Energy level shifts in the three-particle system

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In coll. with H.-W. Hammer, U.-G. Meißner, J.-Y. Pang, F. Romero-López, C. Urbach and J. Wu

arXiv:1806.02367 + some unpublished material

RESQCD 2018, Mainz, August 27-31, 2018

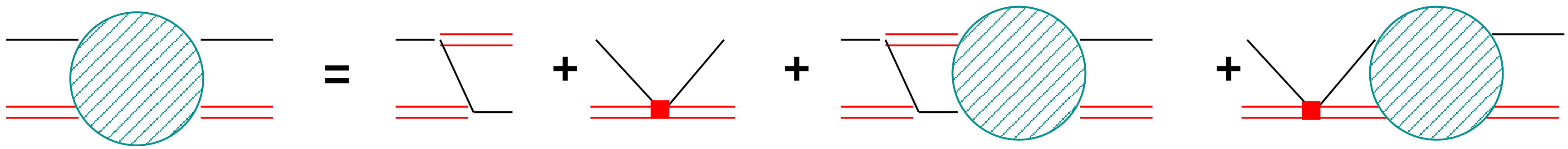
Plan

- Introduction + formalism
  (very briefly, more in H.-W. Hammer's talk)
- Energy shift of the ground state
- Particle-dimer ground state
- Excited states
- Discussion of the results, comparison with other approaches
- Three-body force in the φ^4 theory
- Conclusions, outlook

Formalism: summary

H.-W. Hammer, J.-Y. Pang and AR, JHEP 1709 (2017) 109; JHEP 1710 (2017) 115

Particle-dimer scattering equation in a finite volume:



$$\mathcal{M}_L(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + \frac{8\pi}{L^3}\sum_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{q};E)\tau_L(\mathbf{k};E)\mathcal{M}_L(\mathbf{k},\mathbf{q};E)$$

$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{pq}-mE} + H_0 + H_2(\mathbf{p}^2+\mathbf{q}^2) + \cdots$$

$$\tau_L^{-1}(\mathbf{k};E) = k^*\cot\delta(k^*) - \frac{4\pi}{L^3}\sum_{\mathbf{l}}\frac{1}{\mathbf{k}^2+\mathbf{l}^2+\mathbf{kl}-mE}$$

$\hookrightarrow$ Poles in the amplitude $\rightarrow$ finite-volume energy spectrum

$\hookrightarrow$ $H_0, H_2, \ldots$ should be fitted to the three-particle energies

Reduction of the quantization condition: summary

M. Döring, H.-W. Hammer, M. Mai, J.-Y. Pang, AR and J. Wu, PRD 97 (2018) 114508

- Symmetry in a finite volume: octahedral group O_h, including inversions (rest frame), little groups (moving frames); Γ denotes *irreps*, $T^{(\Gamma)}_{mn}(g)$ are pertinent matrices
- Analog for a sphere $|\mathbf{k}| = \text{const}$ for a cube: *shells*

$$s = \left\{ \mathbf{k} \ : \quad \mathbf{k} = g\mathbf{k}_0 \, , \quad g \in O_h \right\}$$

