Progress on the relativistic three-particle quantization condition

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In collaboration with Tyler Blanton (UW), Raul Briceño (ODU/Jlab), Max Hansen (CERN) and Fernando Romero-Lopez (Valencia)

Mostly based on arXiv:1803.04169 (published in PRD), arXiv:1808:XXXXX, and work in progress

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Outline

- Motivation
- Methodology & status
- Completing the formalism: including resonant subchannels
- Numerical experiments in the isotropic approximation
- Including higher partial waves
- Outlook & open issues

Motivation

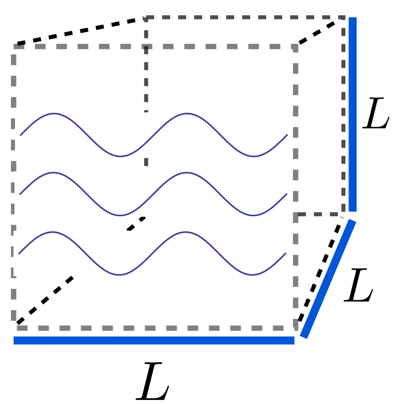
- Studying resonances with three particle decay channels
  - $\omega(782, I^G J^{PC} = 0^- 1^{--}) \to 3\pi$ (no resonant subchannels)
  - $a_2(1320, I^G J^{PC} = 1^- 2^{++}) \to \rho\pi \to 3\pi$
  - $N(1440) \to \Delta\pi \to N\pi\pi$
  - $X(3872) \to J/\Psi\pi\pi$

- Calculating weak decay amplitudes involving 3 or more particles, e.g. K→3π, D→2π, 4π, …

- Determining NNN interactions

Methodology & Status

2 & 3 particle spectrum from LQCD



Quantization conditions

$$\det\left[F_2^{-1} + \mathcal{K}_2\right]$$
$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right]$$

Intermediate scattering quantities

Integral equations in infinite volume

Scattering amplitudes

$$\mathcal{M}_2,\quad \mathcal{M}_3,\quad \mathcal{M}_{23},\ldots$$

Methodology & Status

Quantization conditions

$$\det\left[F_2^{-1} + \mathcal{K}_2\right]$$

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right]$$

Intermediate scattering quantities

Integral equations in infinite volume

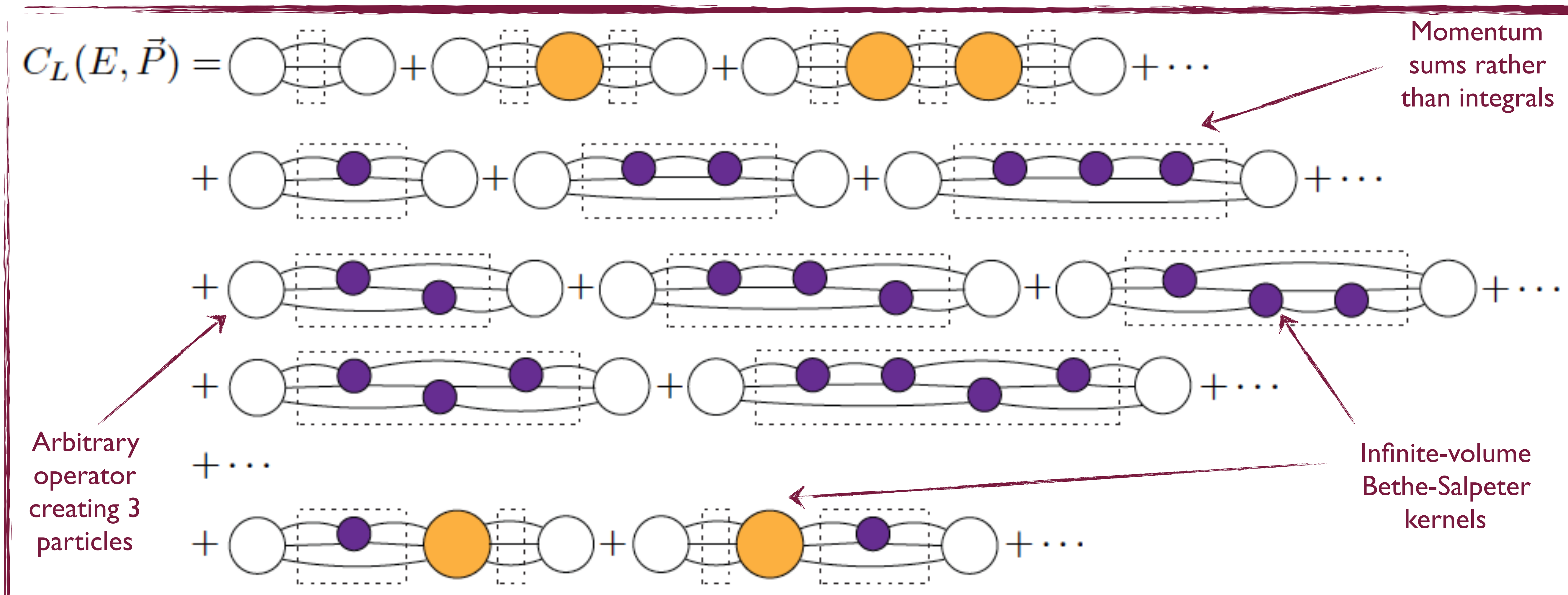
- Three approaches
  - Relativistic [Briceño, Hansen, SRS]
  - NREFT [Hammer, Pang, Rusetsky]
  - Finite-volume Khuri-Treiman [Döring, Mai]
- Each have pros and cons
  - Intermediate scattering quantities differ
  - All require partial-wave truncation
  - Similar challenges for numerical implementation

Our approach (1)

