From two to three-body scattering in the finite volume: The role of unitarity

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Scattering Amplitudes and Resonance Properties from Lattice QCD

27-31 August 2018

Work in collaboration with:
Mai, Molina, Hu, Alexandru, Dehua Guo, Hammer, J-Y.Pang, Pilloni, Rusetsky, Szczepaniak et al.

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MA 7156/1-1

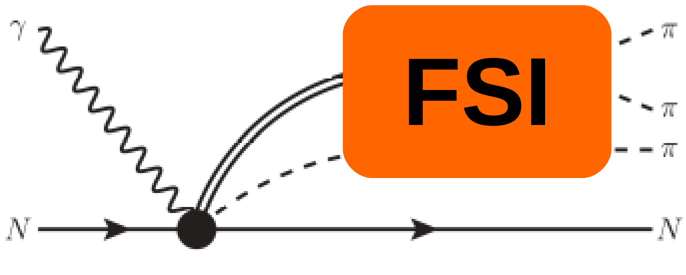
[Many slides from **Maxim Mai**]

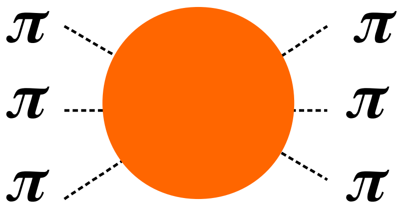
Outline

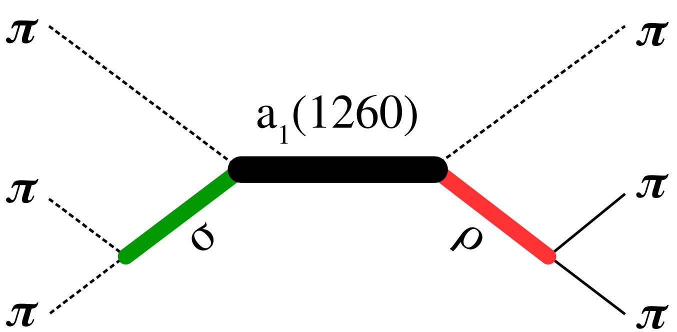
- Motivation
- Three-body dynamics in infinite volume
- Finite-volume problems (application: 2-body)
- Three-body dynamics in finite volume

3-body dynamics for mesons and baryons

Light mesons

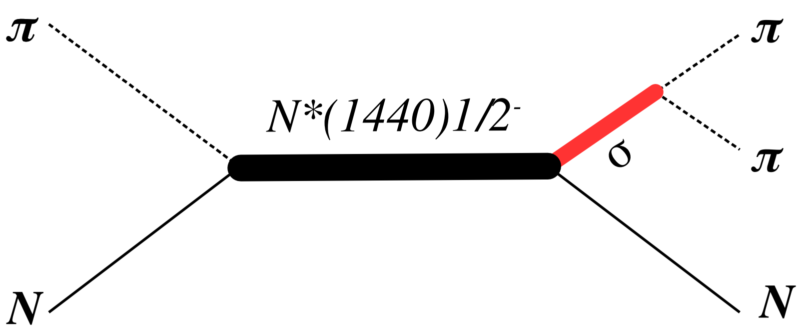






- Important channel in GlueX @ JLab
- Finite volume spectrum from lattice QCD:
  Lang, Leskovec, Mohler, Prelovsek (2014)
  Woss, Thomas et al. [HadronSpectrum] (2018)

Light baryons



- Roper resonance is debated for ~50 years in experiment. Can only be seen in PWA.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

Isobar formulation

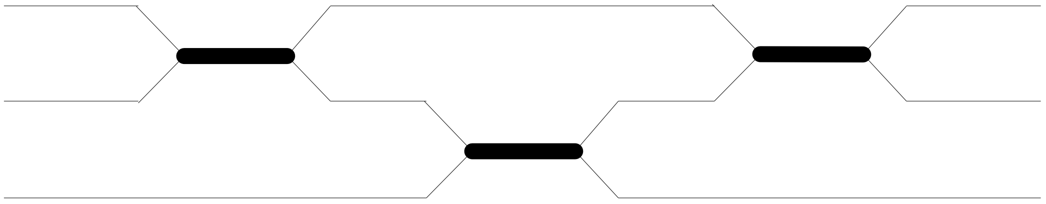
- Understanding of Lattice QCD or experimental searches (BESIII, COMPASS, GlueX) → theory of 3-body scattering problem needed

- **Available tools:**
  - *Faddeev equations (F.E.)* — **Faddeev(1959)**
  - F.E. in fixed-center approximation — **Brueckner(1953)**

    → usefull for ***πd***, ***Kd*** … systems — **Baru et al(2011), Mai et al. (2015)**

  - F.E. in isobar formulation — **Omnes(1964), Aaron(1967)**

    → re-parametrization of two-body amplitude — **Bedaque(1999)**



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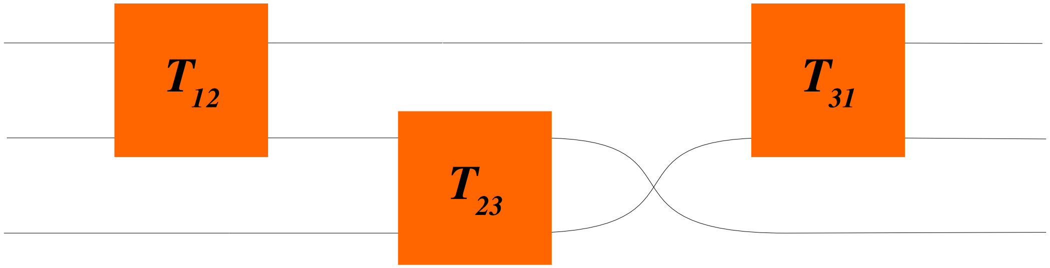


… (+ 3-body force)

Re-ordering of 3-body amplitude in 2-body sub-amplitudes & spectator
→ Not an approximation up to cut in space of allowed quantum numbers

FADDEEV EQUATIONS WITH ISOBARS

Mai, Hu, M. D., Pilloni, Szczepaniak

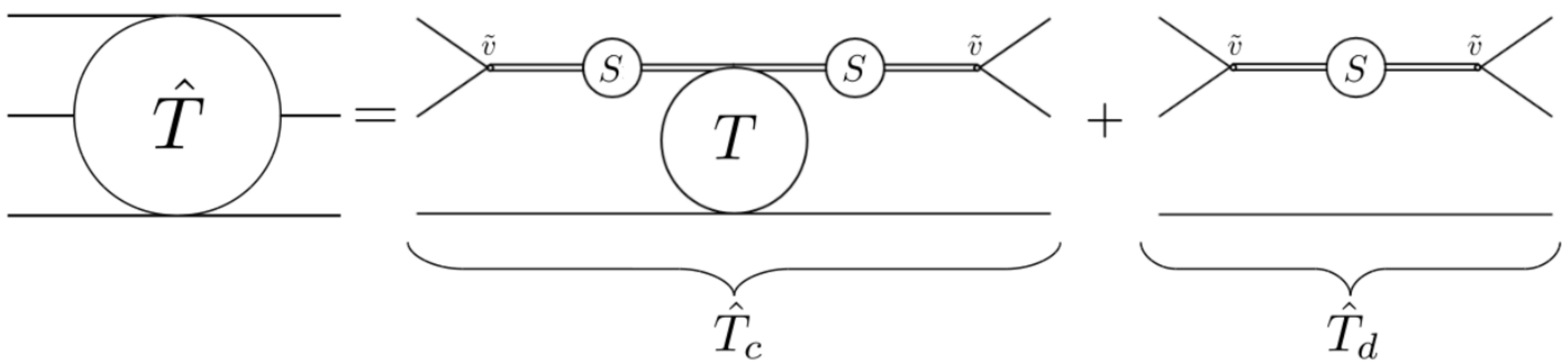
Eur. Phys. J. A53 (2017) 177

FE in isobar parametrization

Original study by Amado/Aaron/Young **AAY(1968)**

- 3-dimensional integral equation from unitarity constraint & BSE ansatz
- valid below break-up energies (*E < 3m*) & analyticity constraints unclear

One has to begin with asymptotic states



- υ a general function without cuts in the phys. region
- two-body interaction is parametrized by an *"isobar"*

= has definite QN and correct r.h.-singularities w.r.t invariant mass

- S and T are yet unknown functions

3-body Unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

$$\times \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2-m^2)}\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)$$

delta function sets all intermediate particles on-shell

3-body Unitarity

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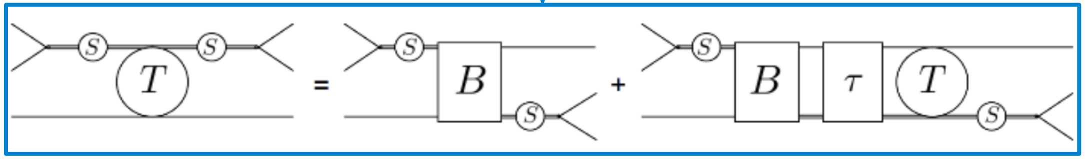

General Ansatz for the isobar-spectator interaction
→ **B & τ** are **new** unknown functions

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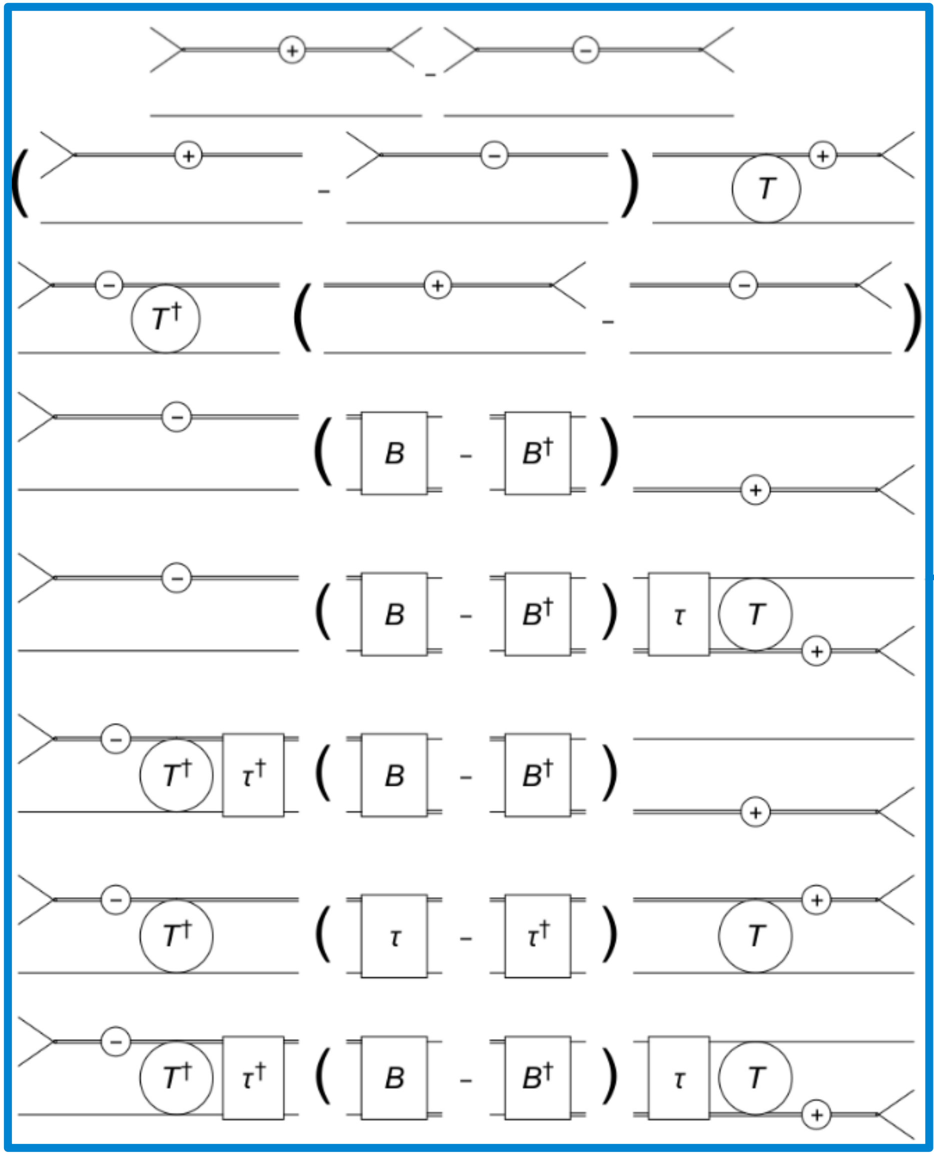

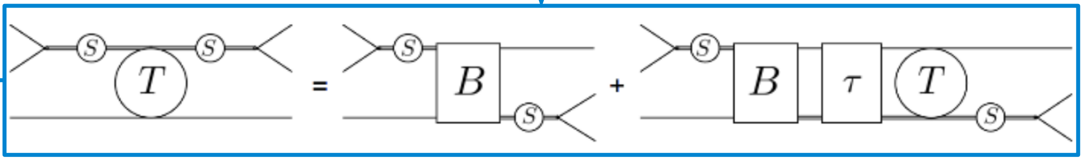

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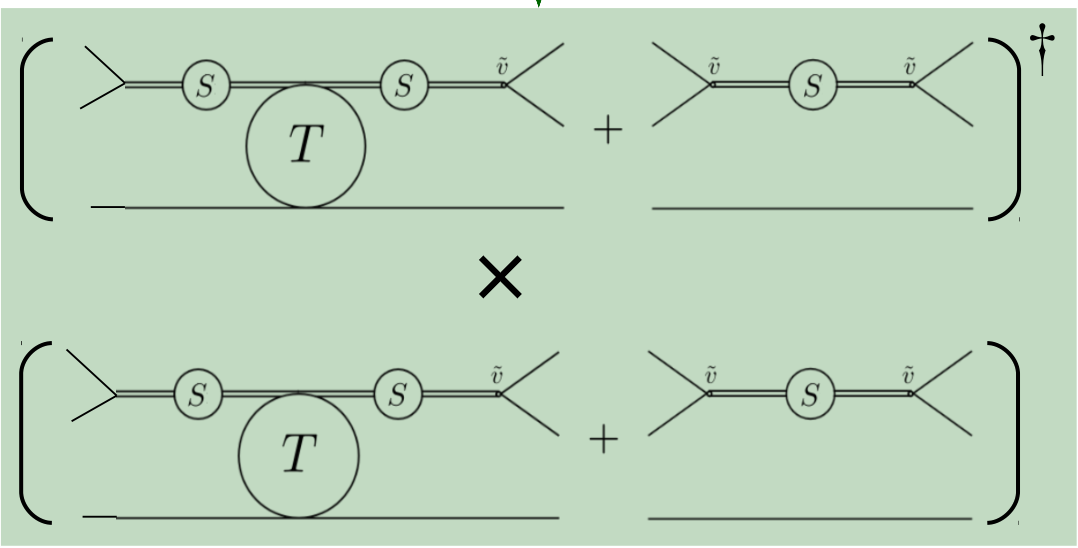

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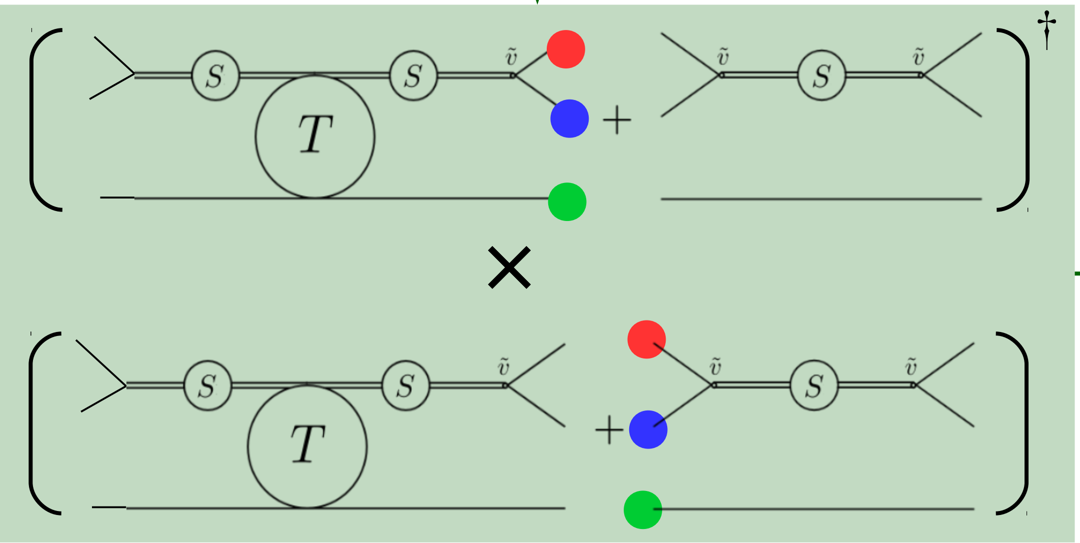