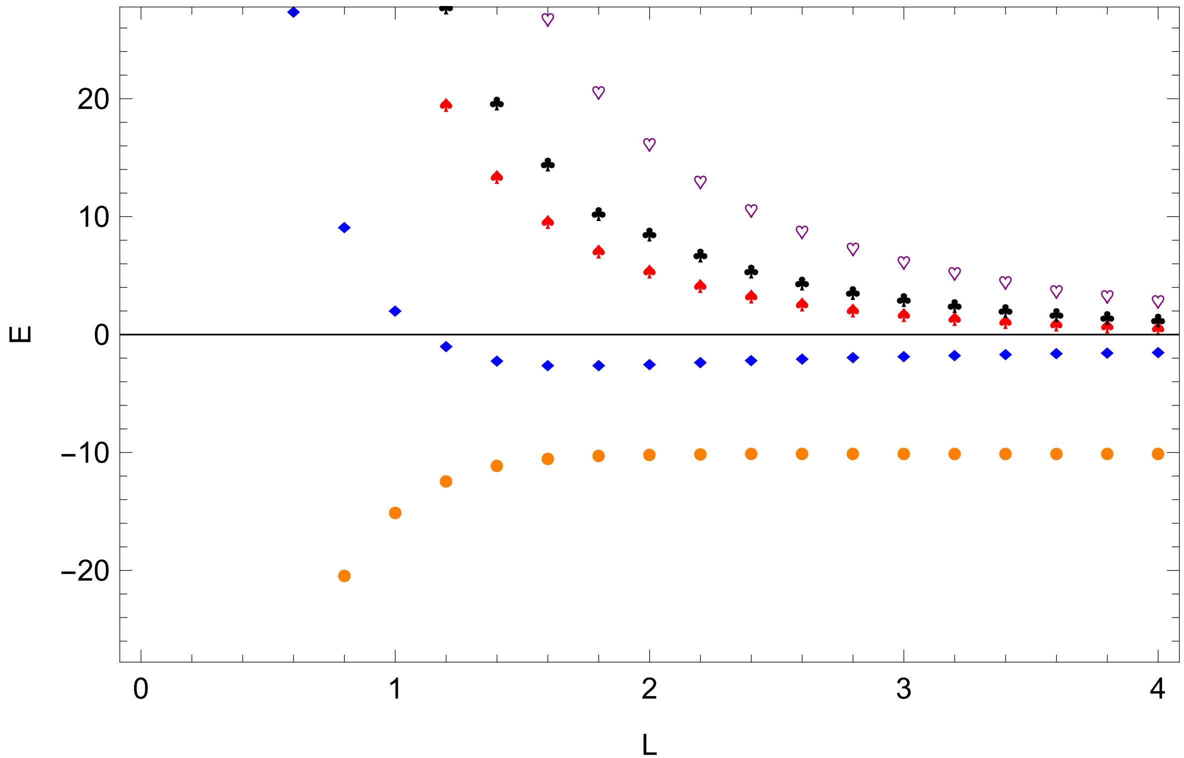
- Projection of a "potential" Z onto a given irrep:

$$Z^{(\Gamma)}_{nm}(r,s) = \sum_{g \in O_h} (T^{(\Gamma)}_{mn}(g))^* Z(g\mathbf{p}_0(r), \mathbf{k}_0(s))$$

- Quantization condition for a given irrep:

$$\det\left(\tau(s)^{-1}\vartheta(s)^{-1}\delta_{rs}\delta_{ij} - \frac{8\pi}{L^3}\frac{1}{G} Z^{(\Gamma)}_{ij}(r,s) \right) = 0 \, .$$

The finite-volume spectrum



The spectrum both below and above the three-particle threshold

M. Döring, H.-W. Hammer, M. Mai, J.-Y. Pang, AR and J. Wu, PRD 97 (2018) 114508

Alternative approaches: Hansen & Sharpe, Briceño & Davoudi, Döring & Mai, . . .

The goals

- How does one obtain the expansion of the energies of the scattering states in $1/L$?
- Is it possible to obtain the expansion of the energy of the excited states?
- In case of a stable dimer: the energy of particle-dimer states?
- Energy shift in different irreps, energy-level splitting, ...
- Apply the result to analyze lattice data

Expansion in cofactors

$$\det\mathcal{A}=\mathcal{A}_{rr}\det\mathcal{A}_{/rr}-\sum_{k,l\neq r}(-1)^{k+l}\mathcal{A}_{rk}\mathcal{A}_{lr}\det\mathcal{A}_{/rk/lr}\,.$$

$$(\mathcal{A}^{-1}_{/rr})_{kl}=(-1)^{k+l}\det\mathcal{A}_{/rr/lk}(\det\mathcal{A}_{/rr})^{-1}\,,$$

$$\hookrightarrow\quad \det\mathcal{A}=\left(\mathcal{A}_{rr}-\sum_{k,l\neq r}\mathcal{A}_{rk}(\mathcal{A}_{/rr})^{-1}_{kl}\mathcal{A}_{lr}\right)\det\mathcal{A}_{/rr}\,.$$