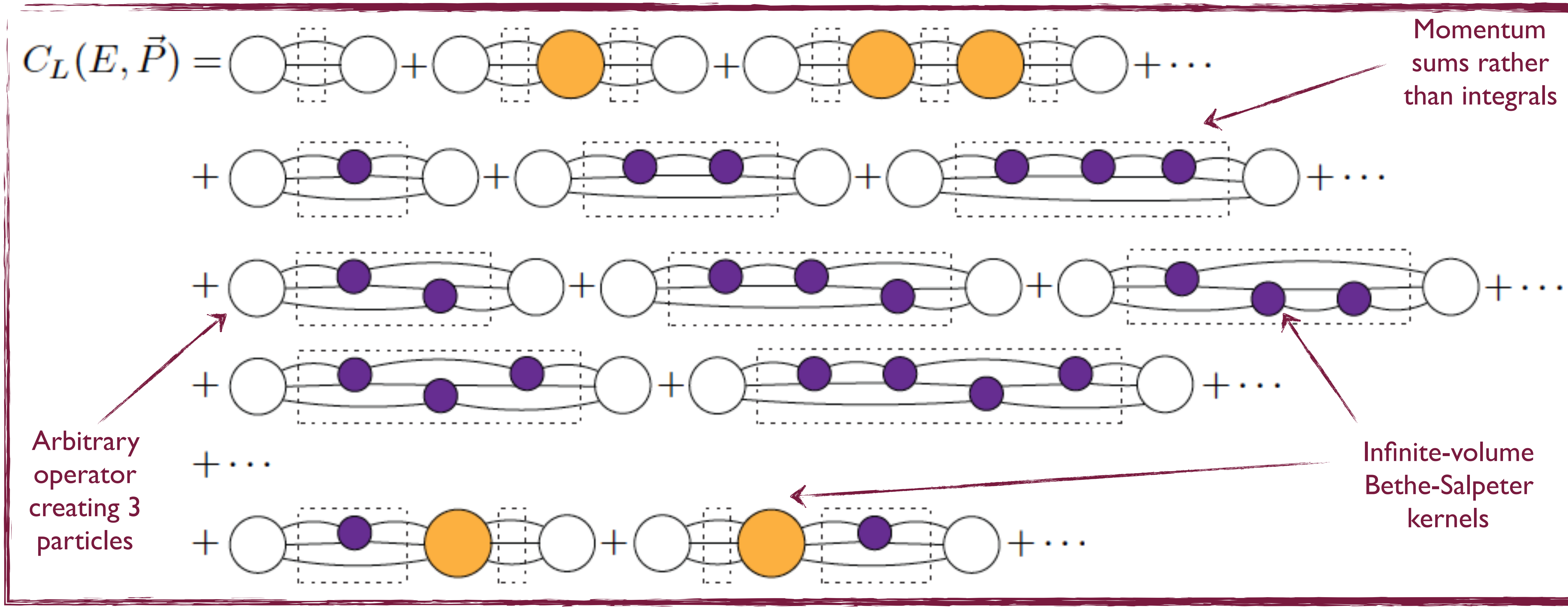
[Hansen & SRS, arXiv:1408.5933]

(1)

- Generic relativistic EFT, working to all orders
  - Do not need a power-counting scheme
  - To simplify analysis: impose a global Z_2 symmetry (G parity) & consider identical scalars
- Obtain spectrum from poles in finite-volume correlator
  - Consider $E_{CM} < 5m$ so on-shell states involve only 3 particles



Our approach (2)



(2)

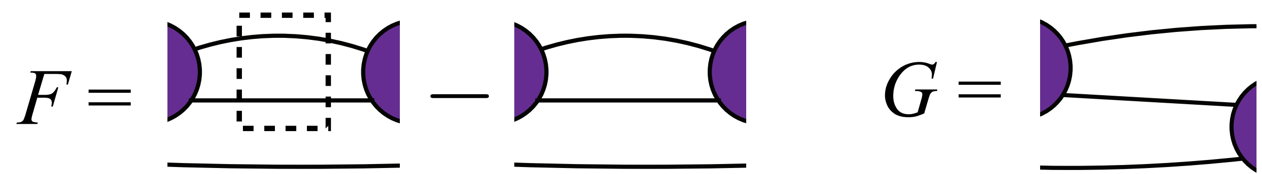
- Replace sums with integrals plus sum-integral differences to extent possible
  - If summand has pole or cusp then difference ~$1/L^n$ and must keep (Lüscher zeta function)
  - If summand is smooth then difference ~ exp(-mL) and drop
- Avoid cusps by using PV prescription—leads to generalized 3-particle K matrix
- Subtract above-threshold divergences of 3-particle K matrix—leads to $\mathcal{K}_{df,3}$

Our approach (3)

(3)
- Reorganize, resum, … to separate infinite-volume on-shell relativistically-invariant non-singular scattering quantities ($\mathcal{K}_2$, $\mathcal{K}_{df,3}$) from known finite-volume functions (F [Lüscher zeta function] & G ["switch function"])

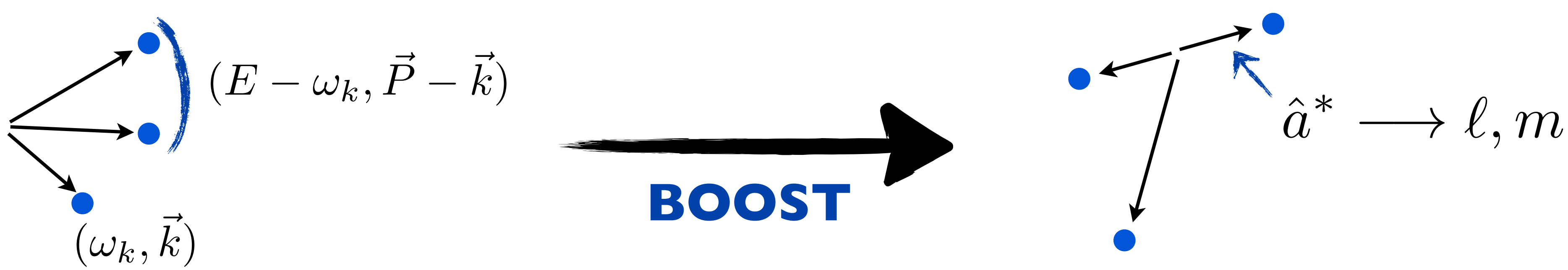
$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right]$$

$$F_3 \equiv \frac{F}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1+[1+\mathcal{K}_2 G]^{-1}\mathcal{K}_2 F}\right]$$



- All quantities are infinite-dimensional matrices with indices describing 3 on-shell particles

[finite volume "spectator" momentum: **k**=2π**n**/L] x [2-particle CM angular momentum: *l,m*]



- For large spectator-momentum **k**, the other two particles are below threshold; we must include such configurations by analytic continuation up to a cut-off at k~m [provided by H(**k**)]

Our approach (4)

[Hansen & SRS, arXiv:1504.04248]