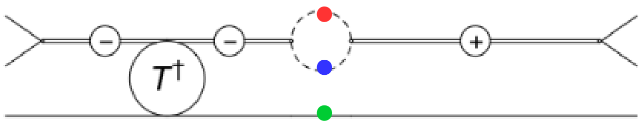

3-body Unitarity

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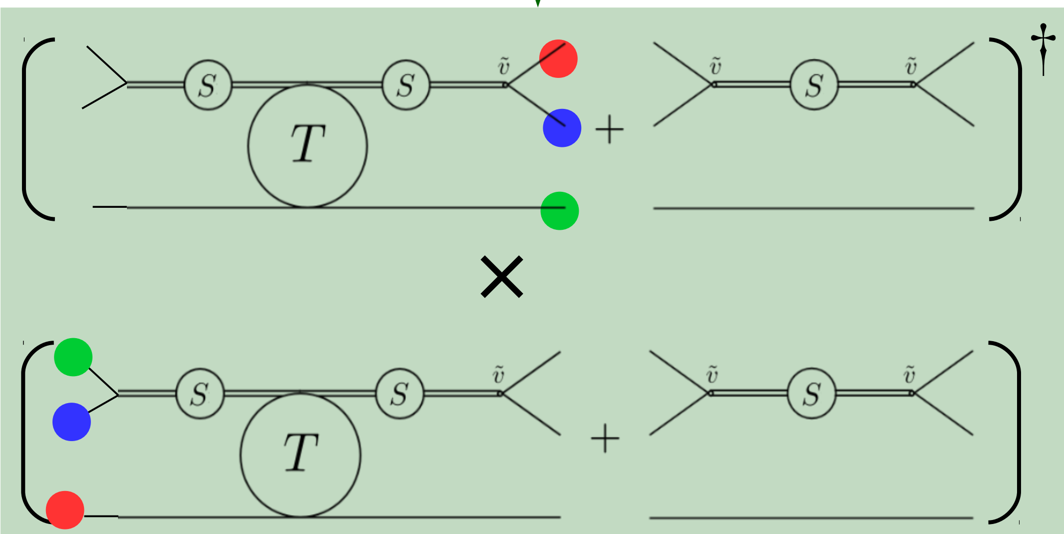

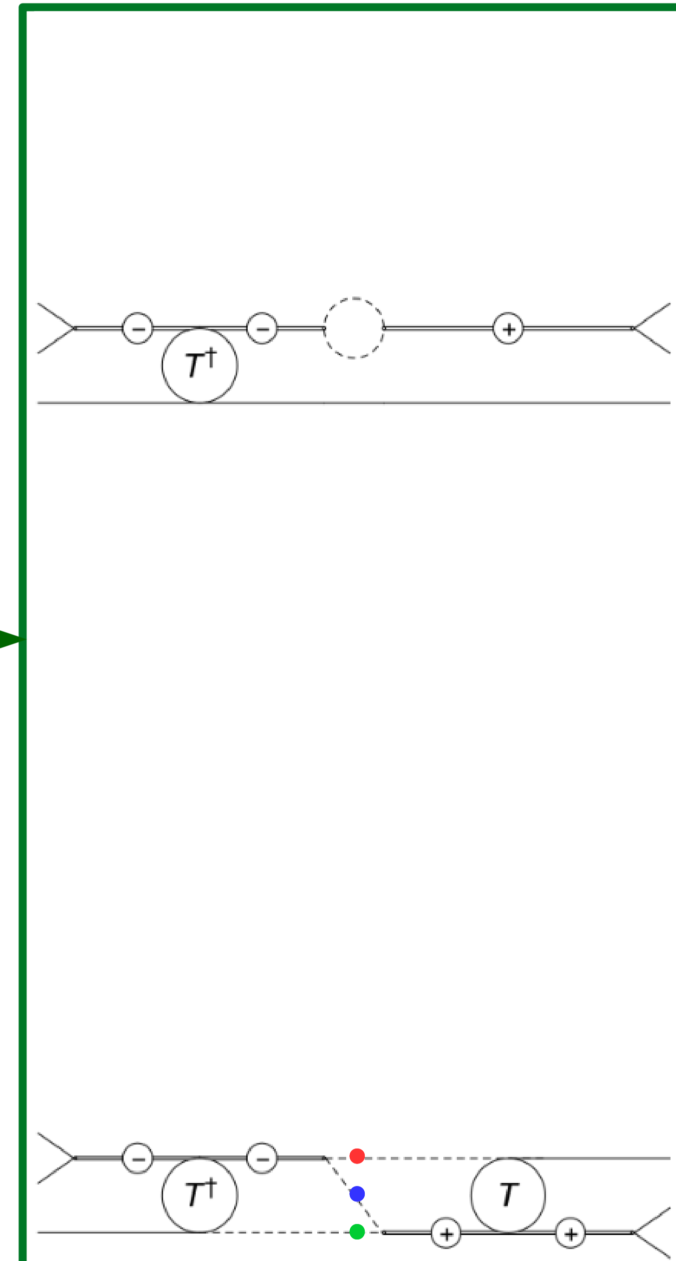

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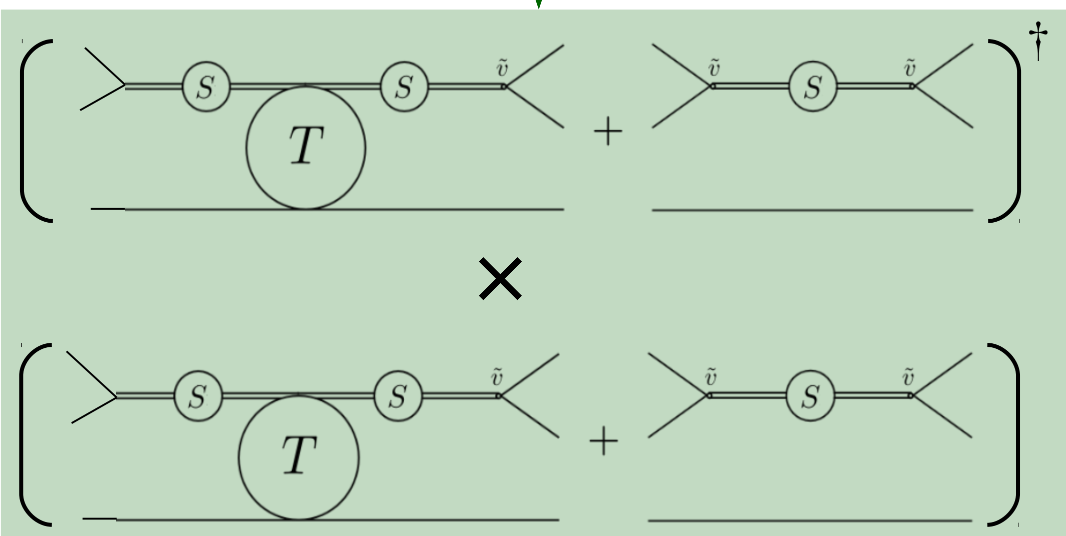
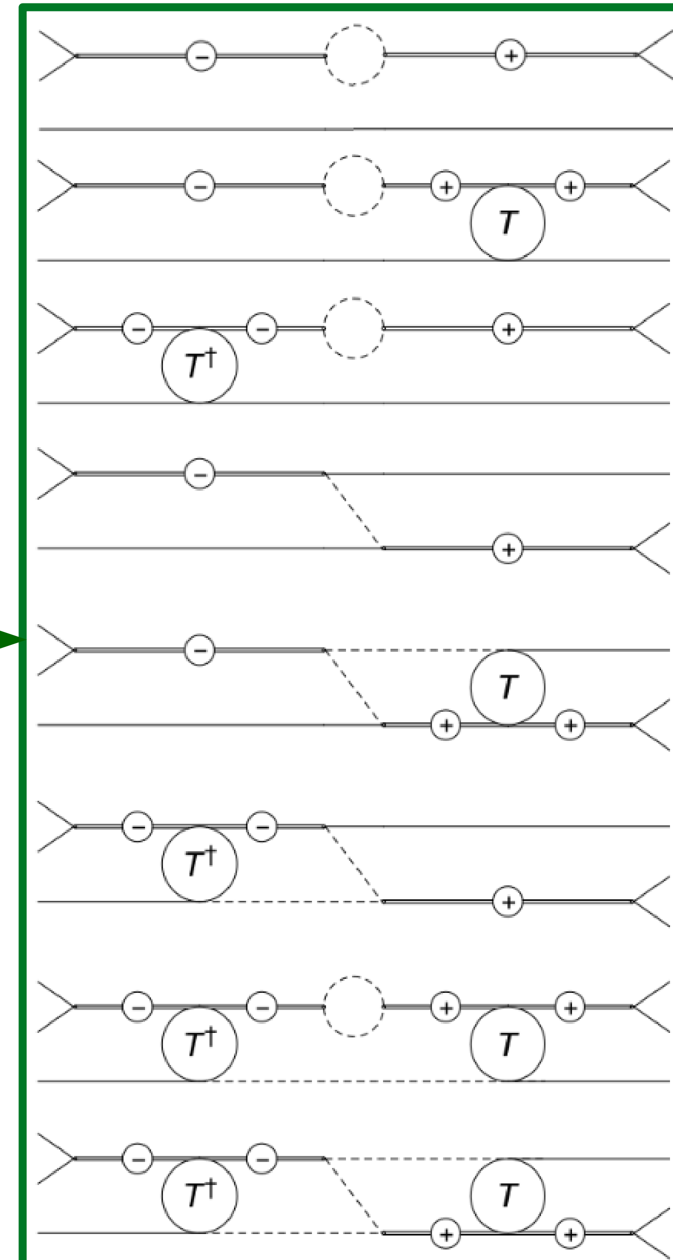


8 top.



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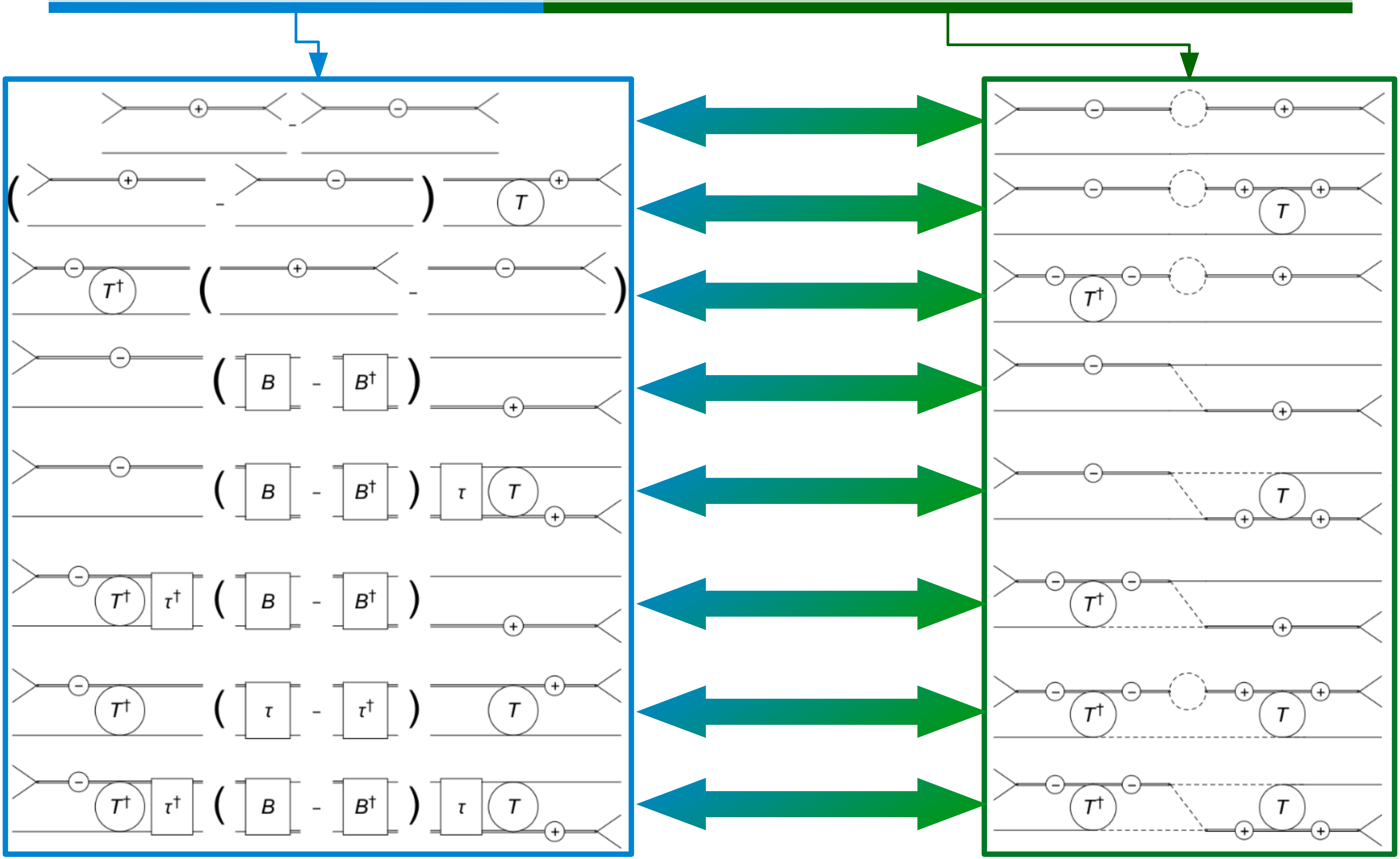
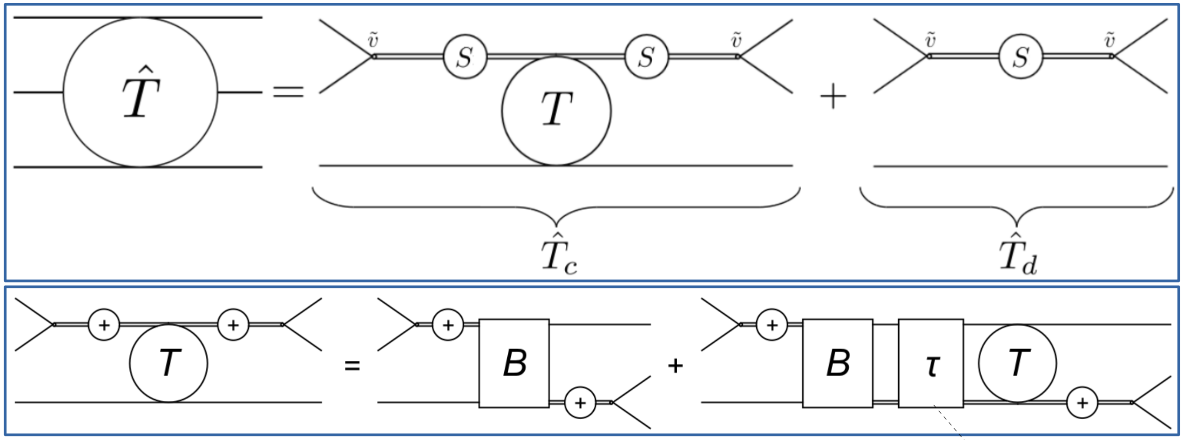


Scattering amplitude

3 → 3 scattering amplitude is a 3-dimensional integral equation



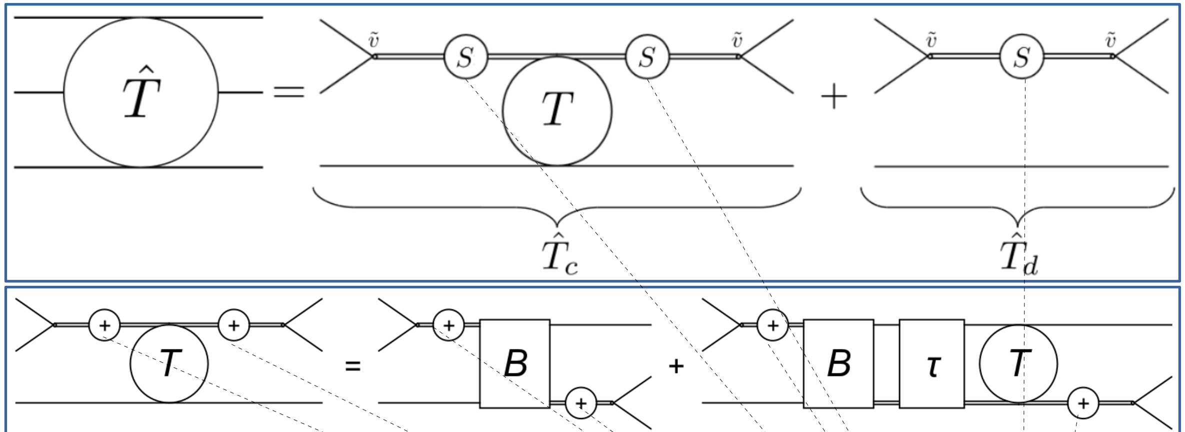
– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***v=λ*** (full relations available)

$$\tau(\sigma(k)) = (2\pi)\delta^+(k^2 - m^2)S(\sigma(k))$$

Scattering amplitude

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***v=λ*** (full relations available)

$$\mathrm{Disc}\,\frac{1}{S} = -\frac{i}{8\pi}\frac{K_{\mathrm{cm}}}{\sqrt{\sigma(k)}}\lambda^2$$

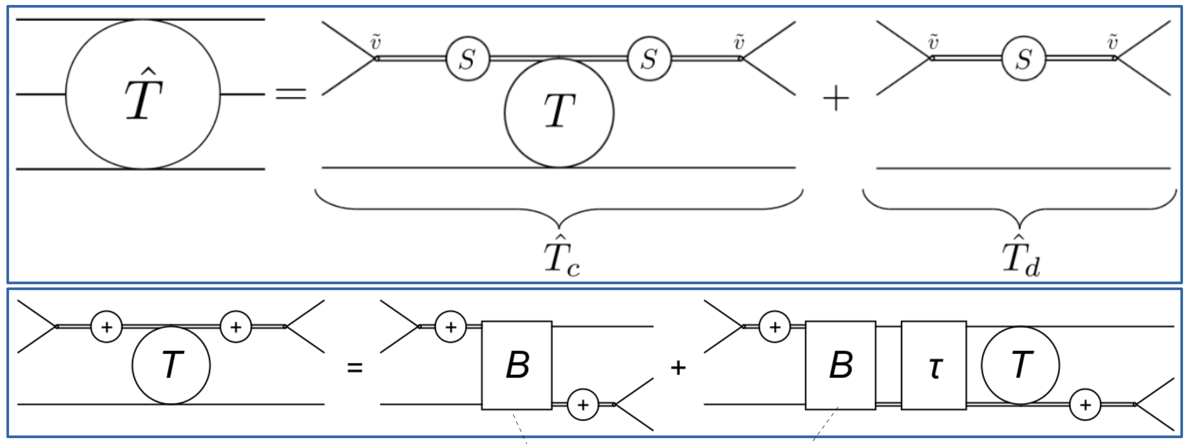
- twice subtracted dispersion relation in invariant mass: ***σ(k)***

$$-\frac{1}{S} = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3}\int d^3\boldsymbol{\ell}\,\frac{\lambda^2}{2E_\ell(\sigma(k)-4E_\ell^2+i\epsilon)}$$

- in the rest-frame of isobar (***Lorentz invariance!***)

Scattering amplitude

3 → 3 scattering amplitude is a 3-dimensional integral equation



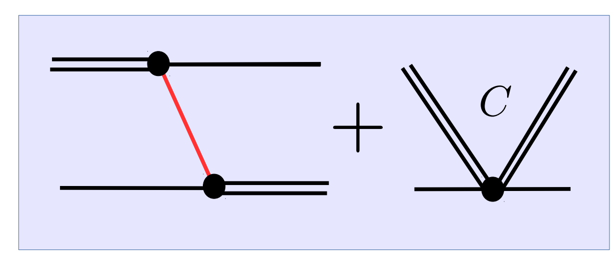
– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***v=λ*** (full relations available)

$$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$



- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

- One can map to field theory, but does not have to. Result is a-priori dispersive.