Perturbative expansion of the quantization condition:

$$\begin{aligned}A_{rr}&=(\tau^{-1}-Z)_{rr}=\sum_{k,l\neq r}Z_{rk}(1-\tau Z)^{-1}_{kl}\tau_l Z_{lr}\\&=\sum_{k\neq r}Z_{rk}\tau_k Z_{kr}+\sum_{k,l\neq r}Z_{rk}\tau_k Z_{kl}\tau_l Z_{lr}+\cdots\doteq\Delta\end{aligned}$$

The ground state: A_1^+

Counting: $q_0^2 = mE = O(L^{-3})$

$$8\pi\tau^{-1}(\mathbf{0};E) = -\frac{1}{a}+\frac{1}{2}\,rq_0^2+\underline{\frac{4\pi}{L^3}\,\frac{1}{q_0^2}}-\frac{1}{\pi L}\,\mathcal{I}-\frac{L}{4\pi^3}\,q_0^2\mathcal{J}-\frac{L^3}{16\pi^5}\,q_0^4\mathcal{K}+\cdots$$

$$8\pi\tau^{-1}(\mathbf{p};E) = -\frac{1}{a}+\sqrt{\frac{3}{4}\,\mathbf{p}^2}-\frac{1}{\pi L}\,\tilde{\mathcal{I}}(\mathbf{n})+\cdots$$

$$\tilde{\mathcal{I}}(\mathbf{n}) = \lim_{N\to\infty}\left\{\sum_{\mathbf{j}}^{N}\frac{1}{\mathbf{j}^2+\mathbf{n}^2+\mathbf{nj}}-4\pi N+2\pi^2\sqrt{\frac{3}{4}\,\mathbf{n}^2}\right\}.$$

$$\Delta = L\Delta_0+\Delta_1\,\ln\frac{mL}{2\pi}+\Delta_2+q_0^2L^3\Delta_3+O(L^{-1})\,.$$

Solving the quantization condition

$$L^3\tau^{-1}\left(\mathbf{0};-\frac{q_0^2}{m}\right)-Z\left(\mathbf{0},\mathbf{0};-\frac{q_0^2}{m}\right)-\Delta=0$$

$$\begin{aligned}q_0^2 &= \frac{12\pi a}{L^3}\left\{1-\left(\frac{a}{\pi L}\right)\mathcal{I}+\left(\frac{a}{\pi L}\right)^2\left(\mathcal{I}^2-3\mathcal{J}-\frac{(2\pi)^3}{a}\Delta_0\right)\right.\\ &- \left.\frac{8\pi a}{L^3}\Delta_1\ln\frac{mL}{2\pi}+\frac{6\pi a^2 r}{L^3}+\frac{X}{L^3}\right\}\end{aligned}$$

$$\begin{aligned}X &= \left(\frac{a}{\pi}\right)^3\left(-\mathcal{I}^3+9\mathcal{I}\mathcal{J}-9\mathcal{K}+\frac{(2\pi)^3}{a}2\mathcal{I}\Delta_0\right)\\ &- 8\pi a\left(\frac{H_0(\Lambda)}{\Lambda^2}+\Delta_2+12\pi a\Delta_3\right)\end{aligned}$$

Expansion of the Δ

$$
\begin{aligned}
\Delta &= S_1 + S_2 + \cdots \\
S_1 &= \frac{8\pi}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}}^{\Lambda} \left(\frac{1}{\mathbf{k}^2 - q_0^2} + \frac{H_0(\Lambda)}{\Lambda^2} \right)^2 \frac{-a}{1 - a\sqrt{\frac{3}{4}\,\mathbf{k}^2} + \frac{a}{\pi L}\,\tilde{\mathcal{I}}(\mathbf{n})} \\
&= S_1^{(0)} + S_1^{(1)} q_0^2 + \cdots
\end{aligned}
$$

