}(F+\overline{F})\;\delta(F-\overline{F})$$

- At every site n there is a weight factor $\mathcal{R}(F+\overline{F})$ and a constraint $\delta(F-\overline{F})$.

- Explicitly the product over the constraints at all sites reads ($d_{n,\nu} = j_{n,\nu} - \overline{j}_{n,\nu}$):

$$\prod_n \delta\Big(\sum_\nu \left[\, d_{n,\nu} - d_{n-\widehat{\nu},\nu}\,\right]\Big) \quad \Leftrightarrow \quad \sum_\nu \left[\, d_{n,\nu} - d_{n-\widehat{\nu},\nu}\,\right] \;=\; \vec{\nabla}\vec{d}_n \;=\; 0 \quad \forall n$$

- Admissible configurations of worldline variables are oriented loops of flux:



Worldline representation - final form

- The original partition function is mapped exactly to a sum over configurations of the worldline variables $j_{n,\nu}, \overline{j}_{n,\nu} \in \mathbb{N}_0$ with $d_{n,\nu} = j_{n,\nu} - \overline{j}_{n,\nu}$.

$$Z \; = \; \sum_{\{j,\overline{j}\}} \mathcal{W}[\,j,\overline{j}\,]\,\mathcal{C}[\,d\,]$$

- $\mathcal{W}[\,j,\overline{j}\,]$: Real and positive weight from radial d.o.f. and combinatorics.

- Constraints from integrating over the symmetry group ($d_{n,\nu} = j_{n,\nu} - \overline{j}_{n,\nu}$):

$$\mathcal{C}[\,d\,] \;\; = \;\; \prod_n \delta\Big(\vec{\nabla}\vec{d}_n\Big)$$

- Particle number N $\Leftrightarrow$ temporal winding number $\omega[d]$ of $d_{n,\nu}$-flux:

$$e^{-\mu \sum_n d_{n,4}} \;\; = \;\; e^{-\mu\, N_t\, \omega[d]} \;\; = \;\; e^{-\mu\, \beta\, \omega[d]} \;\; \equiv \;\; e^{-\mu\, \beta\, N}$$

C. Gattringer, T. Kloiber, Nucl. Phys. B 869 (2013) 56

Condensation thresholds and scattering data

(a study in 2d and 4d ϕ^4 theory)

Determination of the critical values μ_n

- Using worldlines we compute $\langle N \rangle$ as function of μ.

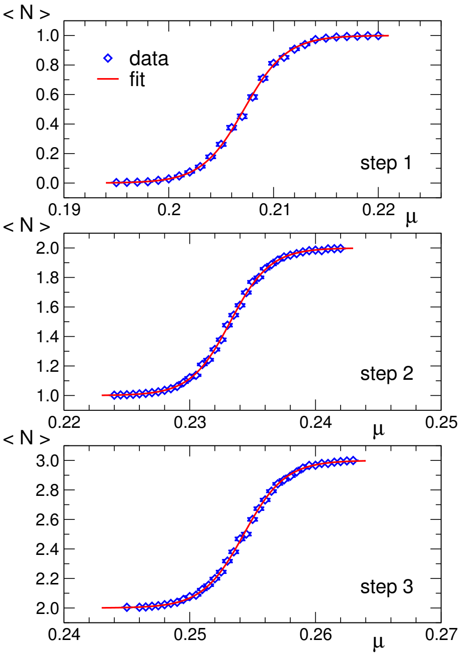
- Near the steps we fit $\langle N \rangle$ with a logistic function:

$$\langle N \rangle \sim \frac{1}{1 + e^{-a_n(\mu - \mu_n)}} + n - 1$$

- The μ_n are obtained as fit parameters.

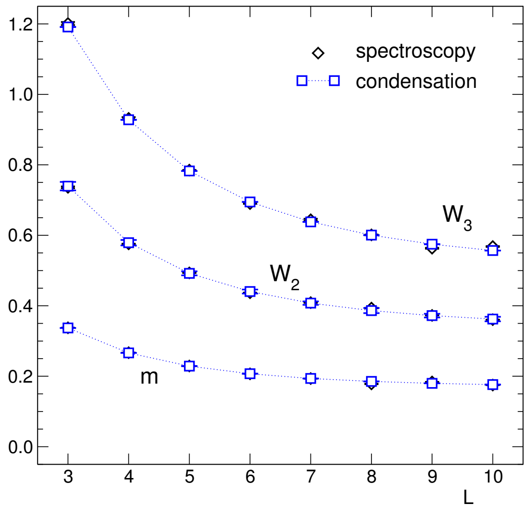
- From the $\mu_n(L)$ at different L we obtain

$$\begin{aligned} m(L) &= \mu_1(L) \\ W_2(L) &= \mu_1(L) + \mu_2(L) \\ W_3(L) &= \mu_1(L) + \mu_2(L) + \mu_3(L) \end{aligned}$$



Cross-check for the interpretation of the thresholds

We compute $m(L), W_2(L)$ and $W_3(L)$ also in a conventional spectroscopy calculation from 2-, 4-, and 6-point functions.



The good agreement confirms the interpretation of the $\mu_n(L)$ in terms of multi-particle energies, and their determination from the fits.