Scattering amplitude – analytic expression

External on-shell
2-body interaction

Recasting in on-shell
2→2 amplitudes +
real 3-body forces

$$\langle q_1, q_2, q_3|\hat{T}_c(s)|p_1, p_2, p_3\rangle =$$
$$\frac{1}{3!}\sum_{n=1}^{3}\sum_{m=1}^{3} T_{22}(\sigma(q_n))\langle q_n|T(s)|p_m\rangle T_{22}(\sigma(p_m))$$

with

Real three-body force

Exchange force

$$\langle q|T(s)|p\rangle = \langle q|C(s)|p\rangle + \frac{1}{m^2-(P-p-q)^2-i\epsilon}$$
$$-\int\frac{\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3}\frac{1}{2E_\ell}T_{22}(\sigma(\ell))\left(\langle\ell|C(s)|q\rangle\right.$$
$$\left.+\frac{1}{m^2-(P-p-\ell)^2-i\epsilon}\right)\langle\ell|T(s)|p\rangle$$

<u>On-shell</u> 2→ 2 interaction
(even within integral, but
without left-hand cuts)

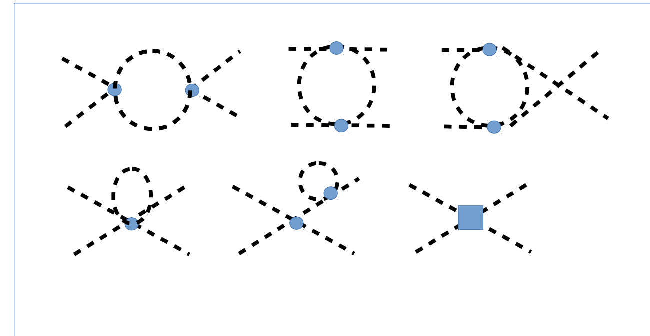
From two to three particles in finite volume

Finite-volume & chiral extrapolations

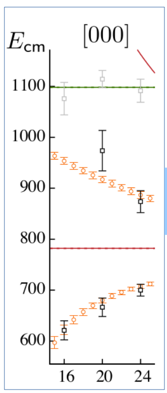
QCD calculations in finite volume

- unphysical pion mass
- (periodic) boundary conditions

→ <u>discrete momenta</u> & <u>discrete spectrum</u>

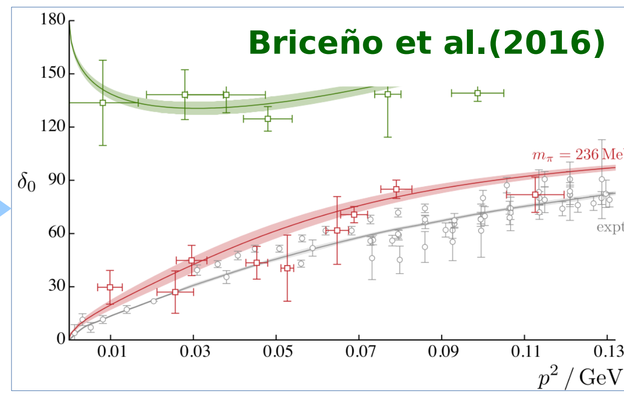


Recipe for 2 → 2 scattering (e.g. *I=J=0* ππ scattering)



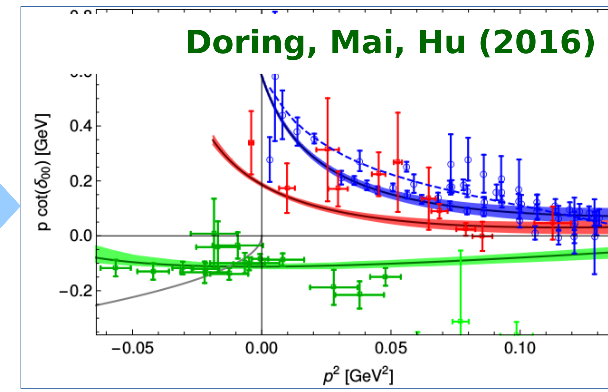
HSC(2016)

step 1



(This step can be skipped)

step 2



<u>LÜSCHER(1986)</u>

- 1 eigenenergy ↔ 1 phase-shift in infinite volume
- also with coupled channels **He et al. (2005)**

Doring, Prelovsek, HSC

<u>CHIRAL EXTRAPOLATIONS</u>

- M_π dependence from NLO ChPT (IAM)

Gasser, Leutwyler(1981)

Dobado, Pelaez (1997)

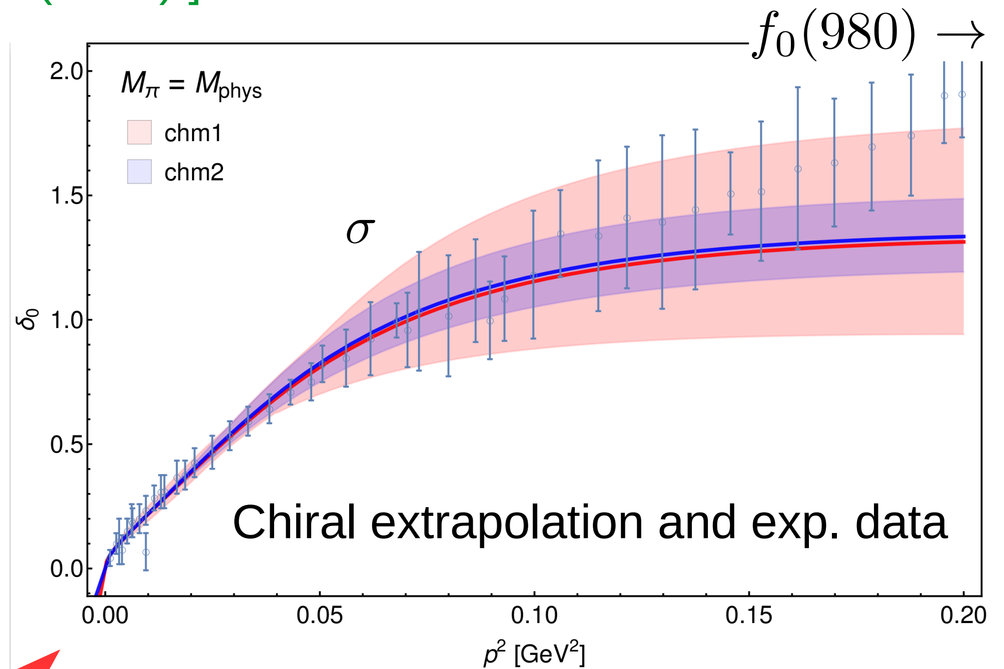
- Extrapolation in flavor

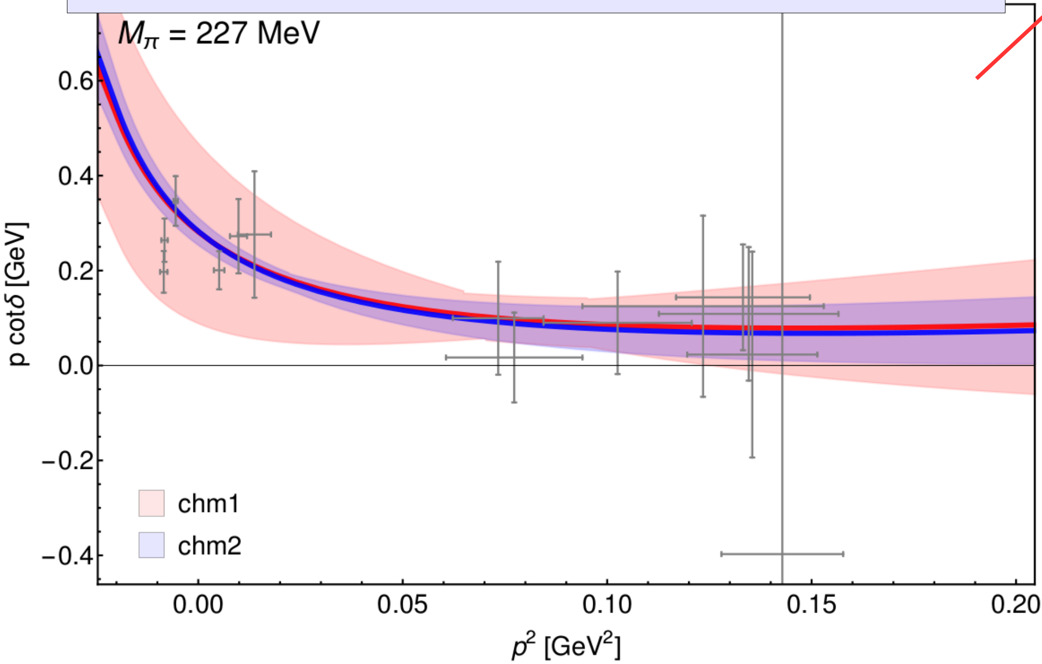
B. Hu, MD, R. Molina M. Mai et al. (2016)

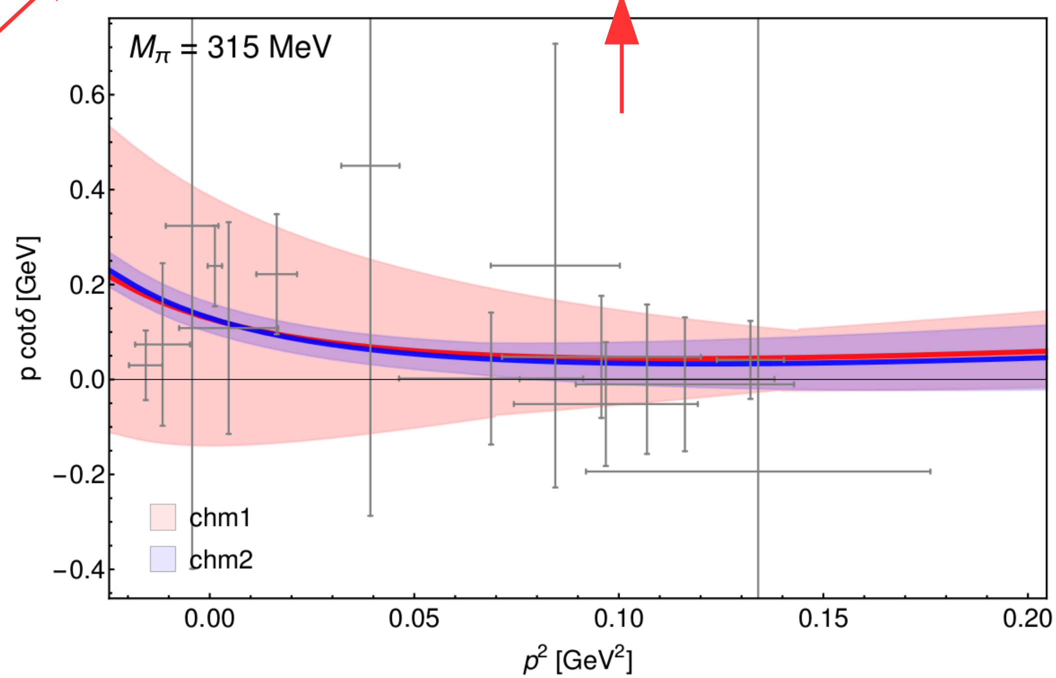
GWU lattice group: the isoscalar sector

[Guo, Alexandru, Molina, M.D., M. Mai, PRD (2018)]

- nHYP-smeared clover fermions with mass-degenerate quark flavors (N_f = 2)
- M_π=227 MeV and 315 MeV
- 3 elongated boxes
- Large variational basis including several meson-meson operators
- Moving frames
- Conformal mapping for σ pole extraction
- Unitarized Chiral Perturbation Theory fits for chiral extrapolation:
  chm1: $I = L = 0,\ M_\pi = 227, 315\,\mathrm{MeV}$
  chm2: $I = L = 0,\ 1,\ M_\pi = 227, 315\,\mathrm{MeV}$







Chiral extrapolation of σ pole

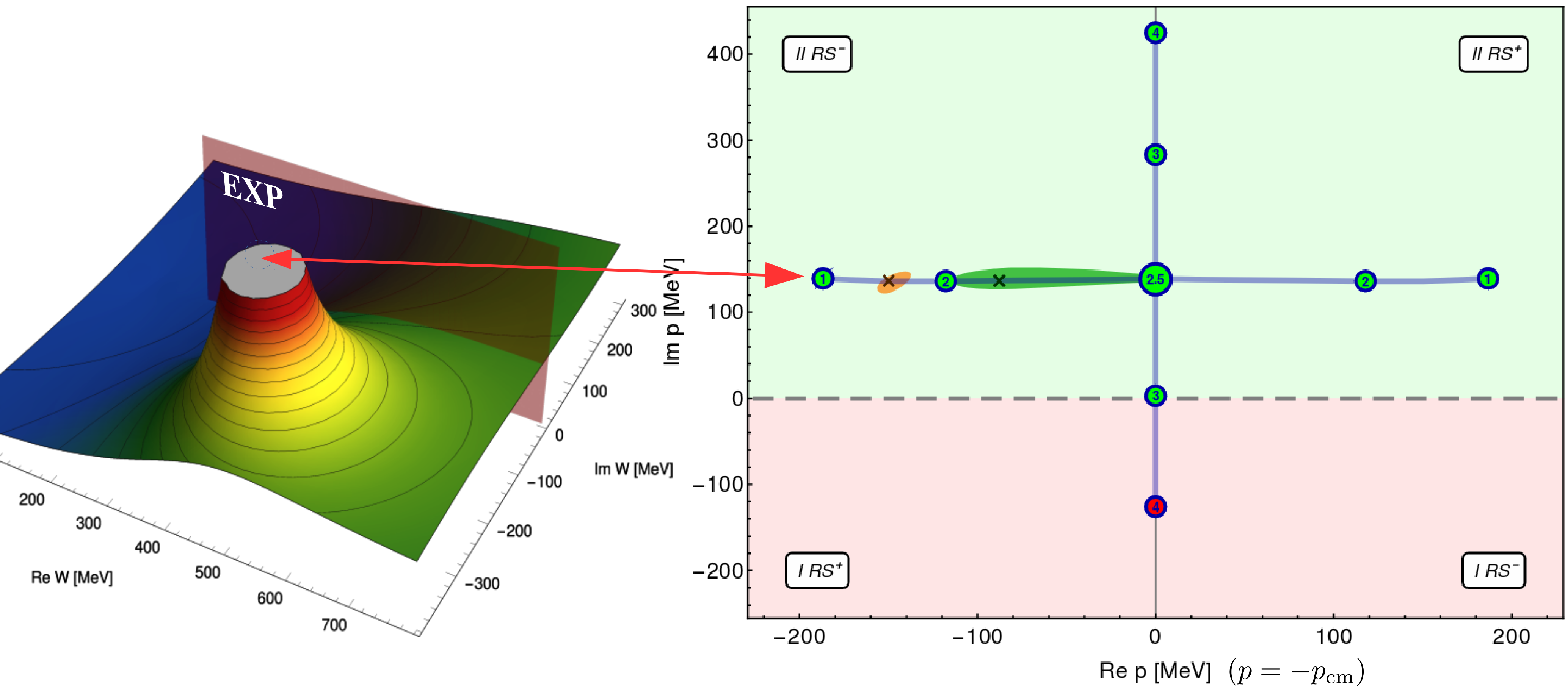
Parametrization	Fitted data	$M_\pi = 138$ MeV Re z^*	$-$ Im z^*	$\lvert g \rvert$
chm1	$\sigma_{227,315}$	440^{+60}_{-90}	240^{+20}_{-50}	$3.0^{+0.2}_{-0.6}$
chm2	σ_{227} ρ_{227}	430^{+20}_{-30}	250^{+30}_{-30}	$3.0^{+0.1}_{-0.1}$
chm2	σ_{315} ρ_{315}	460^{+10}_{-15}	210^{+40}_{-30}	$3.0^{+0.1}_{-0.1}$
chm2	$\sigma_{227,315}$ $\rho_{227,315}$	440^{+10}_{-16}	240^{+20}_{-20}	$3.0^{+0.0}_{-0.0}$
Ref. [1]	experimental	449^{+22}_{-16}	275^{+12}_{-12}	$3.5^{+0.3}_{-0.2}$

[1] J. R. Pelaez, Phys. Rept. **658**, 1 (2016), arXiv:1510.00653 [hep-ph].