Infinite-volume counterpart has an infrared singularity

$\rightarrow$ threshold amplitude (singularities subtracted)

$$
\int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\mathbf{k}^4} \rightarrow \lim_{\varepsilon\to 0} \left(\int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{(\mathbf{k}^2 + \varepsilon^2)^2} - \frac{1}{8\pi\varepsilon} \right)
$$

$$
\hookrightarrow \quad S_1 = \hat{S}_1 + aL + b\ln\frac{mL}{2\pi} + c + dq_0^2 + \cdots
$$

Higher orders

- Only S_1, S_2 should be considered explicitly at $O(L^{-6})$. In the remaining terms, the limit $L \to \infty$ can be directly performed

$$
\begin{aligned}
q_0^2 &= \frac{12\pi a}{L^3}\left\{1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2 \left(\mathcal{I}^2 + \mathcal{J}\right)\right. \\
&+ \left.\left(\frac{a}{\pi L}\right)^3 16\pi^3 \left(\sqrt{3} - \frac{4\pi}{3}\right) \ln \frac{mL}{2\pi} + \frac{6\pi a^2 r}{L^3} + \frac{X}{L^3}\right\},
\end{aligned}
$$

$$
X = \left(\frac{a}{\pi}\right)^3 \left(-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} - \pi B\right) - 8\pi a \hat{\mathcal{M}}\,.
$$

The threshold amplitude:

$$
\mathcal{M}\left(\mathbf{0}, \mathbf{0}; -\frac{\varepsilon^2}{m}\right) = \frac{\mathcal{M}_{-2}}{\varepsilon^2} + \frac{\mathcal{M}_{-1}}{\varepsilon} + \mathcal{M}_0 \ln \frac{\varepsilon}{m} + \hat{\mathcal{M}} + \cdots\,,
$$

Comparison with known results

- Agrees with Hansen & Sharpe (modulo relativistic corrections)

Example: H& S :
$$\mathcal{C}_4 = 4\sqrt{3}\pi^2 \sum_{\mathbf{n}\neq\mathbf{0}} \frac{H^3(p/m)}{|\mathbf{n}|^3} - 16\sqrt{3}\pi^3 \ln\frac{mL}{2\pi}$$

our :
$$\mathcal{D}_4 = 4\sqrt{3}\pi^2 \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int_1^\infty\right) \frac{1}{|\mathbf{n}|^3}$$

$$\mathcal{C}_4 = \mathcal{D}_4 + 32\sqrt{3}\pi^5 \lim_{\varepsilon\to 0}\left(\int_\varepsilon^\infty \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{H^3(p/m)}{p^3} + \frac{\ln(\varepsilon/m)}{2\pi^2}\right) \to \mathcal{M}_{3,df}$$

- Generally agrees with Beane, Detmold and Savage at the first order in PT

$$\hat{\mathcal{M}} = \#\,\eta_3^r(\mu) + \cdots$$

- Small differences need to be settled (in progress)

The shift of the lowest particle-dimer energy level

Assume stable dimer ...

- The energy shift from the unperturbed value:

$$E = E_d + \Delta_d = -\frac{1}{ma^2} + \Delta_d$$

$$8\pi\tau^{-1}(E) = 8\pi K^{-1}(E) + \sqrt{-mE} + \cdots = -\frac{1}{a} + \cdots + \sqrt{-mE}$$

- The quantization condition:

$$8\pi K^{-1}(E_d + \Delta_d) + \sqrt{-m(E_d + \Delta_d)} = \frac{8\pi}{L^3} \underbrace{(Z + Z\tau Z + \cdots)}_{=\mathcal{M}_d}\Big|_{E=E_d+\Delta_d}$$

$$\hookrightarrow \quad \Delta_d = \frac{\mathcal{Z}}{L^3}\mathcal{M}(\mathbf{0},\mathbf{0};E_d) + O(L^{-4})\,, \qquad \mathcal{Z} = \frac{-8\pi}{m/(2\sqrt{-mE_d}) - (8\pi K^{-1})'}$$

- Agrees with the Lüscher equation with a stable dimer!