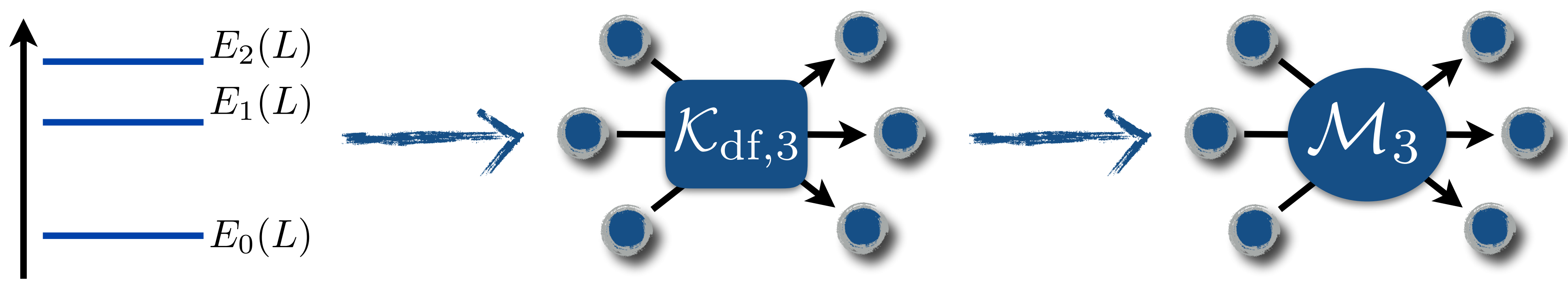
(4)

- Relate $\mathcal{K}_{df,3}$ to $\mathcal{M}_3$ by taking infinite-volume limit of finite-volume scattering amplitude
  - Results in infinite-volume integral equations involving $\mathcal{M}_2$ & cut-off function H
  - Can formally invert equations to show that $\mathcal{K}_{df,3}$ (while unphysical) is relativistically invariant and has same properties under discrete symmetries (P,T) as $\mathcal{M}_3$

Status of relativistic approach

- Original work applied to scalars with G-parity & no subchannel resonances [Hansen, SRS: 1408.5933 & 1504.04248]

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right]$$

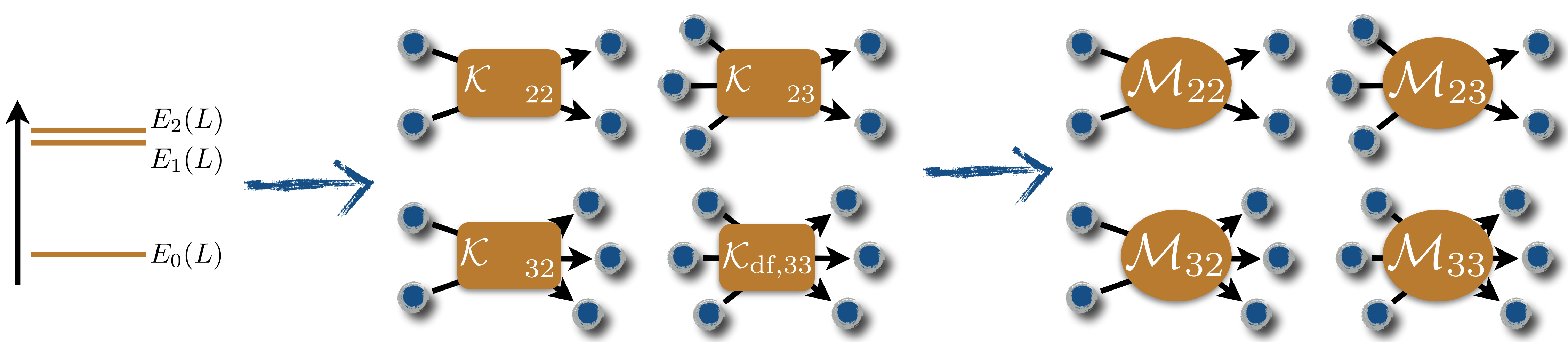


Status of relativistic approach

- Second major step: removing G-parity constraint, allowing 2↔3 processes [Briceño, Hansen, SRS: 1701.07465]

F_2 appears in 2-particle quantization condition

$$\det\left[\begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathscr{K}_{22} & \mathscr{K}_{23} \\ \mathscr{K}_{32} & \mathscr{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$



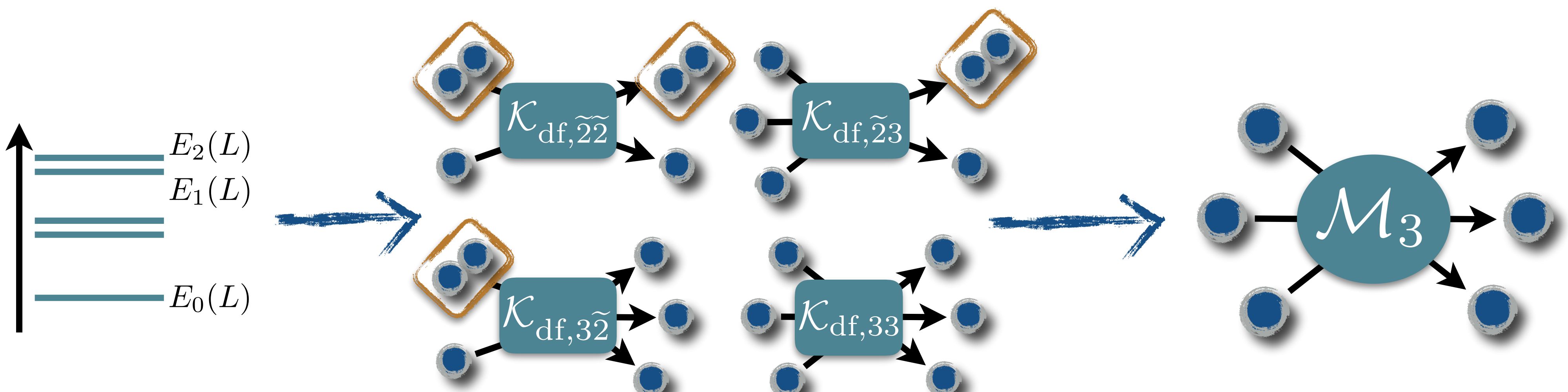
Completing the formalism (1)

- Final major step: allowing subchannel resonance (i.e. pole in $\mathcal{K}_2$)
  [Briceño, Hansen, SRS: 1808.XXXXX]

Determined by K_2 & Lüscher finite-volume zeta functions

resonance + particle channel (not physical)

$$\det\left[\begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathscr{K}_{\mathrm{df},\tilde{2}\tilde{2}} & \mathscr{K}_{\mathrm{df},\tilde{2}3} \\ \mathscr{K}_{\mathrm{df},3\tilde{2}} & \mathscr{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$



Completing the formalism (2)

$$\det\left[\begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{\mathrm{df},\tilde{2}\tilde{2}} & \mathcal{K}_{\mathrm{df},\tilde{2}3} \\ \mathcal{K}_{\mathrm{df},3\tilde{2}} & \mathcal{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$