Finite volume analysis in 4d

Finite volume relations: $(\mathcal{I} = -8.914, \mathcal{J} = 16.532)$

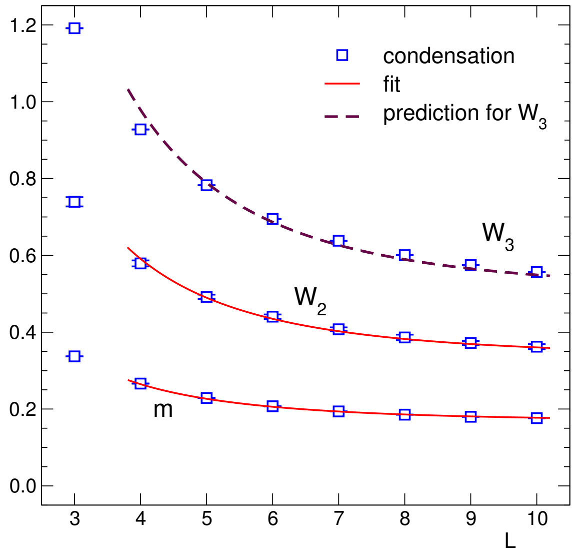
(K. Huang, C.N. Yang, M. Lüscher, S.R. Beane, W. Detmold, M.J. Savage, S.R. Sharpe, M.T. Hansen)

$$\begin{aligned} m(L) &= m_\infty + \frac{A}{L^{\frac{3}{2}}}\, e^{-L\, m_\infty} \,, \\ W_2(L) &= 2m + \frac{4\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 - \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right], \\ W_3(L) &= 3m + \frac{12\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 + \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right]. \end{aligned}$$

- From fitting the mass data we obtain m_∞.
- In $W_2(L)$ we use $m(L)$ for m on the rhs. and obtain a as a fit parameter.
- In $W_3(L)$ we use $m(L)$ and a from $W_2(L)$ to "predict" the 3-particle data.

For our couplings: $m_\infty = 0.168(1)\ (l.u.),\ \ a = -0.078(7)\ (l.u.),\ \ am_\infty = -0.013(1)$

Comparison of threshold data with the finite volume relations



The good agreement shows that one can describe condensation thresholds with scattering data.

Finite volume analysis in 2d

- 2-particle wave function and energy at zero total momentum: $(p_1 = -p_2 \equiv p)$

$$\psi(r) \;=\; e^{-ipr} \;, \qquad W_2(L) \;=\; 2\sqrt{m(L)^2 + p(L)^2}$$

- Quantization condition from boundary condition:

$$2\,\delta(p) \;=\; -\,p\,L \qquad \Rightarrow \qquad \delta(p(L)) \;=\; -\,p(L)\,L/2 \;\equiv\; \delta(L)$$

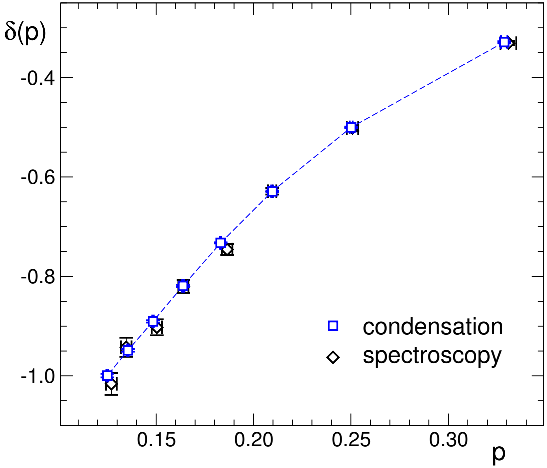
- 3-particle wave function and energy at zero total momentum: $(p_3 = -p_1 - p_2)$

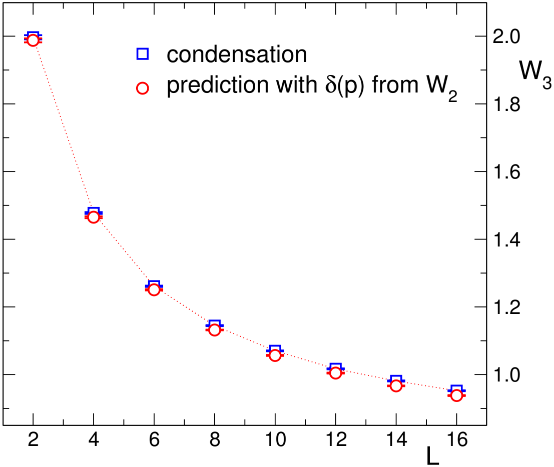
$$\psi(r) \;=\; e^{-ip_1 r_1}\, e^{-ip_2 r_2} \;, \qquad W_3(L) \;=\; \sum_{j=1}^{3} \sqrt{m(L)^2 + p_j(L)^2}$$

- Two quantization conditions from boundary conditions:

$$2\,\delta(p_j) \;=\; -\,p_j\,L \qquad \Rightarrow \qquad p_j(L) \;=\; -\,2\,\delta(L)/L$$

Comparison of threshold data with the finite volume relations





The good agreement shows that one can describe condensation thresholds with scattering data.

Summary

- At low temperature the particle number develops condensation steps as function of μ.
- The critical values $\mu_n(L)$ are related to multi-particle energies. (cross-checked with spectroscopy)
- The multi-particle energies and thus the $\mu_n(L)$ depend on scattering parameters.
- We explored the relation between condensation thresholds and scattering data using worldline simulations of the ϕ^4 field in 2 and 4 dimensions.
- In 4d we computed the scattering length from $\mu_1(L)$ and $\mu_2(L)$ and used it in the relations for $W_3(L)$ to "predict" the 3-particle energy and thus $\mu_3(L)$.
- In 2d we computed the phase shift from $\mu_1(L)$ and $\mu_2(L)$ and used it in the relations for $W_3(L)$ to "predict" the 3-particle energy and thus $\mu_3(L)$.
- In both 2d and 4d we obtained a satisfactory understanding of the $\mu_n(L)$ from scattering data.
- Suggestion: Study QC_2D and QCD with isospin chemical potential.