The first excited state: irrep A_1^+

- Individual momenta: $\mathbf{p}_1=(0,0,1)$, $\mathbf{p}_2=(0,0,-1)$, $\mathbf{p}_3=(0,0,0)$
- Dimers with momenta $|\mathbf{p}|=0,1$ can be formed

$\hookrightarrow$ Shells with $r=1,2$ contain singular denominators

$\longrightarrow$ Diagonalization

$$
\begin{aligned}
8\pi\tau^{-1}(1) &= -\frac{1}{a}+\frac{1}{2}\,rq_0^2-\underline{\frac{6}{1-\kappa^2}}-\frac{1}{\pi L}\,C_1(1)+\frac{1}{\pi L}\,C_2(1)(1-\kappa^2) \\
&- \frac{1}{\pi L}\,C_2(1)(1-\kappa^2)^2+\cdots \\
8\pi\tau^{-1}(2) &= -\frac{1}{a}+\frac{1}{2}\,r\left(q_0^2-\frac{3\pi^2}{L^2}\right)-\underline{\frac{2}{1-\kappa^2}}-\frac{1}{\pi L}\,C_1(2)+\frac{1}{\pi L}\,C_2(2)(1-\kappa^2) \\
&- \frac{1}{\pi L}\,C_2(2)(1-\kappa^2)^2+\cdots
\end{aligned}
$$

$\kappa=\dfrac{q_0L}{2\pi}$ and no singularities in the terms with $r>2$!

The matrix $\mathcal{A}$

$$\mathcal{A} = \left(\begin{array}{cc|cc} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & \cdots \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & \cdots \\ \hline \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

$$\begin{aligned} \mathcal{A}_{11} &= -\frac{6}{\pi L}\,\frac{1}{1-\kappa^2} - \frac{1}{a} + \bar{\mathcal{A}}_{11} \\ \mathcal{A}_{12} = \mathcal{A}_{21} &= -\frac{2\sqrt{6}}{\pi L}\,\frac{1}{1-\kappa^2} + \bar{\mathcal{A}}_{12} \\ \mathcal{A}_{22} &= -\frac{4}{\pi L}\,\frac{1}{1-\kappa^2} - \frac{1}{a} + \bar{\mathcal{A}}_{22} \end{aligned}$$

Diagonalization $\mathcal{A} \to \mathcal{O}\mathcal{A}\mathcal{O}^T$

$$\mathcal{O} = \left(\begin{array}{cc|cc} c & s & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \hline 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right), \qquad c = \sqrt{\frac{3}{5}}, \quad s = \sqrt{\frac{2}{5}}$$

$$\mathcal{A} \to \left(\begin{array}{cc|cc} x + \mathcal{A}'_{11} & sx + \mathcal{A}'_{12} & c\mathcal{A}_{13} + s\mathcal{A}_{23} & \cdots \\ sx + \mathcal{A}'_{21} & s^2 x - \frac{c^2}{a} + \mathcal{A}'_{22} & \mathcal{A}_{23} & \cdots \\ \hline c\mathcal{A}_{31} + s\mathcal{A}_{32} & \mathcal{A}_{32} & \mathcal{A}_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

$$x = -\frac{10}{\pi L}\,\frac{1}{1-\kappa^2} - \frac{1}{a}$$

Energy level splitting possible in higher states ...

The quantization condition for the first excited level

$$-\frac{10}{\pi L}\,\frac{1}{1-\kappa^2}-\frac{1}{a}+\mathcal{A}'_{11}=\sum_{r\neq 1}\mathcal{A}'_{1r}\tau_r\mathcal{A}'_{r1}+\sum_{r,s\neq 1}\mathcal{A}'_{1r}\tau_r Z_{rs}\tau_s\mathcal{A}'_{s1}+\cdots$$

Solving by iterations:

$$\begin{aligned}1-\kappa^2 &= -\frac{10a}{\pi L}\left\{1+\frac{a}{\pi L}\,B_1+\left(\frac{a}{\pi L}\right)^2 B_2+\left(\frac{a}{\pi L}\right)^3 B_3\ln\frac{mL}{2\pi}\right.\\ &+\left.\left(\frac{a}{\pi L}\right)^3 B_4\right\}\end{aligned}$$

- $B_4=\#\hat{\mathcal{M}}+\text{const}$, with the same $\hat{\mathcal{M}}$ as in the ground state
- The effective range contribution is present already in B_2
- Possible contribution from the higher spin dimers at this level? Investigation in progress . . .