- Forced into extra unphysical "ρπ" channel to account for FV effects of poles in $\mathcal{K}_2$ at intermediate stages of derivation
- Positive feature: should allow smooth transition between formalism for resonant and stable ρ as $m_{u,d}$ increased

Completing the formalism (3)

- To-do list
  - Multiple poles in K_2
  - Nondegenerate particles with spin
  - Connecting formalism for resonances to that for stable particles
- All appear straightforward

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Isotropic low-energy approximation

[Briceño, Hansen & SRS, 1803.04169]

- Scalar particles with G parity so no 2⟷3 transitions and no subchannel resonances (e.g. 3 π^+)
- 2-particle interactions are purely s-wave, and determined by the scattering length alone (which can be arbitrarily negative, $a \to -\infty$)
- Point-like three-particle interaction $\mathcal{K}_{df,3}$ independent of momenta, although can depend on $s=(E_{cm})^2$
- Reduces problem to 1-d quantization condition, although intermediate matrices involve finite-volume momenta up to cutoff $|k|\sim m$
- Analog in our formalism of the approximations used in other approaches: [Hammer, Pang, Rusetsky, 1706.07700; Mai & Döring, 1709.08222; Döring et al., 1802.03362; Mai & Döring, 1807.04746]

Isotropic low-energy approximation

[Briceño, Hansen & SRS, 1803.04169]

$$\det\left[F_3^{-1} + \mathcal{K}_{\text{df},3}\right] \longrightarrow 1/\mathcal{K}^{\text{iso}}_{\text{df},3}(E^*) = -F_3^{\text{iso}}[E, \vec{P}, L, \mathcal{M}_2^s]$$

- Relation of $\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$ (matrix equation that becomes integral equation when L→∞)

$$\mathcal{M}_3 = \mathcal{S}\left[\mathcal{D} + \mathcal{L}\frac{1}{1/\mathcal{K}^{\text{iso}}_{\text{df},3} + F^{\text{iso}}_{3,\infty}}\mathcal{R}\right]$$

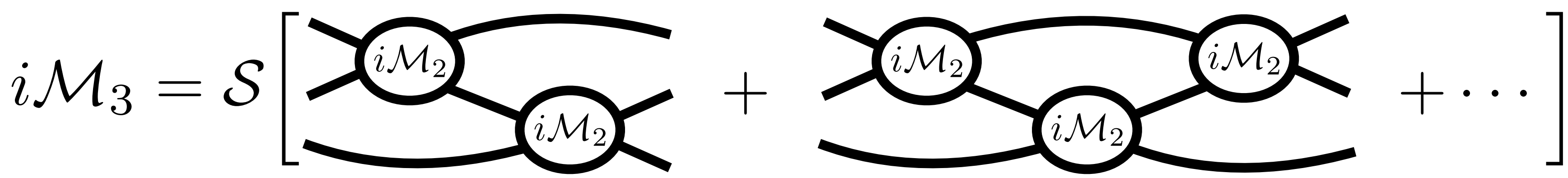
symmetrization

$\mathcal{D}$, $\mathcal{L}$ & $\mathcal{R}$ depend on $\mathcal{M}_2$ & kinematical factors

L→∞ limit of F_3^{iso} depends on $\mathcal{M}_2$ & kinematical factors

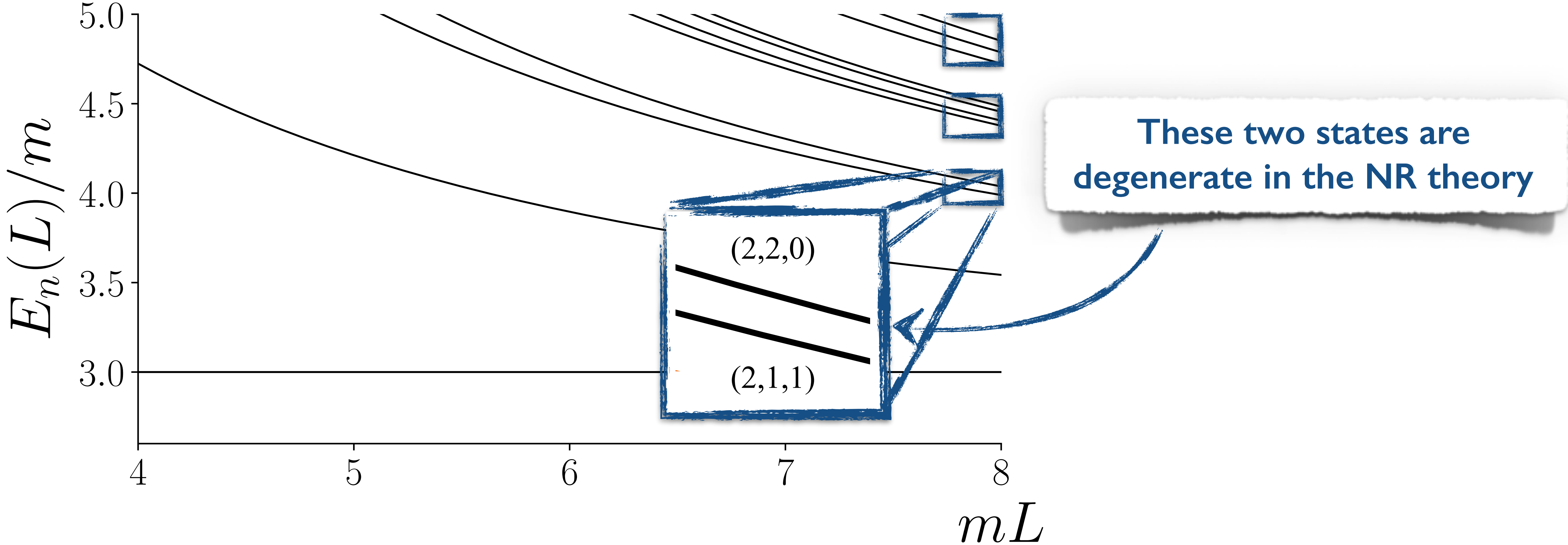
Solutions with $K_{df,3} = 0$

- Useful benchmark: deviations measure impact of 3-particle interaction
  - Caveat: scheme-dependent since $\mathcal{K}_{df,3}$ depends on cut-off function H
- Meaning of limit for $\mathcal{M}_3$:



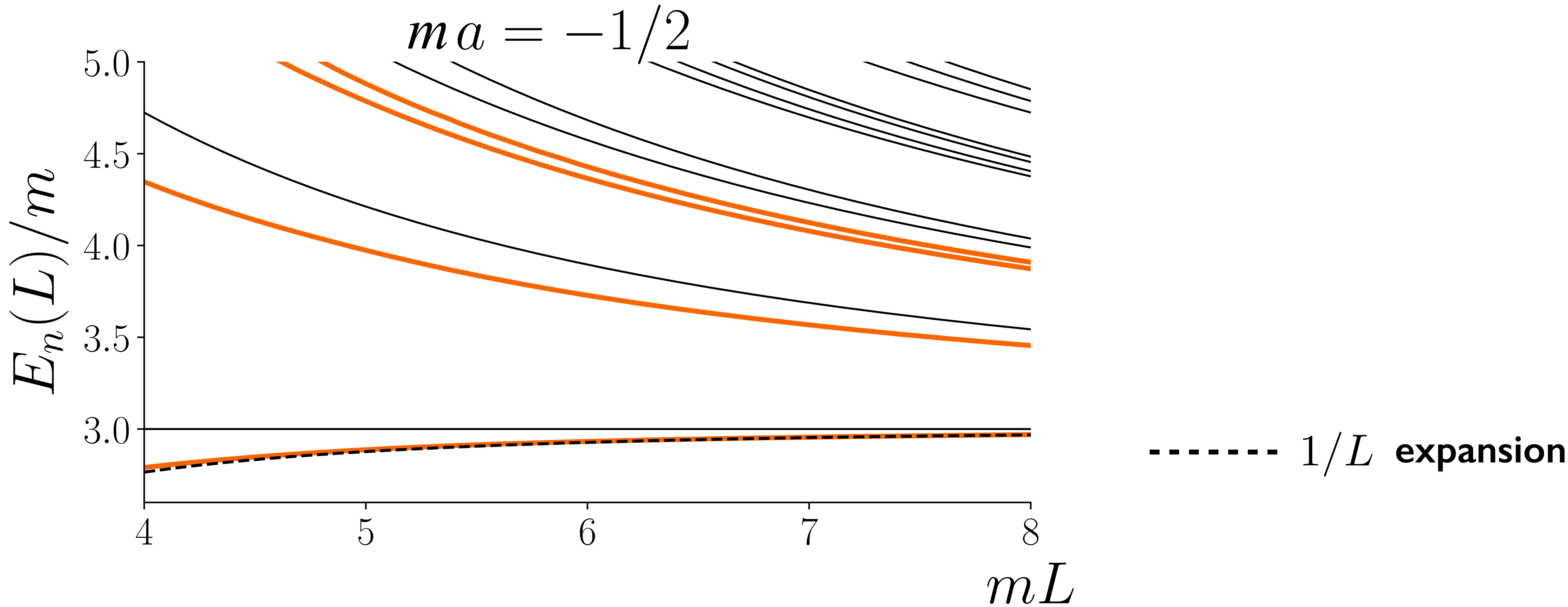
Solutions with $K_{df,3}=0$

- Non-interacting states



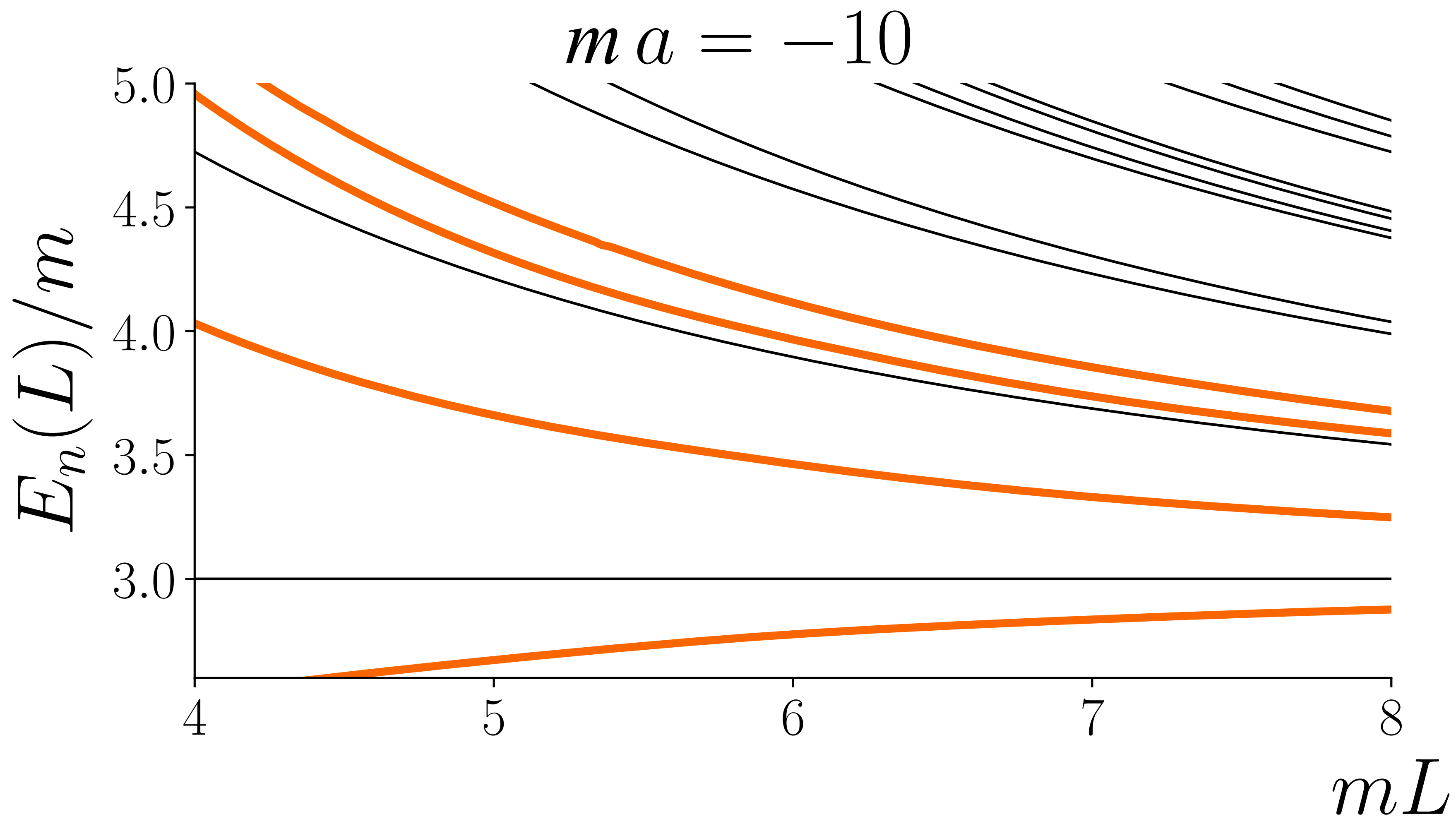
Solutions with $K_{df,3}=0$

- Weakly attractive two-particle interaction