[Consistent with conformal-mapping amplitude parametrization (model-independent, not shown)]

Pole trajectory

First prediction: Hanhart, Pealez, Rios, PRL (2008)



→ σ becomes a (virtual) bound state @ M_π = (345) 415 MeV

THREE-BODY AMPLITUDE IN A BOX

M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]

Overview

Lüscher-like formalism in 3 → 3 case is under investigation

Polejaeva/Rusetsky (2012)

Briceño/Hansen/Sharpe (2014, 2015, 2016, 2017)

Non-relativistic approaches based on dimer picture & effective field theory

Kreuzer, Griesshammer(2012), Hammer et al. (2016, 2x)

F. Romero, Rusetsky, Urbach et. al. (2018)

Requirements

- 3-body systems involve (resonant) two-body sub-amplitudes: Construct such that 2-body information can be included
- Need extrapolations between different energies (problem of underdetermination)
- Allow for systematic improvement by allowing more and more quantum numbers as lattice data improve (problem of underdetermination)
- At least, **all** possible intermediate on-shell configurations must be identified and included to ensure all power-law finite-volume effects are taken account of.
- Formulation that lattice practice can connect to → isobars

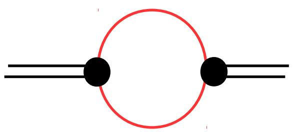
$\Longrightarrow$ This work: **Quantization condition from 3-body unitarity in isobar formulation**

How to derive the 2-body quanti-zation condition

Two-body unitarity

↓

On-shell condition



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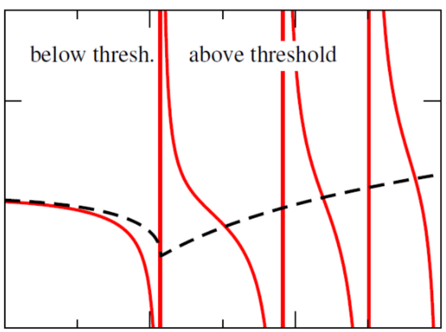
Imaginary parts



Infinite → Fin. Vol

↓

Power-law fin-vol. effects

↓

Lüscher

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

How to derive the 2-body quantization condition

Two-body unitarity

↓

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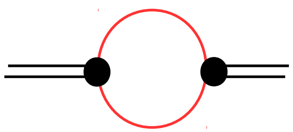

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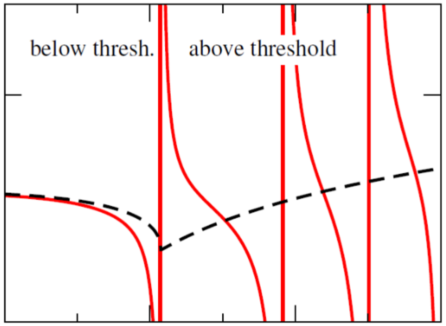

Infinite → Fin. Vol

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Power-law fin-vol. effects

↓

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$$\boxed{p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)}$$

.

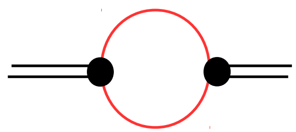
Three-body?

Analogously!

Two-body unitarity

↓

On-shell condition



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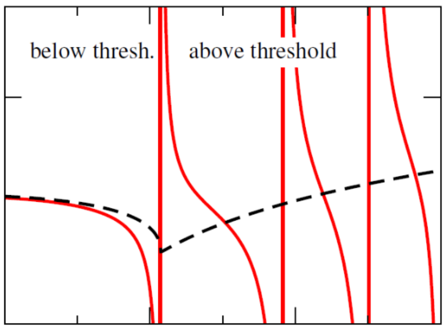

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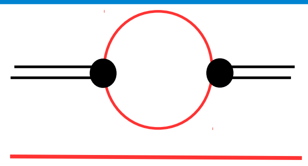
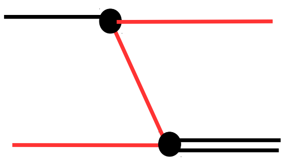
Three-body unitarity

↓

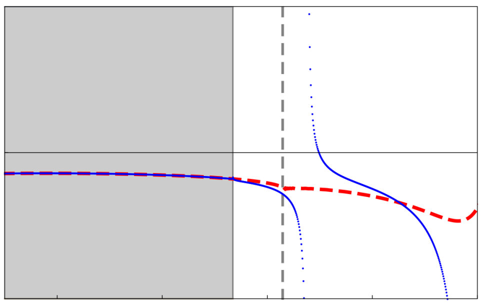
On-shell condition

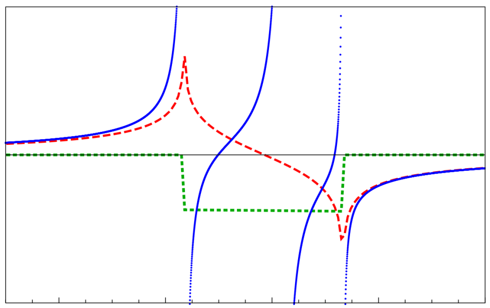




↓

Imaginary parts





↓

Power-law fin-vol. effects

↓

Quantization Condition

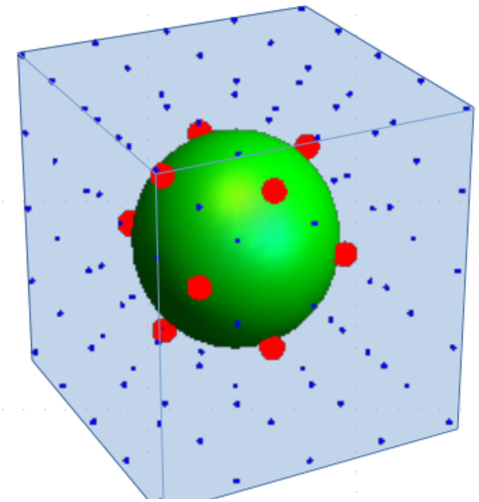
$$\boxed{\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}}$$

.

Projection to irreps

[M.D.,Hammer,Mai,Pang,Rusetsky,Wu(2018)]

- **Lüscher formalism relies on regular 2→ 2 potentials**
  - Now: manifestly singular interactions
  - Find generalization that projects also the interactions to the irreps of cubic symmetry, not only propagation
- **Separation of variables**
  - shells = sets of points related by O_h
  - Analogous to radial coordinate in infinite volume
- **Find the orthonormal basis for arbitrary functions defined on each point of a given shell.**



$$q_i = \frac{2\pi}{L} n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \to \frac{1}{L^3} \sum_s \sum_{i=1}^{\vartheta(s)}$$

- **J (inf. volume) → irreps (finite volume):** $\Gamma \in \{A_1^\pm, A_2^\pm, E^\pm, T_1^\pm, T_2^\pm\}$
- **<u>Partial wave projection</u> (inf. Volume) ➡ <u>Irrep. projection</u> (fin.)**

$$f(\mathbf{p}) = \sqrt{4\pi} \sum_{\ell m} Y_{\ell m}(\hat{p}) f_{\ell m}(p)$$

$$f_{\ell m}(p) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y^*_{\ell m}(\hat{p}) f(\mathbf{p})$$

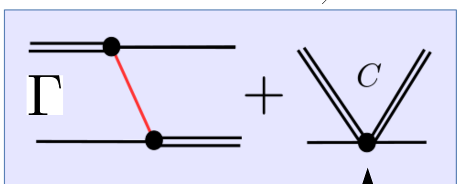
$$f^s(\hat{p}_j) = \sqrt{4\pi} \sum_{\Gamma\alpha} \sum_a f_a^{\Gamma\alpha s} \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

$$f_a^{\Gamma\alpha s} = \frac{\sqrt{4\pi}}{\vartheta(s)} \sum_{j=1}^{\vartheta(s)} f^s(\hat{p}_j) \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

(a is index u in quantization condition; Quantization condition has projection in incoming AND outgoing basis states with indices u, u')

Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{s}\mathbf{s}'}_{\mathbf{u}\mathbf{u}'}(\mathbf{W}^2) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W}^2)^{-1}\delta_{\mathbf{s}\mathbf{s}'}\delta_{\mathbf{u}\mathbf{u}'}\right) = \mathbf{0}$$



Fix to 3→3 data

W – total energy

s/s' - shell index

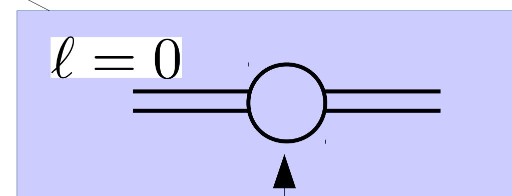
u/u' - basis index

ϑ – multiplicity

L – lattice volume

E_s – spect. energy

Determinant of (s,u) x (s',u') matrix at fixed W, Γ, L



Fix to 2→2 data:

$T_{22} = v\,\tau\,v$

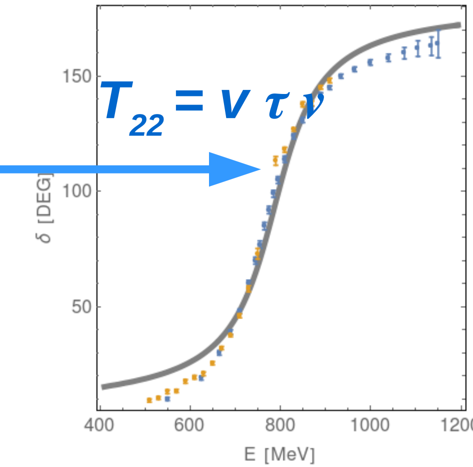
- Not a Lüscher-like equation ("left": infinite volume, "right": finite volume)
- Instead: Fix parameters to lattice eigenvalues
- With parameters fixed, evaluate infinite-volume amplitude
- Same workflow as in many 2-body coupled-channel fits (see, e.g., M.D., Meißner, Oset, Rusetsky, EPJA (2012))

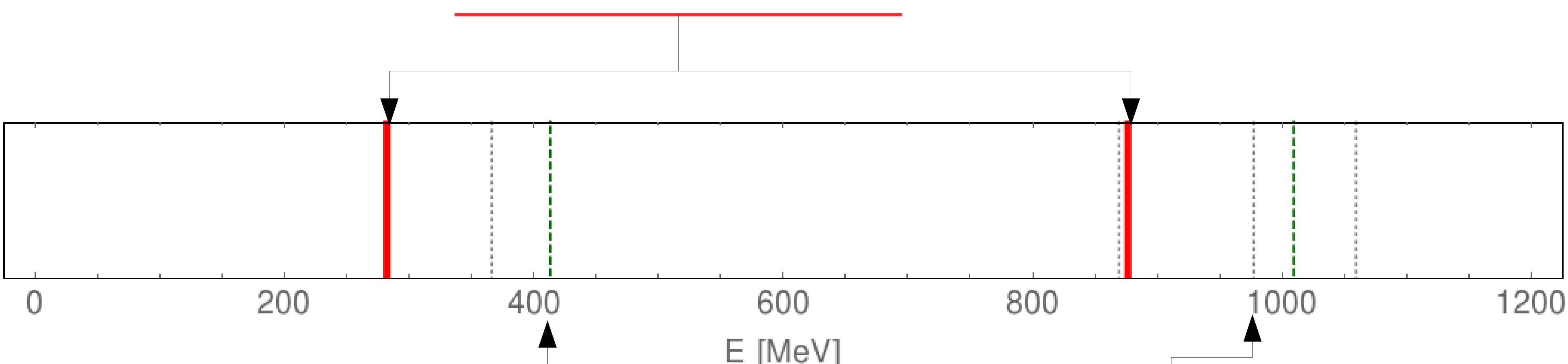
Numerical demonstration

[M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]]

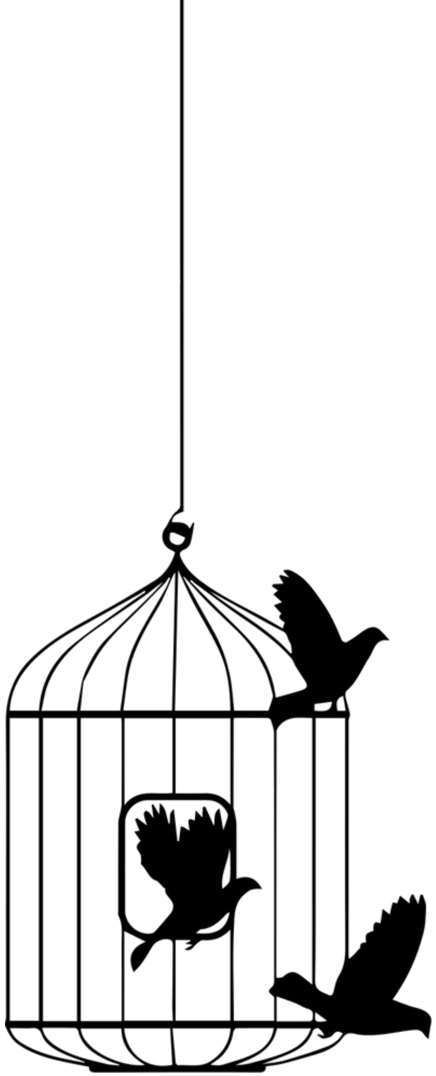
- Numerical demonstration of three-body finite volume formalism
- 3 particles in finite volume: ***m=138 MeV, L=3 fm***
- one S-wave isobar → two unknowns:
  - – vertex(Isobar→2 stable particles)
  - – subtraction constant (~mass)
- **Project to $\Gamma = A^{1+}$**

  → prediction of 3body energy-eigenlevels (C=0)





unphysical lvls cancel out (exact proof available)



A physical system:

$\pi^+\pi^+\pi^+$

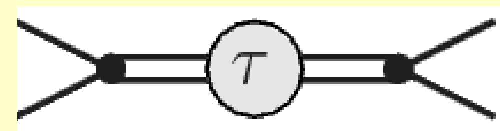
Mai, M.D., arXiv:1807.04746

Three positive pions

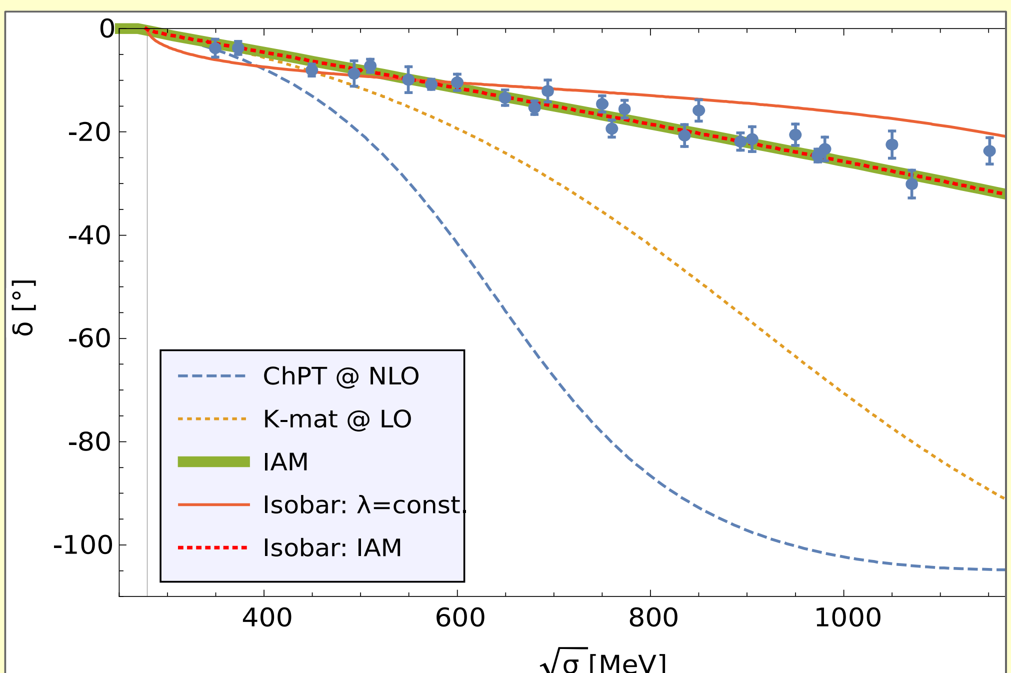
- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → ***Q:*** *does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

I. 2-body subchannel:

- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$\frac{T^2_{\rm LO}}{T_{\rm LO}-T_{\rm NLO}}$$



1) ~~**Fix λ, M_0 to exp. data**~~

☹ incoorrect m_π behavior!