The irrep E^+

- The shell $r = 1$ drops completely out, no need for the diagonalization
- The energy shift:

$$1 - \kappa^2 = -\frac{4a}{\pi L}\left\{1 + \frac{a}{\pi L}\,D_1 + (\frac{a}{\pi L})^2\,D_2 + (\frac{a}{\pi L})^3\,D_3\right\} + \cdots$$

  - No logarithm
  - D_4 does not contain the amplitude $\hat{\mathcal{M}}$ (S-wave)
  - Higher partial waves in particle-dimer scattering contribute to higher orders in $1/L$
- Irreps $A_2,\ T_1,\ T_2$: no energy levels near $\kappa^2 = 1$

Energy shift in the φ^4 theory

F. Romero-López, A. Rusetsky and C. Urbach, arXiv:1806.02367

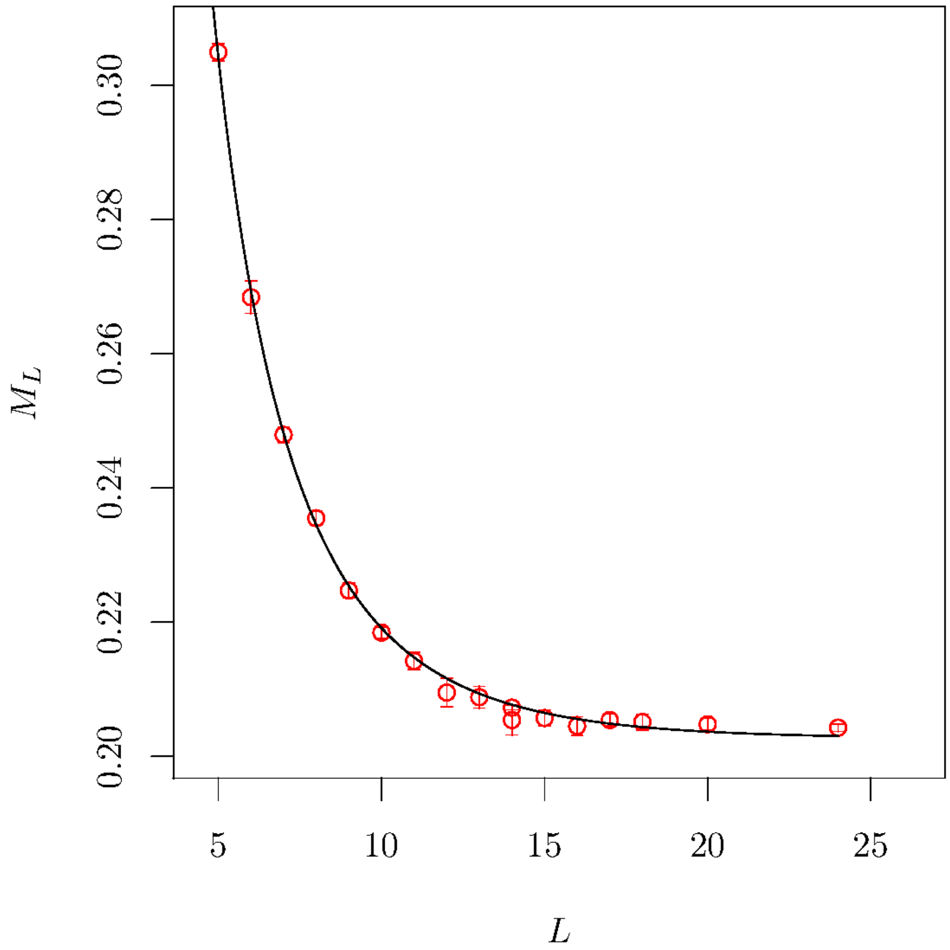
$$S = \sum_x \Big(-\kappa \sum_\mu (\varphi_x^* \varphi_{x+\mu} + c.c.) - \lambda(|\varphi_x|^2 - 1)^2 + |\varphi_x|^2 \Big)$$

- The calculations are performed for different values of L
- For our choice of parameters λ and κ: perturbative, the phase shift does nor exceed few degrees
- Single particle mass: perfectly fits the one-loop expression:

$$M(L) - M = \text{const}\,\frac{K_1(ML)}{(ML)^{1/2}} \sim \text{const}\,\frac{\exp(-ML)}{(ML)^{3/2}}$$

- Extracting H_0 at small L: does one have control over exponentially suppressed contributions?