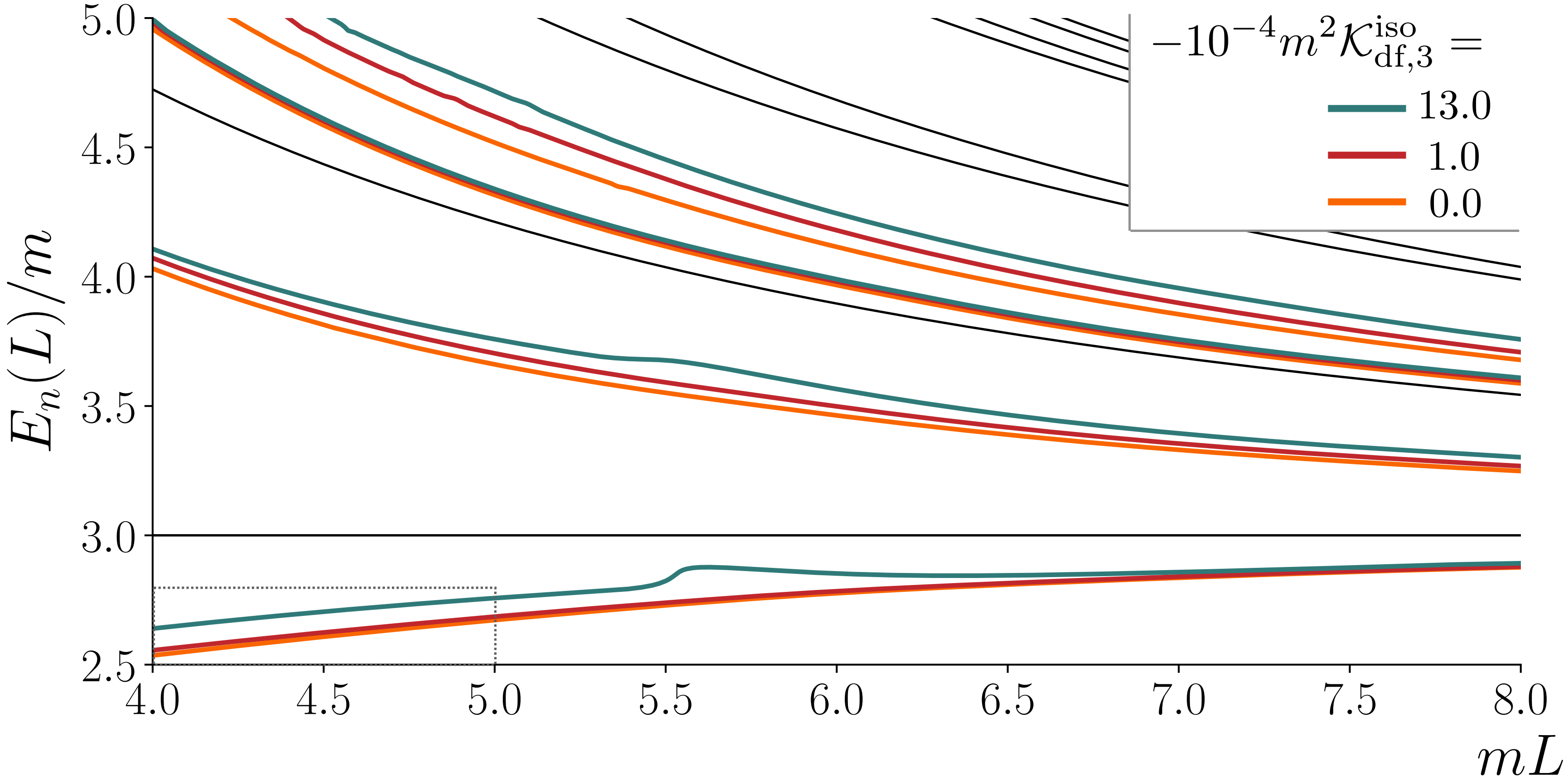
Solutions with $K_{df,3}=0$

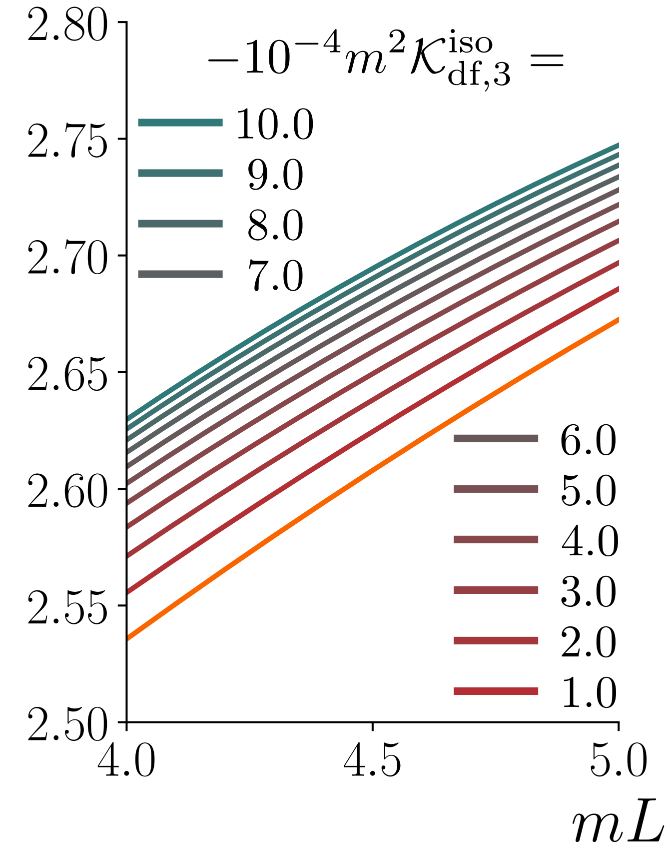
- Strongly attractive two-particle interaction



Impact of $K_{df,3}$

$ma = -10$ (strongly attractive interaction)