2) ~~**Chiral NLO & K-matrix**~~

☹ works badly for high energies

3) **Inverse Amplitude** Truong(1988)

☺ correct σ & m_π behavior

☺ parameters known Gasser/Leutwyler(1984)

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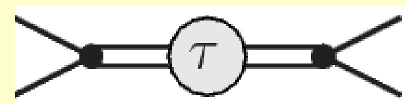
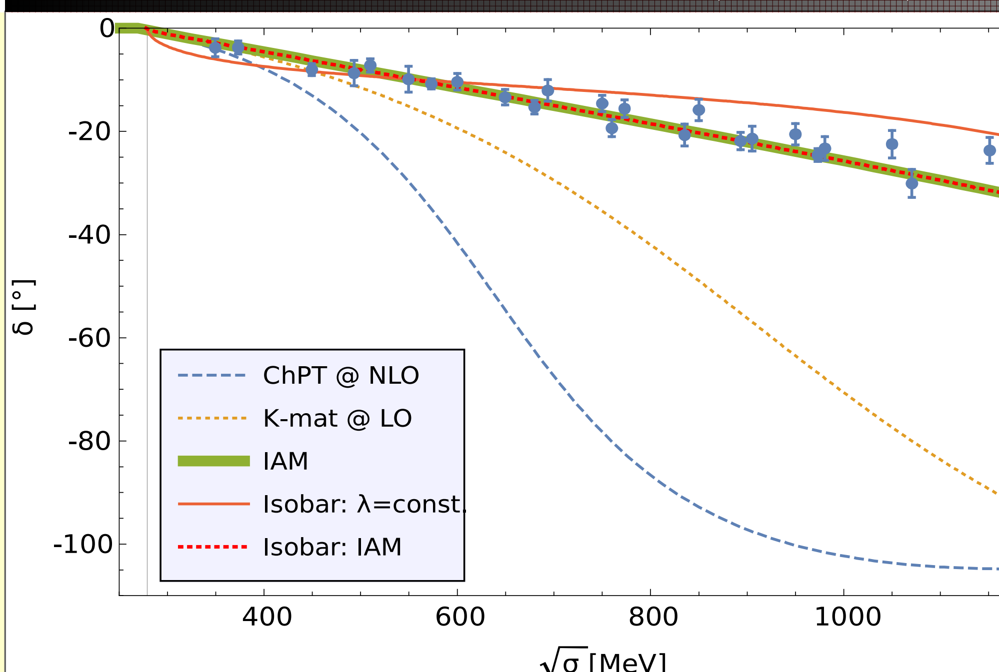
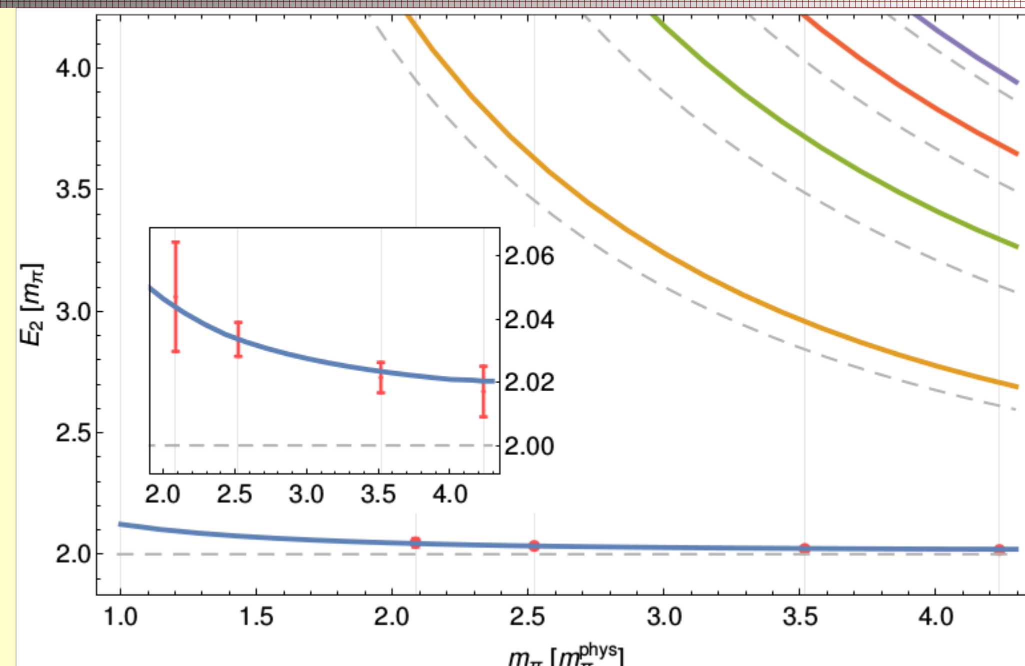


$$\frac{T_{\rm LO}^2}{T_{\rm LO}-T_{\rm NLO}}$$

discretize (Lüscher) → predicted fin-vol. spectrum





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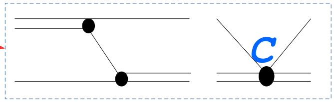
II. 3-body spectrum

Remaining unknown: ***C***

QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma\mathbf{ss'}}_{\mathbf{uu'}}(\mathbf{W^2})+\frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right)=\mathbf{0}$$



- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $\boxed{C_{\boldsymbol{qp}}=c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})}$

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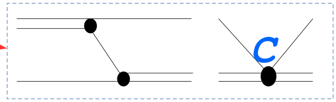

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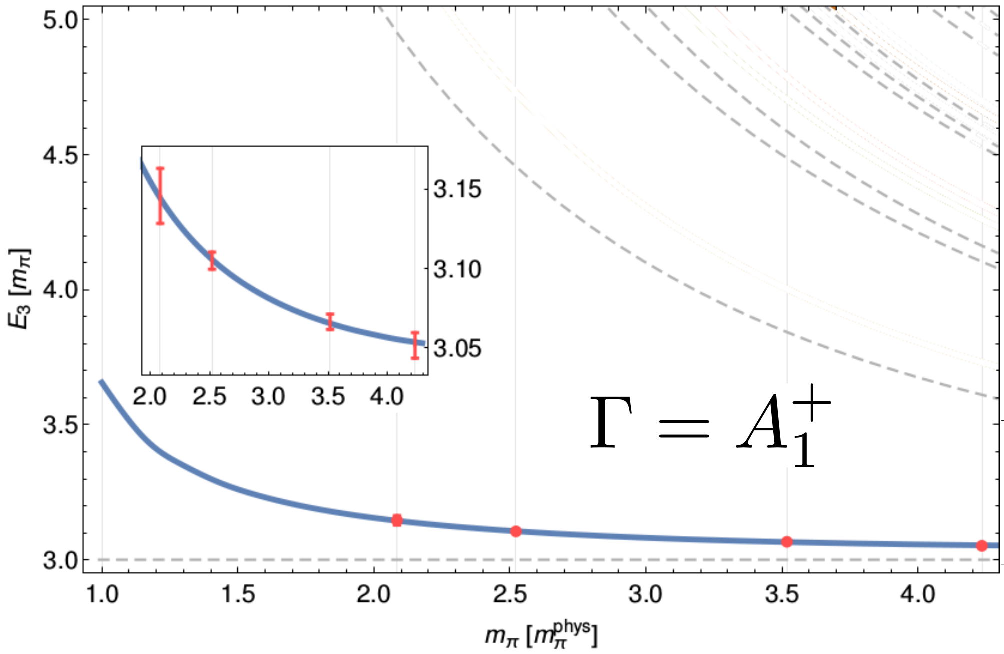

Fit *C* to NPLQCD ground state level

→ ***C=(0.2 ± 1.5)·10^{-10}***

First prediction of excited levels for physical system

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Detmold et al. (2008)

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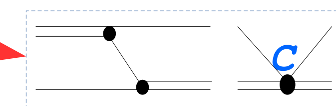

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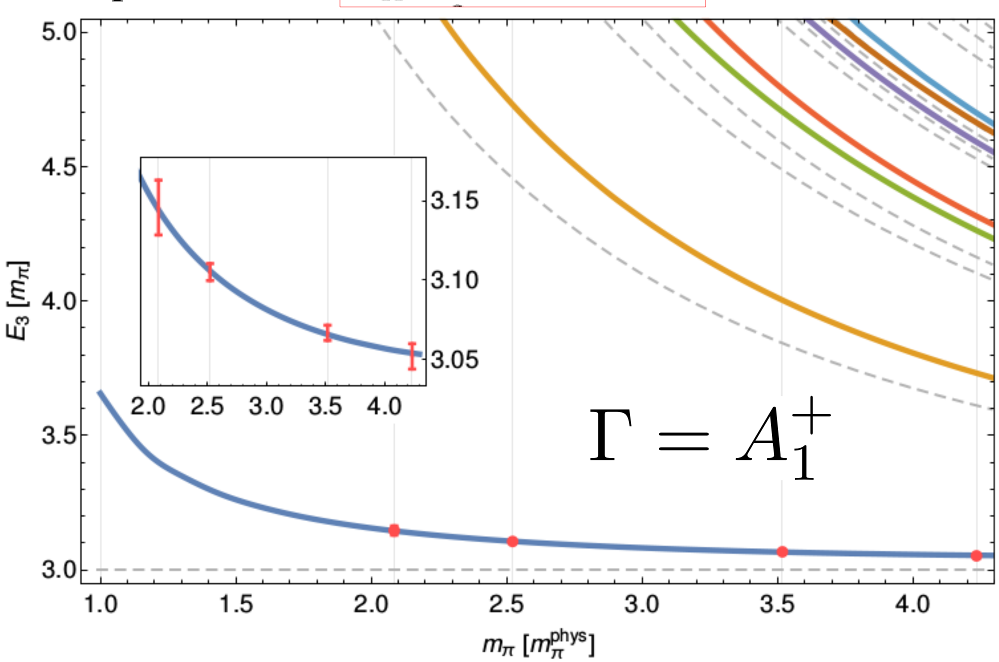


Predict excited spectrum:

→ novel pattern

1/1 of interacting/non-interacting lvls

→ all QC-poles are simple

→ chiral extrapolation to phys point

(under assumptions)

Summary

3-body amplitude in infinite volume

- 3-body unitarity dictates on-shell condition (driving term & isobar propagator)
- Result: 3-dim. relativistic integral equations, explicit proof of 3B unitarity above threshold
- Equivalent to Khuri-Treiman equations*

Finite volume investigation:

- On-shell condition dictates leading, power-law finite-volume effects
- Quantization condition
- Bare-bone, stripped-down infinite-volume extrapolation tool (in spirit of Lüscher equation)
- First numerical application to physical system $\pi^+\pi^+\pi^+$

OUTLOOK

→ include spin isobars & multiple isobars

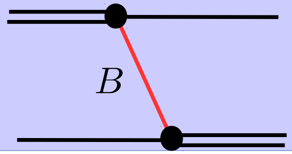
→ unequal masses

→ practical studies: $a_1(1260)$, Roper...

* Ian Aitchison, private communication

SPARES

The Power of Unitarity

Question: Does  provide full imaginary part of all possible 3→ 3 transitions?

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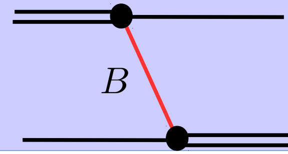
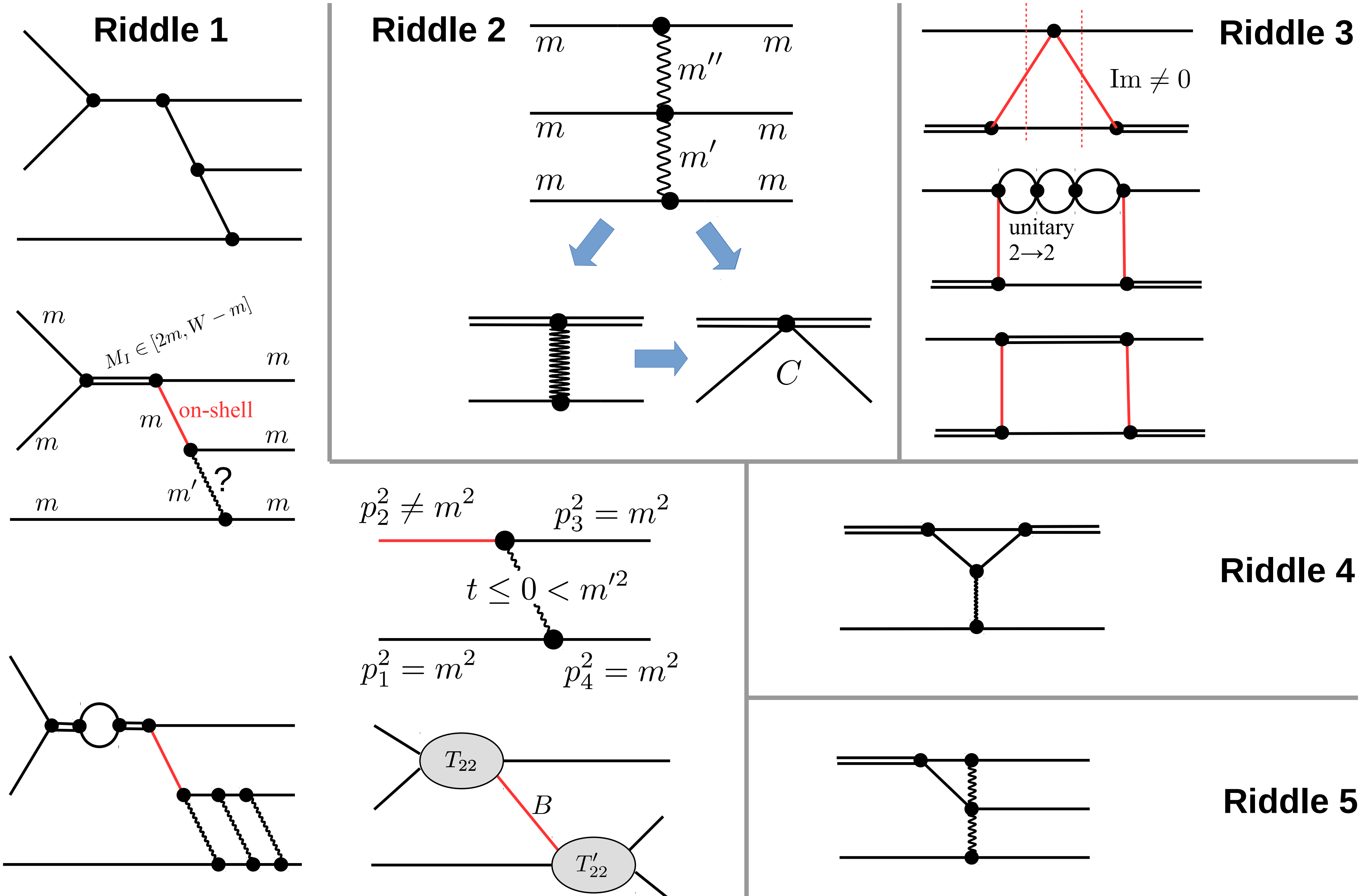
Question: Does  provide full imaginary part of all possible 3→ 3 transitions?



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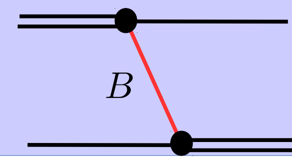
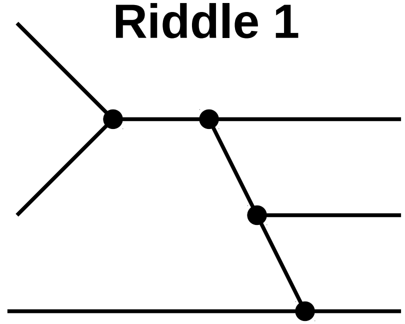
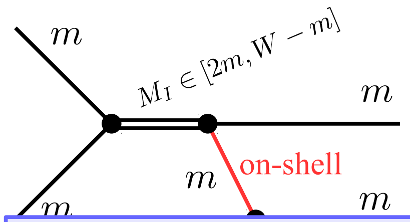
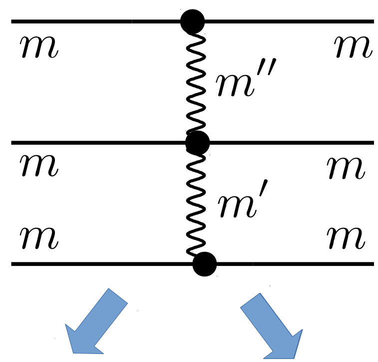
Question: Does  provide full imaginary part of all possible 3→ 3 transitions?