The volume-dependent one-particle mass



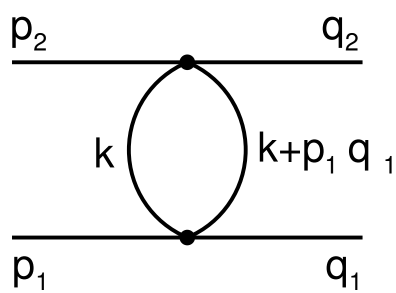
Exponentially suppressed contribitions: 2-body levels

Using quasi-potential reduction of the Bethe-Salpeter equation...

$$E_2 - 2M(L) = \frac{1}{L^3}\, T_L(\mathbf{0}, \mathbf{0}, E_2)$$

$$T_L = \bar{T}_L + \bar{T}_L(g'_L - g_\infty)T_L\,, \qquad \bar{T}_L = V_L + V_L g_\infty \bar{T}_L$$

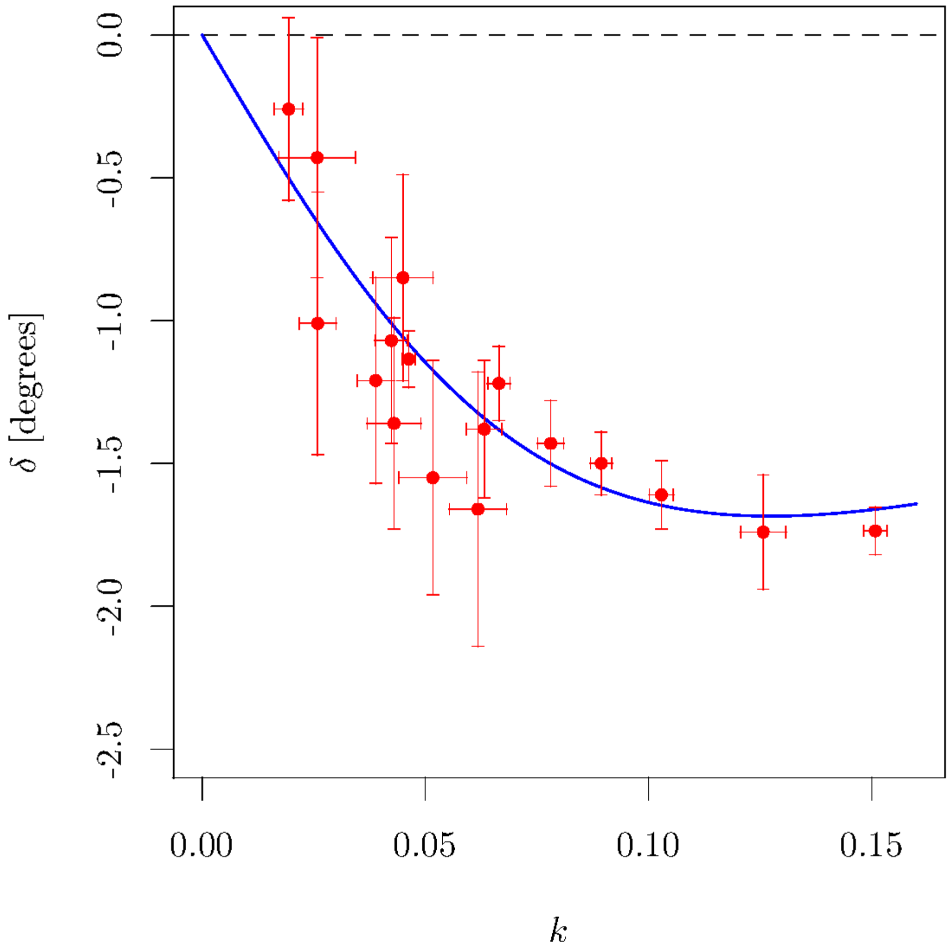
Leading exponentially suppressed term:



$$V_L - V_\infty \sim \frac{\exp(-ML)}{(ML)^{1/2}} \quad \hookrightarrow \quad E_2 - 2M(L)\Big|_{\text{exp}} \sim \frac{\exp(-ML)}{(ML)^{7/2}} + \cdots$$