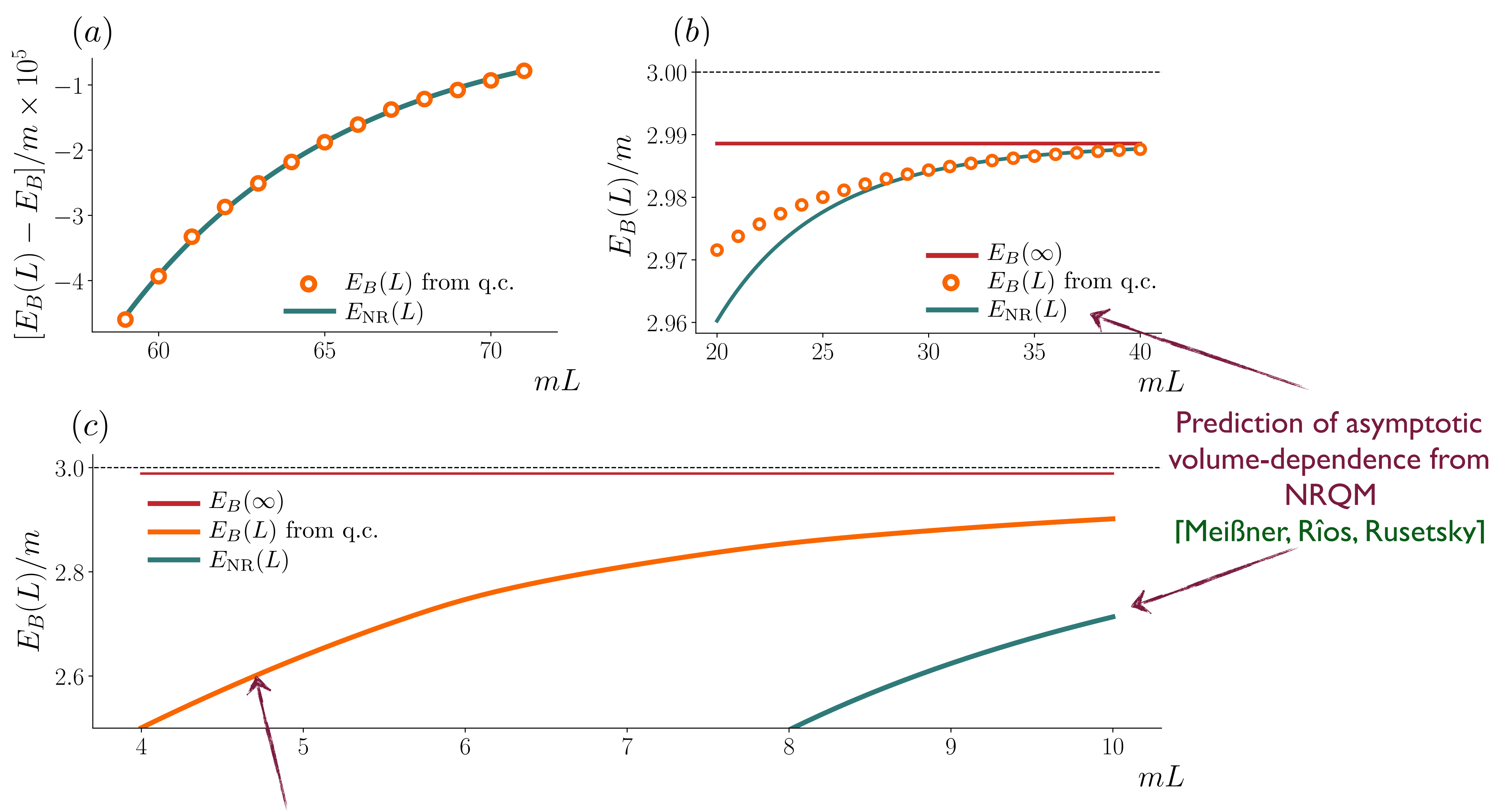




Local 3-particle interaction has significant effect on energies, especially in region of simulations (mL<5), and thus can be determined

Volume-dependence of 3-body bound state

am=−10^4 & $m^2 K_{df,3}^{iso}$=2500 (unitary regime)



Need quantization condition to determine finite-volume effects for realistic values of mL

Bound state wave-function

- Work in unitary regime (ma=−10⁴) and tune $\mathcal{K}_{\text{df},3}$ so 3-body bound state at E_B=2.98858 m
- Solve integral equations numerically to determine $\mathcal{M}_{\text{df},3}$ from $\mathcal{K}_{\text{df},3}$
- Determine wavefunction from residue at bound-state pole

$$\mathcal{M}_{\text{df},3}^{(u,u)}(k,p) \sim -\frac{\Gamma^{(u)}(k)\Gamma^{(u)}(p)^*}{E^2 - E_B^2}$$

- Compare to analytic prediction from NRQM in unitary limit [Hansen & SRS, 1609.04317]

$$|\Gamma^{(u)}(k)_{\text{NR}}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2 + 3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\frac{\sqrt{3}k}{2\kappa}\right)}{\sinh^2 \frac{\pi s_0}{2}}$$