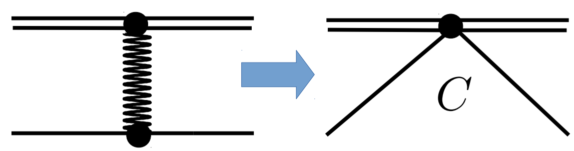
Riddle 1



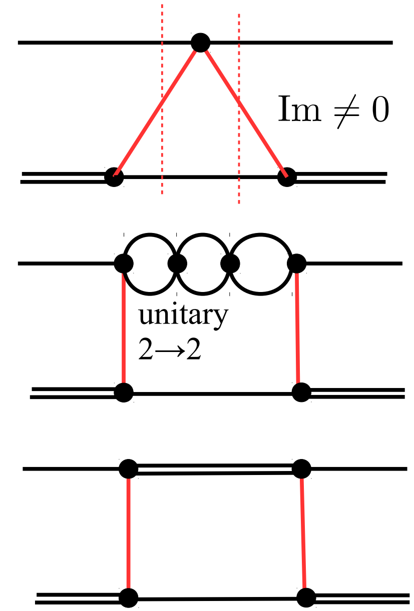


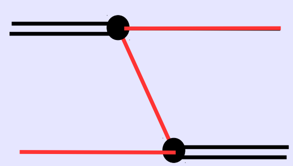
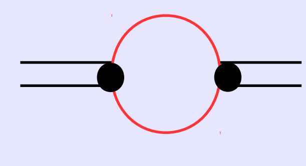
Riddle 2



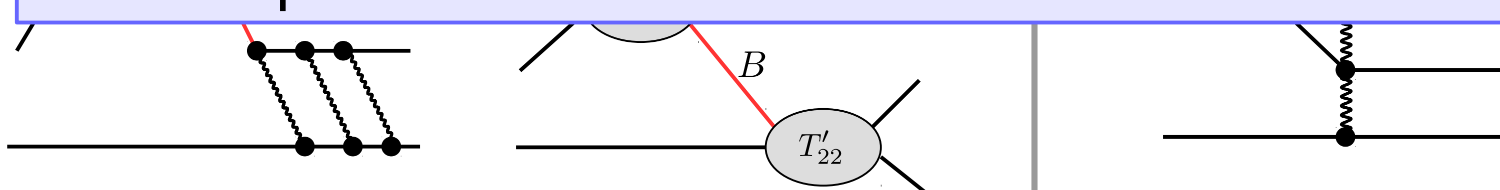


Riddle 3



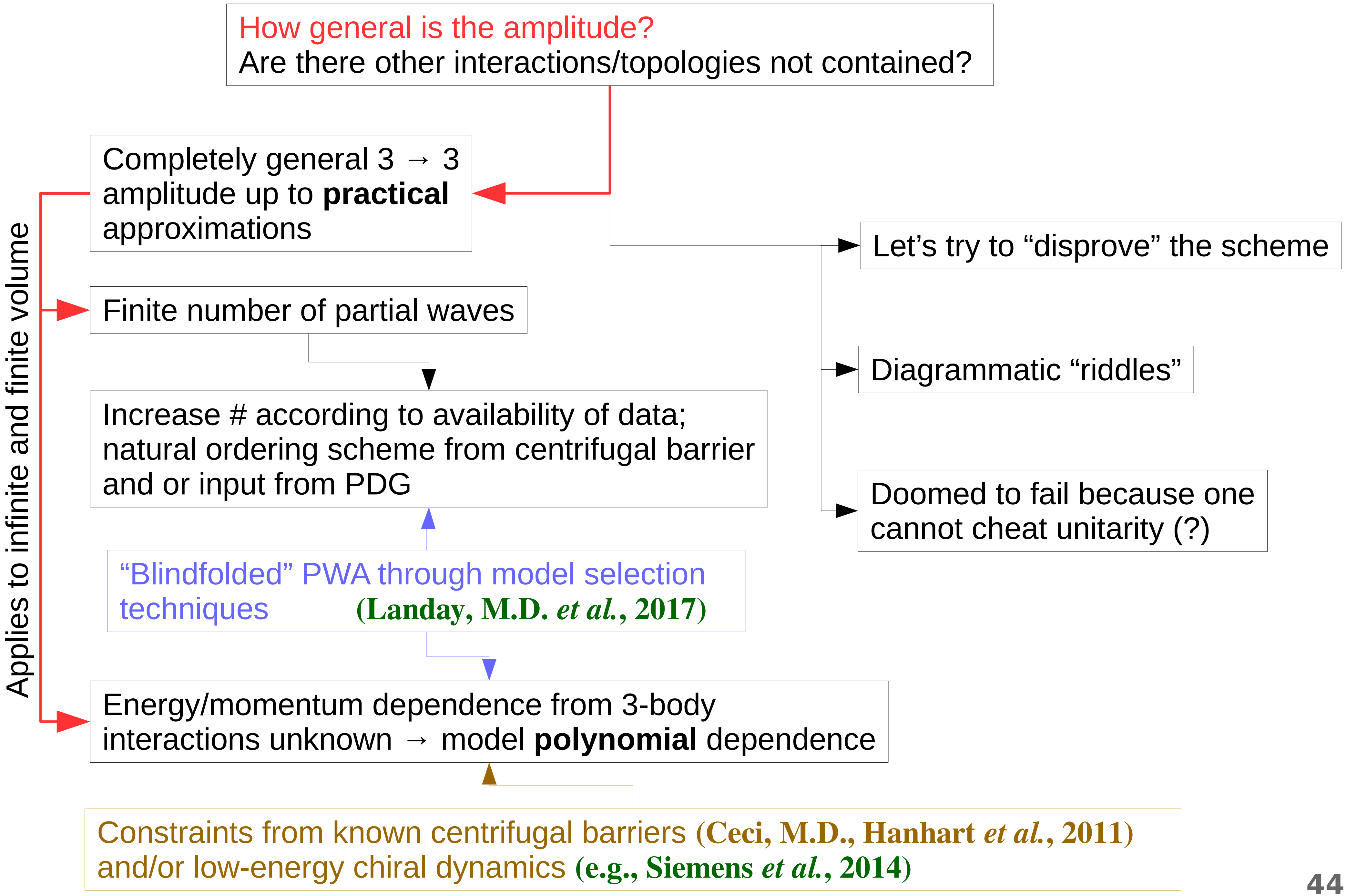
Answer: Yes.  and 

are the only on-shell configurations in physical region. Three-body unitarity avoids many artificial complications of diagrammatic expansions.

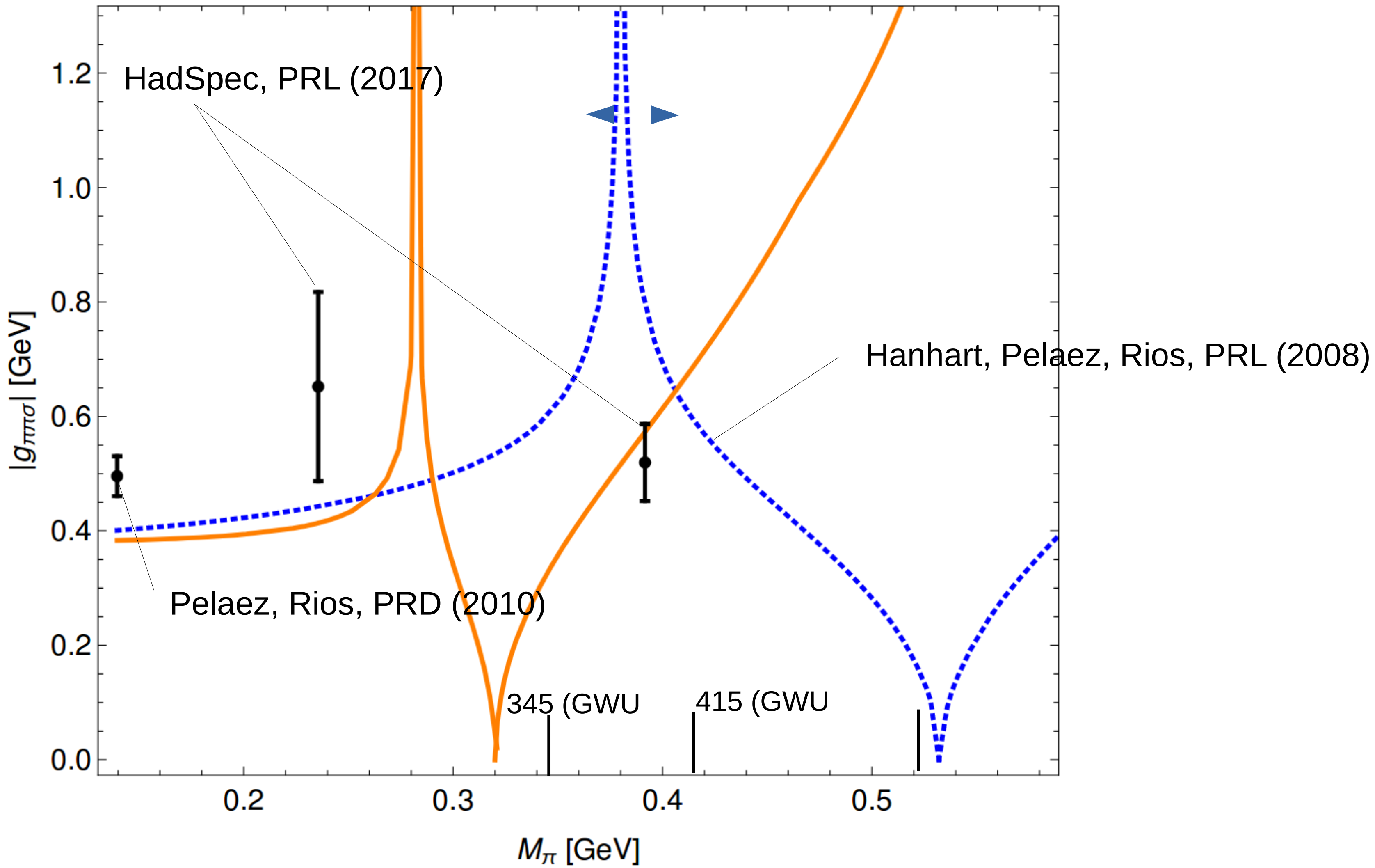


Riddle 5

The Power of Unitarity



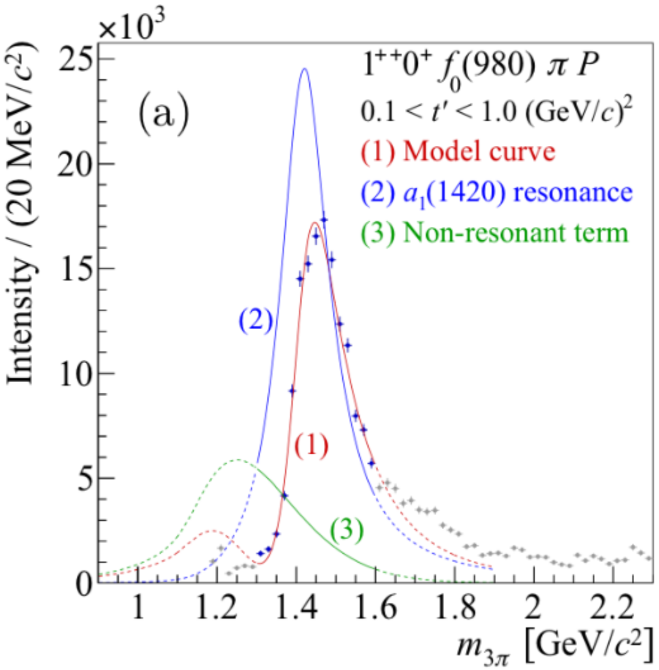
Residues



HadSpec, PRL (2017)
Hanhart, Pelaez, Rios, PRL (2008)
Pelaez, Rios, PRD (2010)
345 (GWU
415 (GWU
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.3
0.4
0.5
$|g_{\pi\pi\sigma}|$ [GeV]
M_π [GeV]

Connection to triangle diagrams

Interesting application: a1(1420)
- observed in COMPASS
- in $f_0(980)\pi$ final state

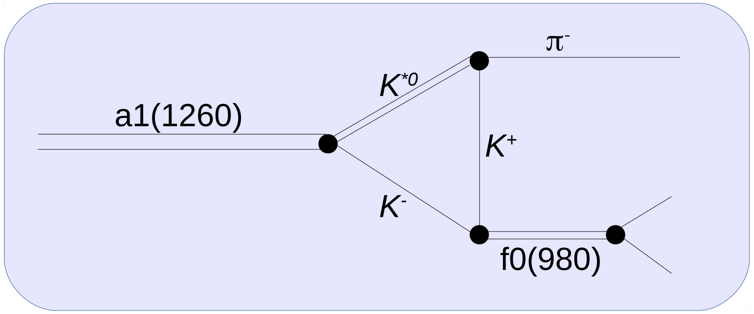


ONE EXPLANATION:
Log-like behavior of the "triangle-diagram"

Mikhasenko/Ketzer/Sarantsev(2015)

Aceti/Dai/Oset(2016)

- **Q:** Does such a feature exist in full 3b-unitary FSI?



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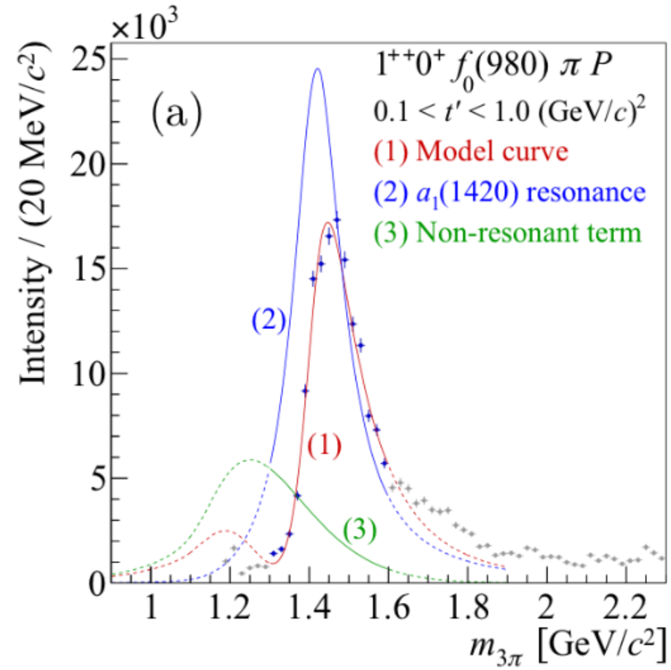

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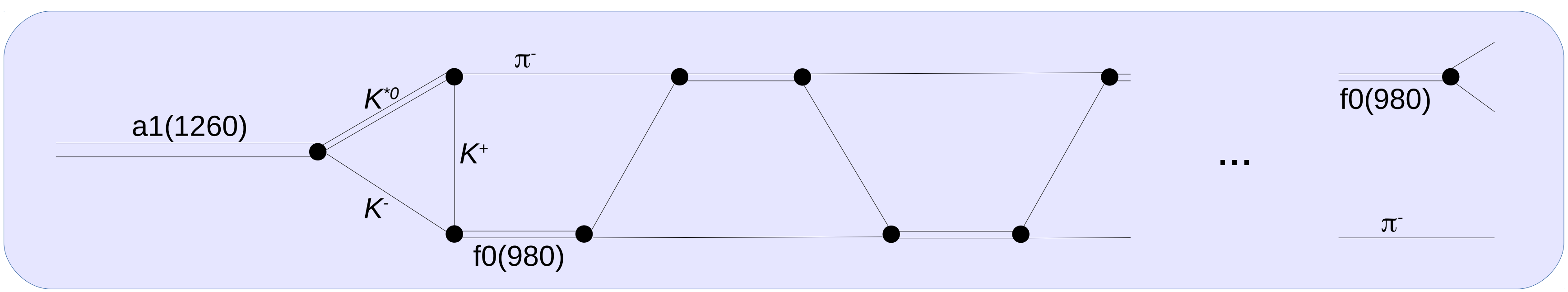


$\pi\pi\pi \leftrightarrow \pi K\bar{K}$

Coupled-channel problem

Sadasivan, M. Mai, M.D. in progress...