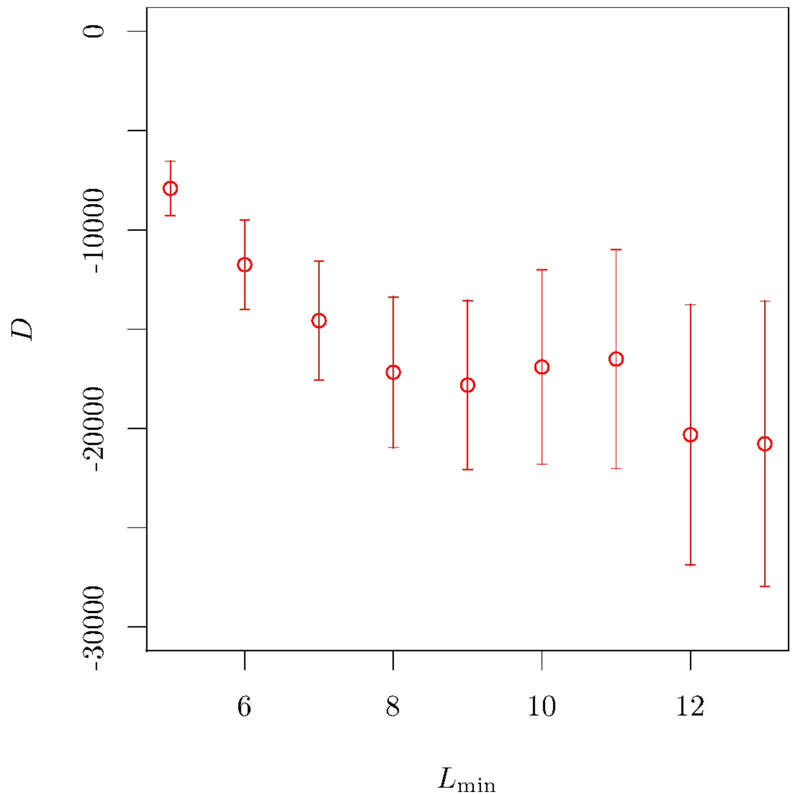
$\hookrightarrow$ The difference $E_2 - 2M(L)$ already captures the leading exponentially suppressed contribution. The correction coming from the potential is suppressed by an additional factor L^{-2}

The two-particle phase shift



solid line: ERE with best fit parameters for a and r

The three-body force D



$$\Delta E_3 = \frac{12\pi a}{mL^3}\left(1 + d_1\left(\frac{a}{\pi L}\right) + d_2\left(\frac{a}{\pi L}\right)^2 + \frac{6\pi r a^2}{L^3} + d_3\left(\frac{a}{\pi L}\right)^3 \ln\frac{mL}{2\pi}\right) - \frac{D}{48m^3L^6}$$

D is nonvanishing at 4σ for $L_{min} = 9$

Conclusions/outlook

- From our quantization condition we have derived the perturbative shifts of the three-particle energy levels at $O(L^{-6})$ and for the particle-dimer ground level at $O(L^{-3})$
- For the three-particle ground state, the result agrees with the known ones. The rest of results is new
- A caveat: P-wave dimer might contribute to the excited energy at this order, need to be included *(in progress)*
- Relativistic formulation should be worked out *(in progress)*
- We used the framework in practice, extracting the three-body force in the φ^4 theory

$\hookrightarrow$ Particle-dimer energy level shift contains the three-body force at the leading order: $\pi\sigma$, $\pi\rho$ scattering at heavier quark masses?