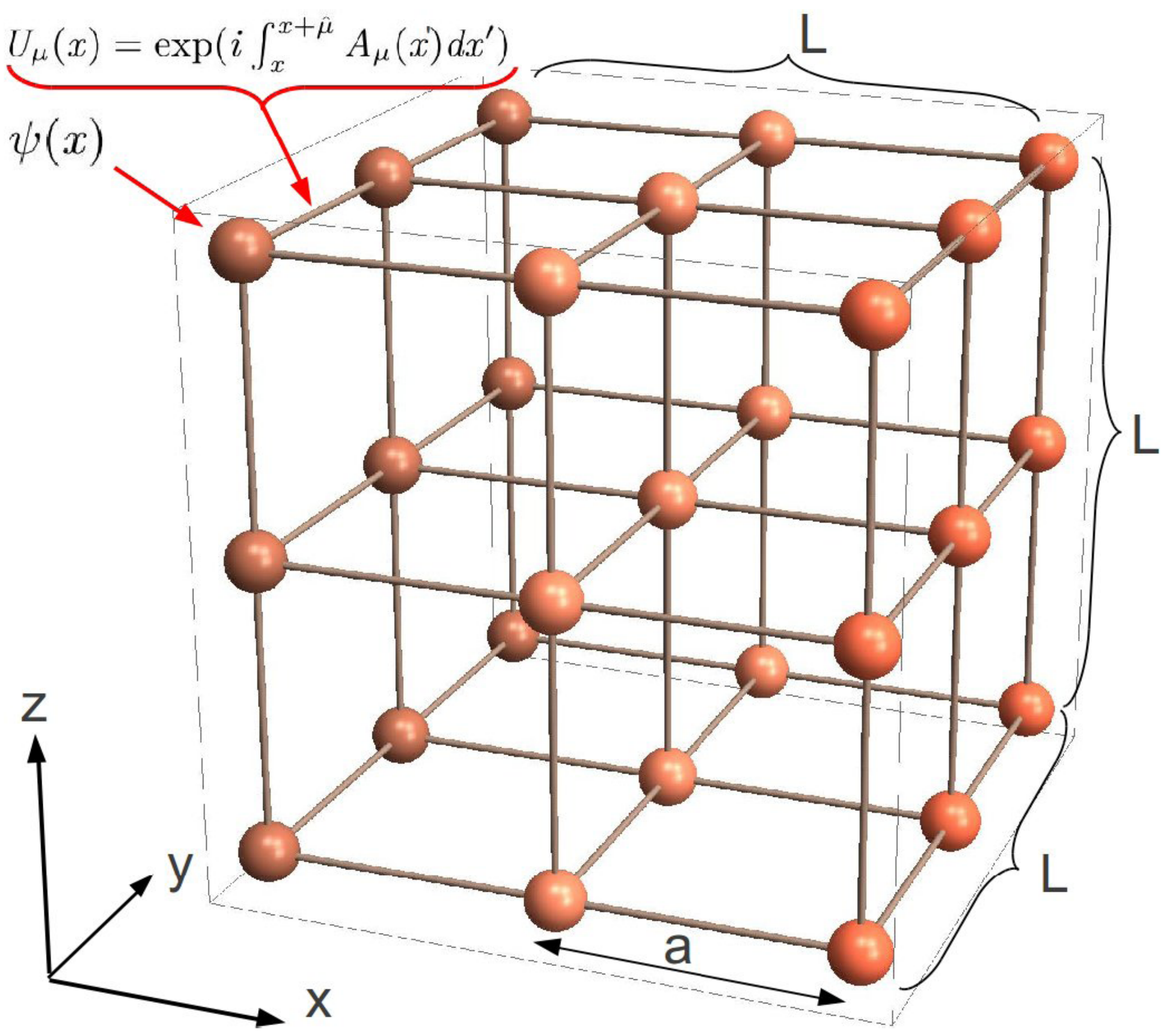
It can be shown that this is equivalent
To Khuri Treiman equations

I. Aitchison, private communication

Two-body scattering on lattice

Input for 3-body

The cubic lattice



- Side length L, periodic boundary conditions $\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i L)$ $\rightarrow$ finite volume effects $\rightarrow$ Infinite volume $L \rightarrow \infty$ extrapolation
- Lattice spacing a $\rightarrow$ finite size effects Modern lattice calculations: $a \simeq 0.07$ fm $\rightarrow p \sim 2.8$ GeV $\rightarrow$ (much) larger than typical hadronic scales;

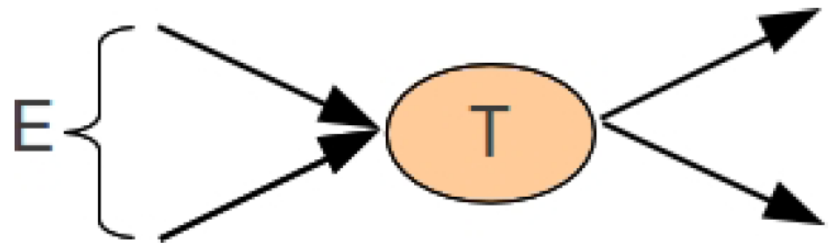
  not considered here.
- Unphysically large quark/hadron masses $\rightarrow$ (chiral) extrapolation required.

Two body scattering

In the infinite volume

- Unitarity of the scattering matrix S: $SS^\dagger = \mathbb{1}$ $\quad\quad [S = \mathbb{1} - i\,\frac{p}{4\pi E}\,T]$.

$$\mathrm{Im}\, T^{-1}(E) = \sigma \equiv \frac{p}{8\pi\, E}$$



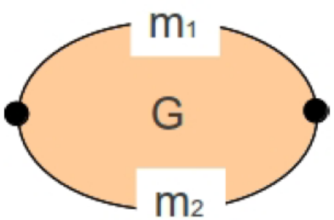
- $\rightarrow$ Generic (Lippman-Schwinger) equation for unitarizing the T-matrix:

$$T = V + V\,G\,T \qquad \mathrm{Im}\, G = -\sigma$$

V: (Pseudo)potential, σ: phase space.

- G: Green's function:

$$\begin{aligned} G &= \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon}, \\ \omega_{1,2}^2 &= m_{1,2}^2 + \vec{q}^{\;2} \end{aligned}$$



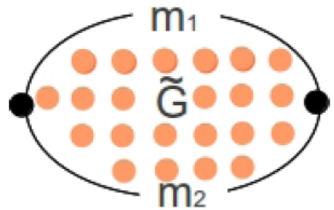
Discretization

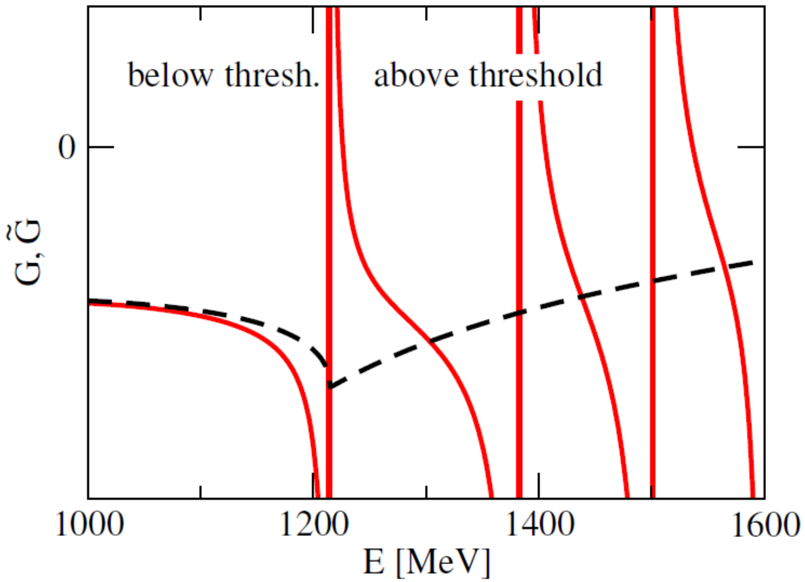
Discretized momenta in the finite volume with periodic boundary conditions

$$\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L) = \exp\left(i\, L\, q_i\right) \Psi(\vec{x}) \quad \Longrightarrow \quad q_i = \frac{2\pi}{L}\, n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

$$\boxed{\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}\,|^2) \to \frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L}\, \vec{n}, \quad \vec{n} \in \mathbb{Z}^3}$$

$$G \to \tilde{G} = \frac{1}{L^3} \sum_{\vec{q}} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2}$$





- $E > m_1 + m_2$: $\tilde{G}$ has poles at *free energies in the box*, $E = \omega_1 + \omega_2$
- $E < m_1 + m_2$: $\tilde{G} \to G$ exponentially with L (regular summation theorem).

Finite → infinite volume: the Lüscher equation

Warning: rather crude re-derivation

- Measured eigenvalues of the Hamiltonian (tower of *lattice levels* $E(L)$)
  → Poles of scattering equation $\tilde{T}$ in the finite volume → determines V:

$$\tilde{T} = (1 - V\tilde{G})^{-1}V \;\rightarrow\; \quad V^{-1} - \tilde{G} \stackrel{!}{=} 0 \;\rightarrow\; V^{-1} = \tilde{G}$$

- The interaction V determines the T-matrix in the infinite volume limit:

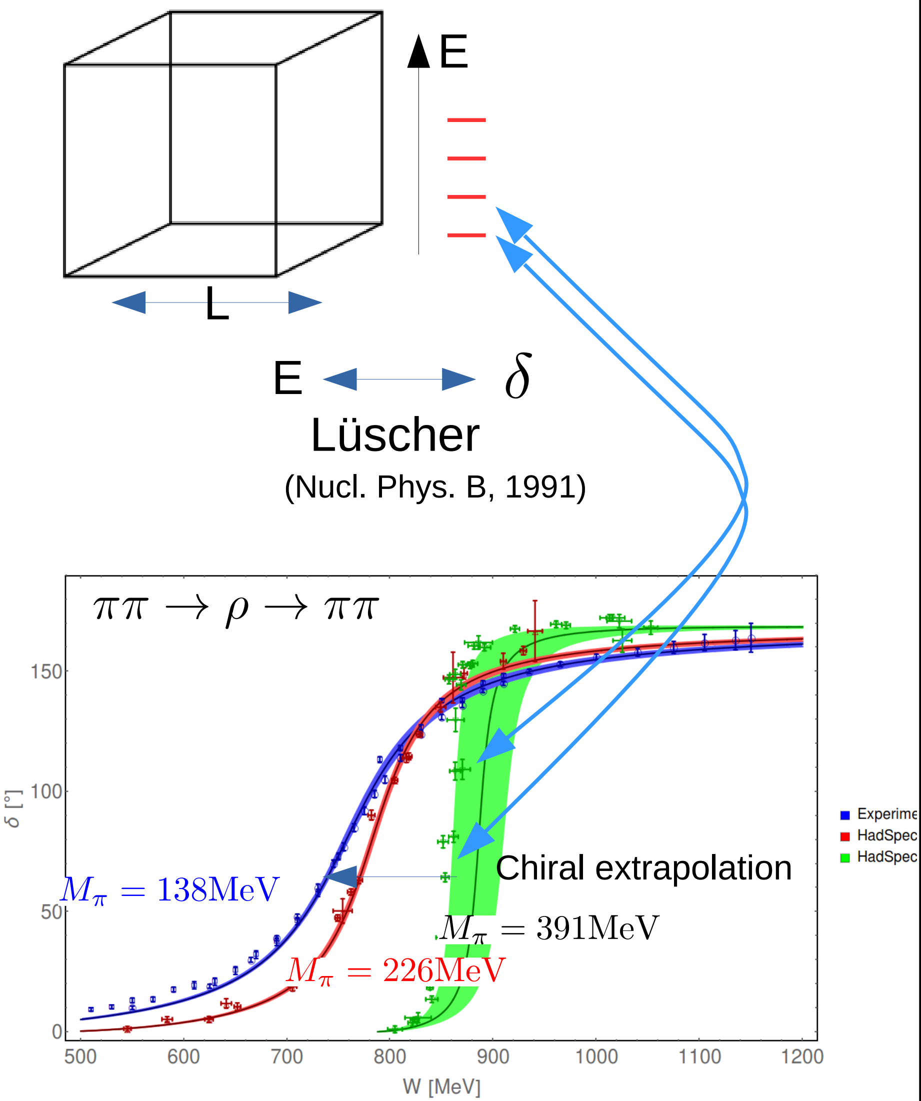
$$T = \left(V^{-1} - G\right)^{-1} = \left(\tilde{G} - G\right)^{-1}$$

- Re-derivation of Lüscher's equation (T determines the phase shift δ):

$$\boxed{p \cot \delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)}$$

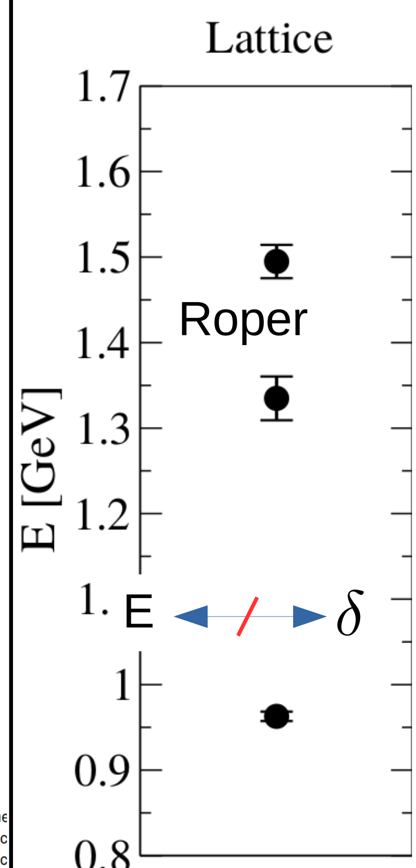
  - V and dependence on renormalization have disappeared (!)
  - p: c.m. momentum
  - E: scattering energy
  - $\tilde{G} - \mathrm{Re}G$: known kinematical function
    ($\simeq \mathcal{Z}_{00}$ up to exponentially suppressed contributions)
  - One phase at one energy.

Two-body vs. Three-body



Data: HadronSpectrum (Dudek, PRD 2013,Briceño PRL 2016);
Analysis: M.D., B. Hu, M. Mai, PLB (2018)
See also: Bolton, Briceno, Wilson, Phys.Lett. B757 (2016) 50

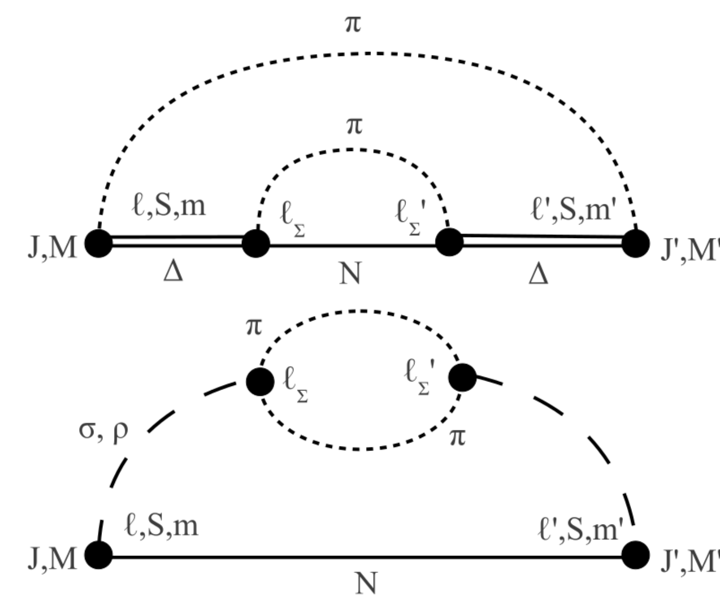
$$M_\pi \approx 156\,\mathrm{MeV}$$



Channels:

$$\pi N, \eta N, \pi\pi N(\sigma N, \pi\Delta, \dots)$$

Genuine three-body dynamics



Data: [Lang et al., Phys.Rev. D95 (2017), 014510]

Three-body methods:

- Briceño, Hansen, Sharpe PRD96 (2017)
- Hammer, Pang, Rusetsky JHEP (2017)
- ...

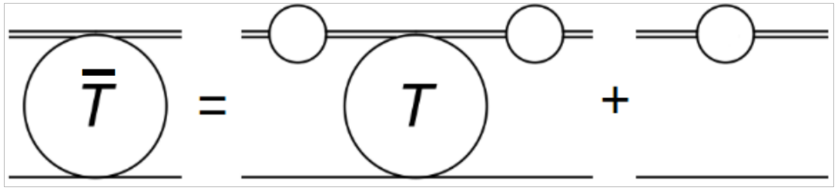
Large # of d.o.f. require efficient parametrizations

Example: The coupled-channel 2→2, 2→3, 3→3 meson-baryon system

μ	$J^P =$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^-$	$\frac{5}{2}^+$	$\frac{7}{2}^+$	$\frac{7}{2}^-$	$\frac{9}{2}^-$	$\frac{9}{2}^+$
1	πN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
2	$\rho N(S=1/2)$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
3	$\rho N(S=3/2, \|J-L\|=1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
4	$\rho N(S=3/2, \|J-L\|=3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
5	ηN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
6	$\pi\Delta(\|J-L\|=1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
7	$\pi\Delta(\|J-L\|=3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
8	σN	P_{11}	S_{11}	D_{13}	P_{13}	F_{15}	D_{15}	G_{17}	F_{17}	H_{19}	G_{19}
9	$K\Lambda$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
10	$K\Sigma$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}

including 3-body dynamics [Julich-Bonn; ANL-Osaka].

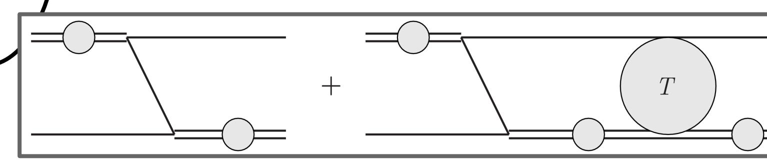
Cancellations



→ fin. vol. normalization of δ-distribution!

$$\bar{T}^{A_1^+}_{nm}(s) = \tau_n(s) T^{A_1^+}_{nm}(s)\tau_m(s) - 2E_n\tau_n(s)\frac{L^3}{\vartheta(n)}\delta_{nm}$$

$$T^{A_1^+}_{nm}(s) = B^{A_1^+}_{nm}(s) - \frac{1}{L^3}\sum_{x\in sets_8}\vartheta(x)B^{A_1^+}_{nx}(s)\frac{\tau_x(s)}{2E_x}T^{A_1^+}_{xm}(s)$$



$B^{A_1^+}$ singular at $W^+ = E_m + E_n + E(\boldsymbol{q}_{nj} + \boldsymbol{p}_{mi})$

τ_m^{-1} singular at $W^{\pm\pm} = E_m \pm E((2\pi/L)\boldsymbol{y}) \pm E((2\pi/L)\boldsymbol{y} + \boldsymbol{p}_{mi})$ for $\boldsymbol{y}\in\mathbb{Z}^3$

– when isobar-momenta are discretized in the 3-body cms momenta

$$\tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3}\int d^3\boldsymbol{\ell}\,\frac{\lambda^2}{2E_\ell(\sigma(k)-4E_\ell^2+i\epsilon)}$$

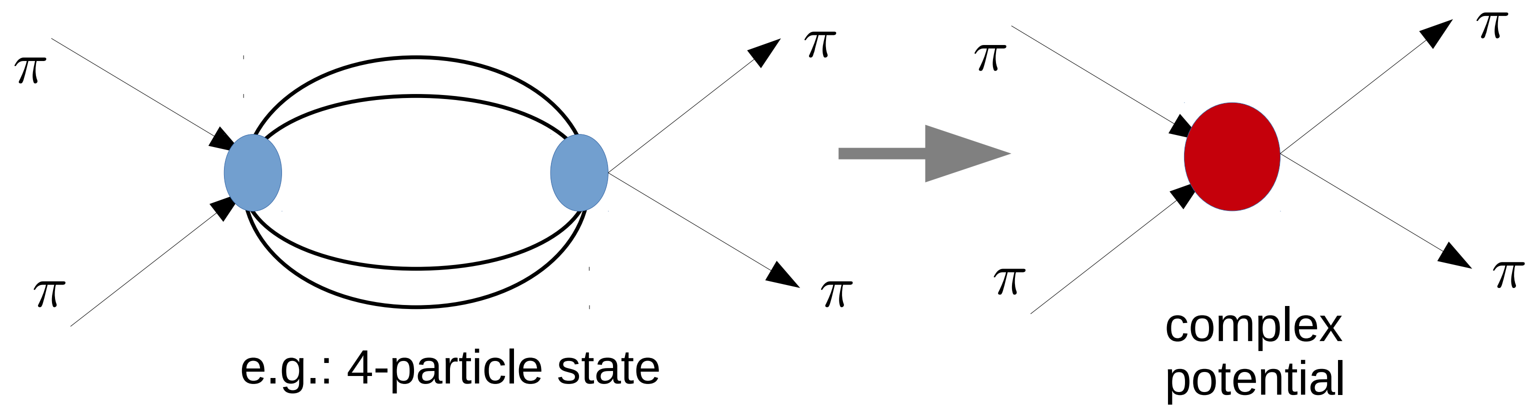
Also: all 2nd order singularities in determinant cancel → All consequence of Manifest three-body unitarity

Effective method for multi-particle states

The Optical potential [D. Agadjanov, M.D., M. Mai, U.-G. Meißner, A. Rusetsky, JHEP (2016)]

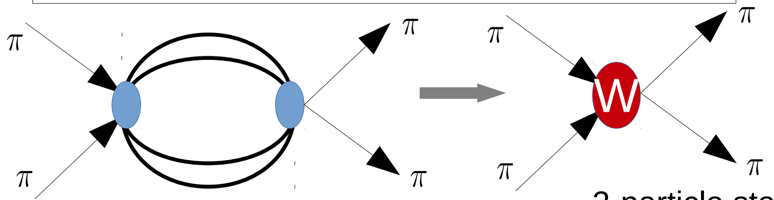
Optical potential in finite volume

- Finite-volume corrections for complex hadronic systems.
- Example: The optical potential on the lattice



- It is not always necessary to explicitly parameterize complicated intermediate states → Absorb all “uninteresting” dynamics in a complex-valued optical potential

Optical potential: The formal rewriting of a complicated scattering problem



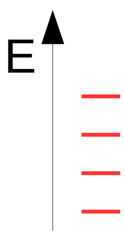
n-particle states
m coupled channel

2-particle state
1 channel
<u>BUT</u>
complex potential
with full dynamics

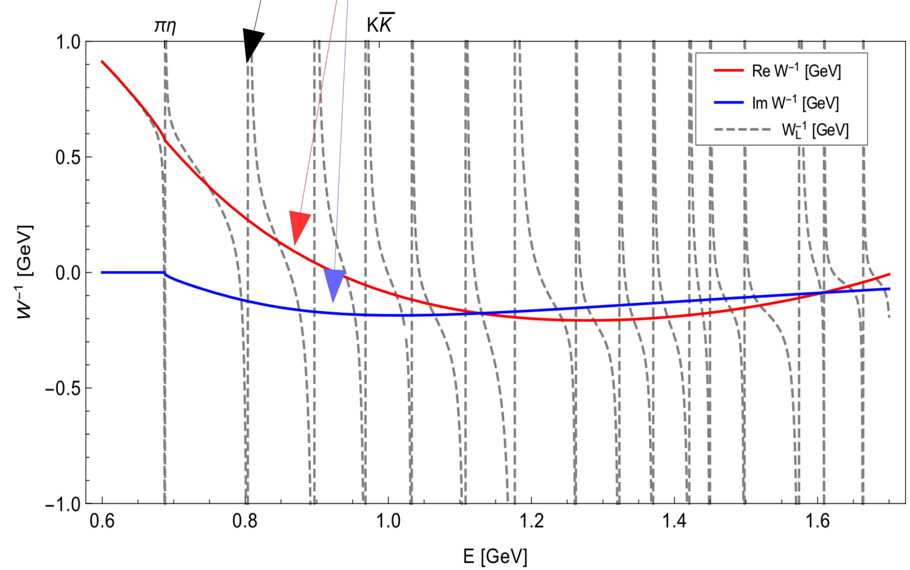
- Measured finite-volume optical potential
- Poles/functional form contain full multi-channel/ multi-particle dynamics
- How to efficiently measure this function → later

How to reconstruct true OP (complex) from finite volume OP (real)?

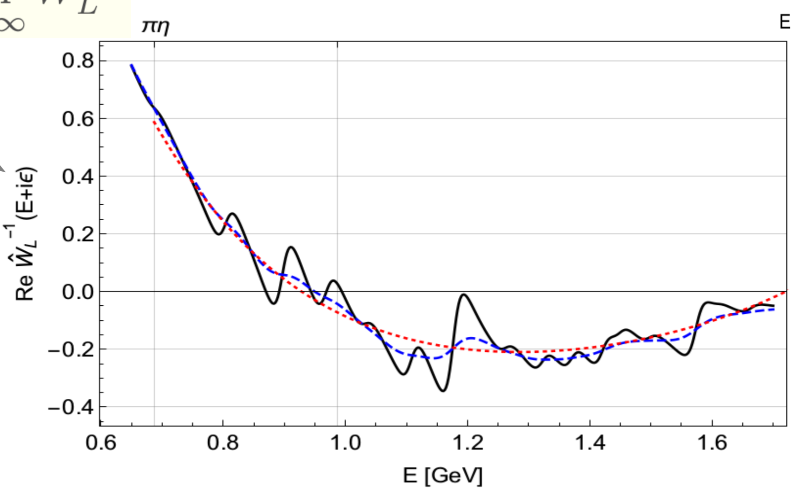
Lattice: measure eigenvalues, map to the optical potential

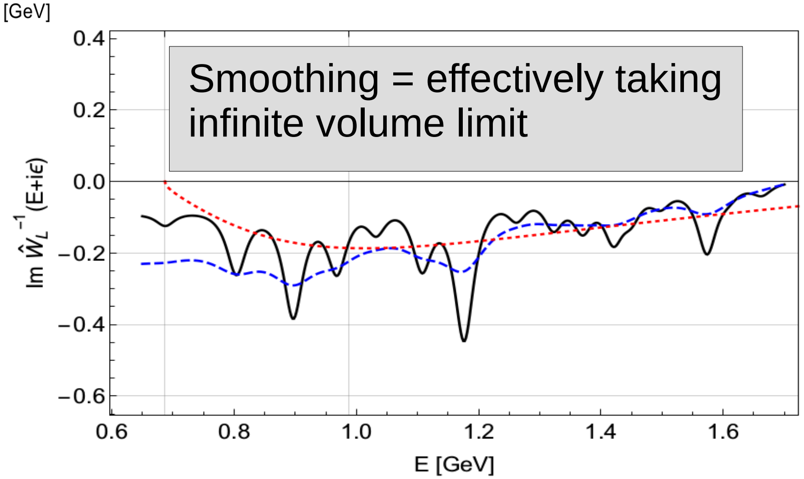


$$W_L^{-1}(E) \doteq \frac{2}{\sqrt{\pi}L} Z_{00}(1; q^2_{K\bar{K}})$$



$$W^{-1} = \lim_{\epsilon \to 0} \lim_{L \to \infty} W_L^{-1}$$



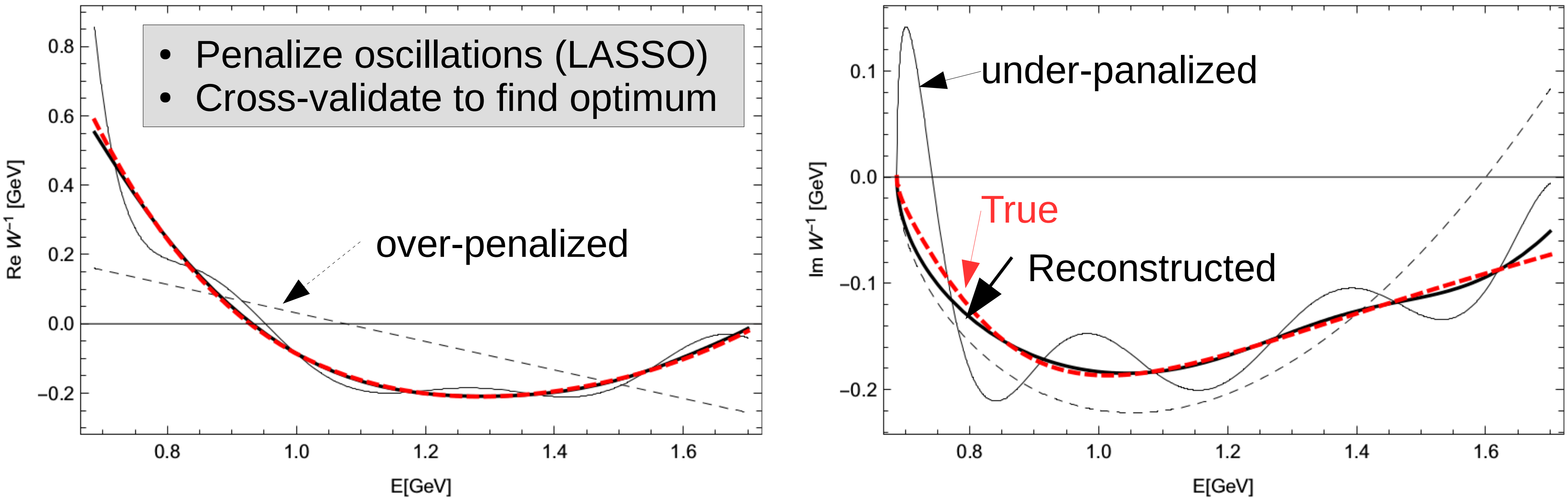


Smoothing = effectively taking infinite volume limit

Minimize: $\chi^2 = \sum_{k=1}^{m} \frac{\left|\hat{W}^{-1}(E_k) - \hat{W}_L^{-1}(E_k)\right|^2}{\sigma_k^2} + P_i(a_j, b_j)$

$$P_1(a_j, b_j) = \lambda^4 \int_{E_{\min}+i\varepsilon}^{E_{\max}+i\varepsilon} dE \left|\frac{\partial^2 \hat{W}^{-1}(E)}{\partial E^2}\right|$$

The reconstructed infinite-volume limit [LASSO + Cross Validation]

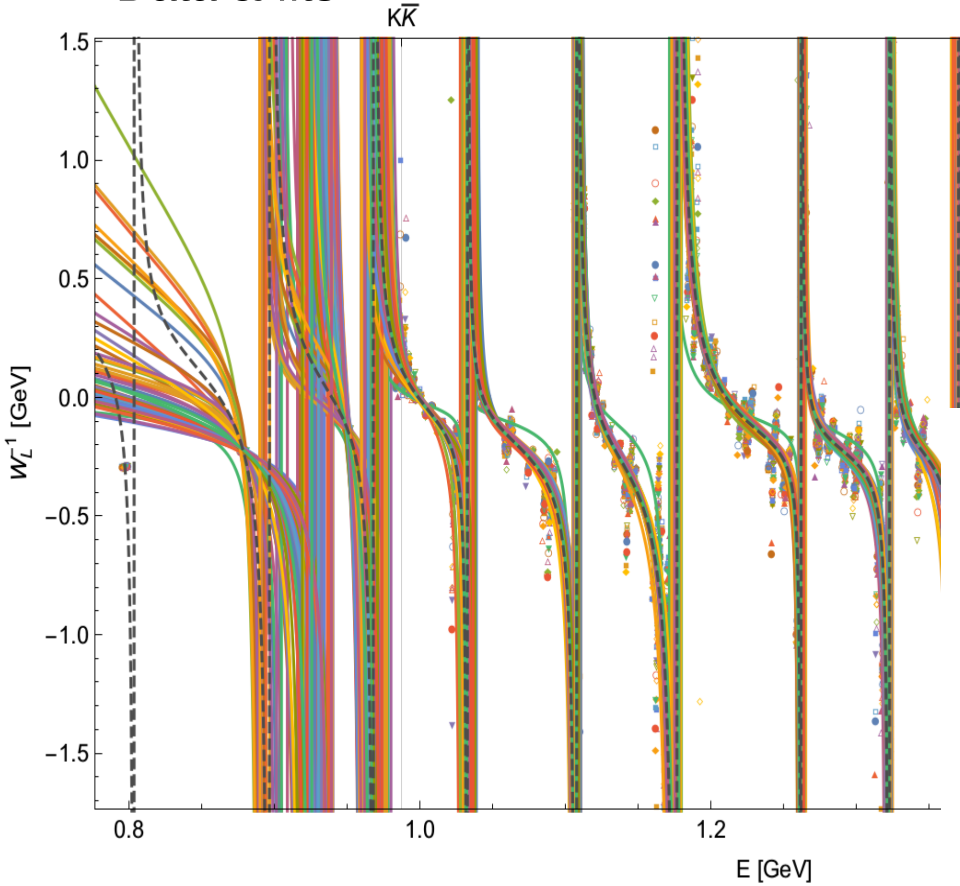


Correct Choice of penalization parameter λ through cross validation:

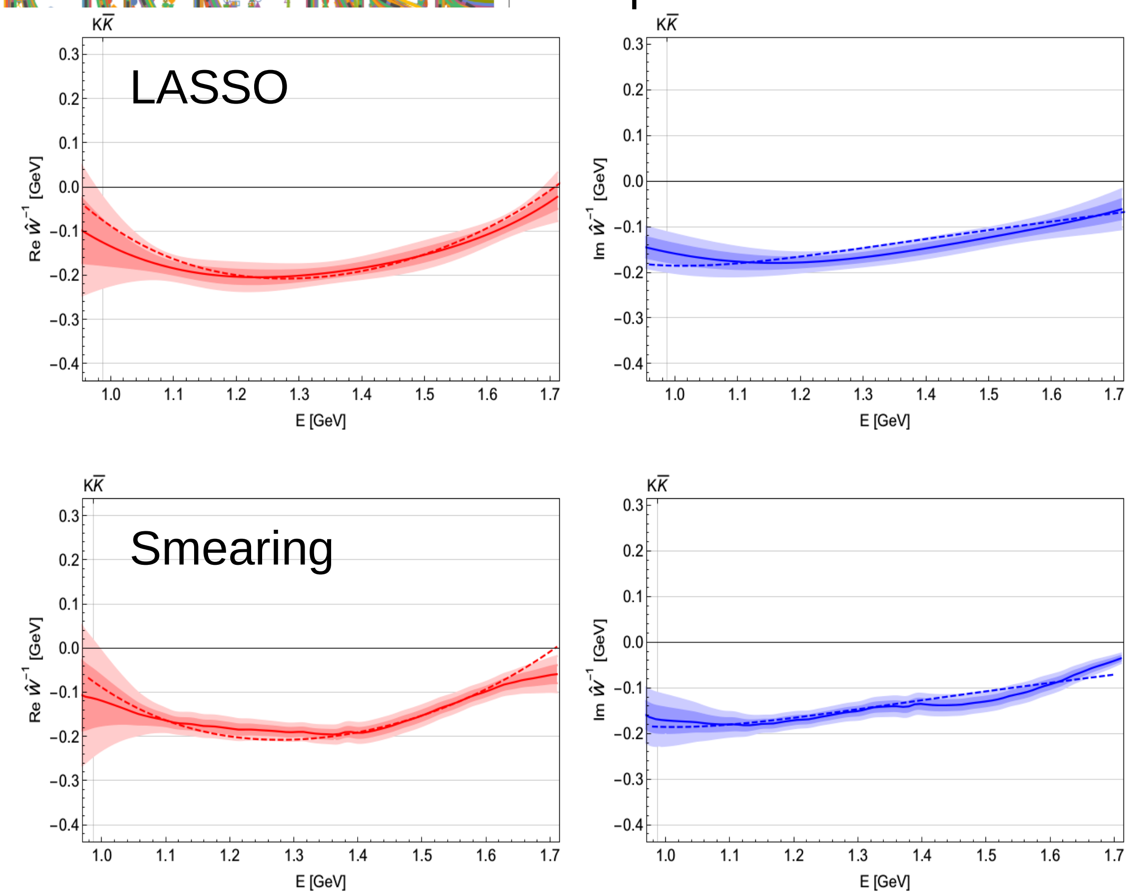
Fit at finite ϵ, validate at different ϵ' ($E \to E + i\epsilon$).

Numerical simulation

Data & fits



Reconstructed potential



Unitarity & Matching

- 3-body Unitarity (normalization condition ↔ phase space integral)

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$