Known constant

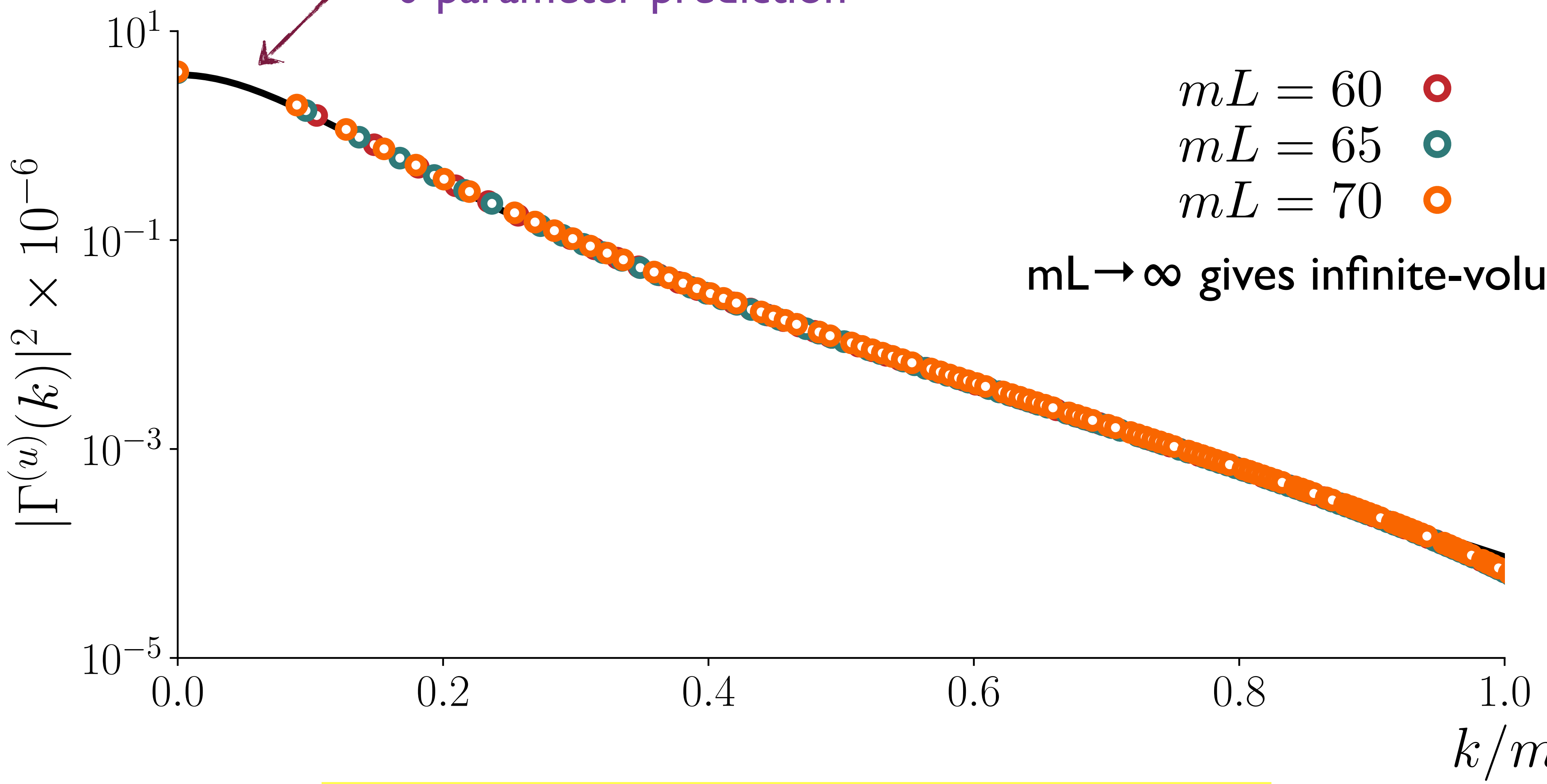
Known constant

Determined by fit to volume-dependence of bound-state energy

Bound state wave-function

$$|\Gamma^{(u)}(k)_{\rm NR}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2+3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\frac{\sqrt{3}k}{2\kappa}\right)}{\sinh^2\frac{\pi s_0}{2}}$$

0-parameter prediction

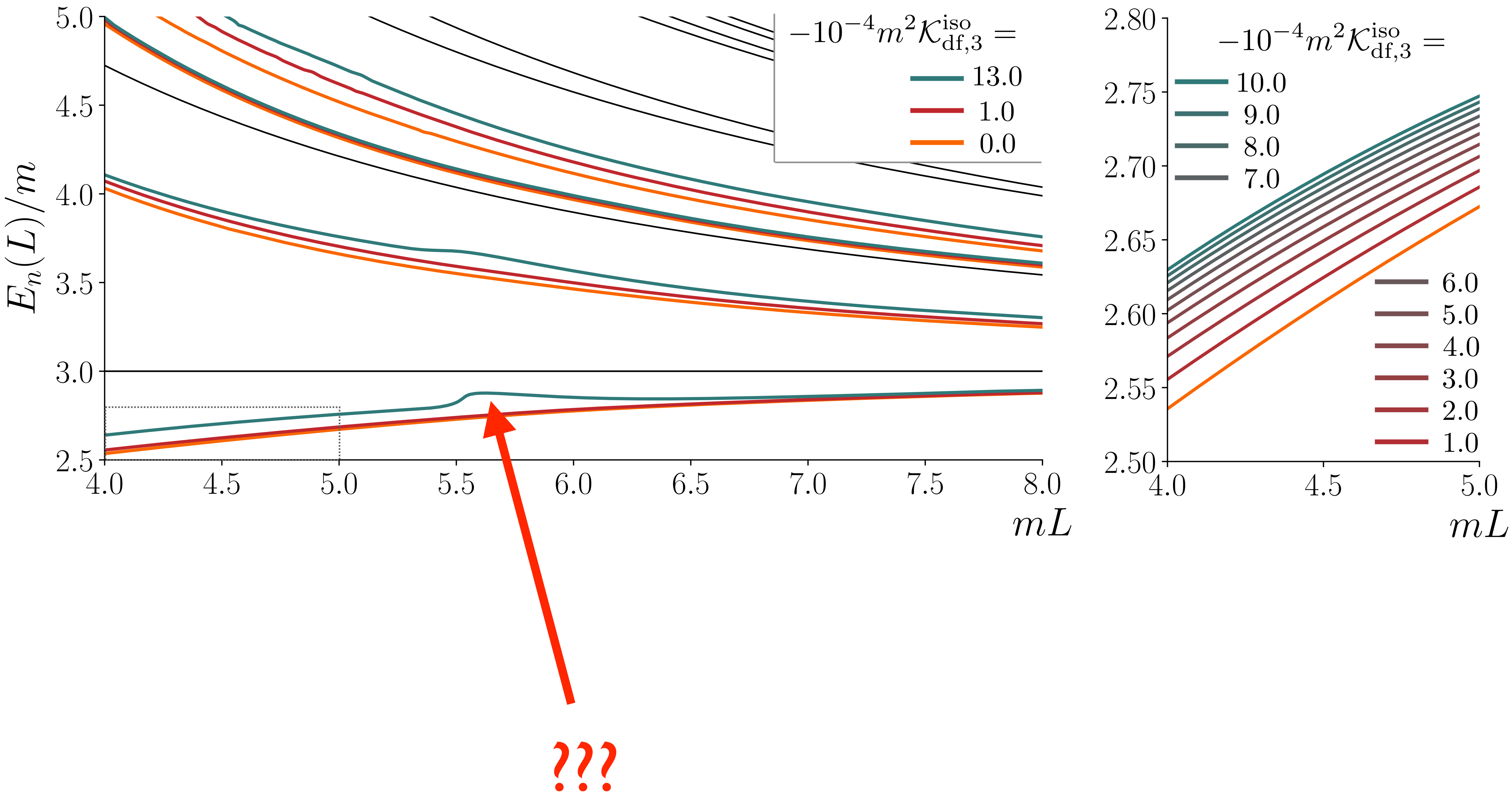


mL→∞ gives infinite-volume result

Works over many orders of magnitude
to expected accuracy

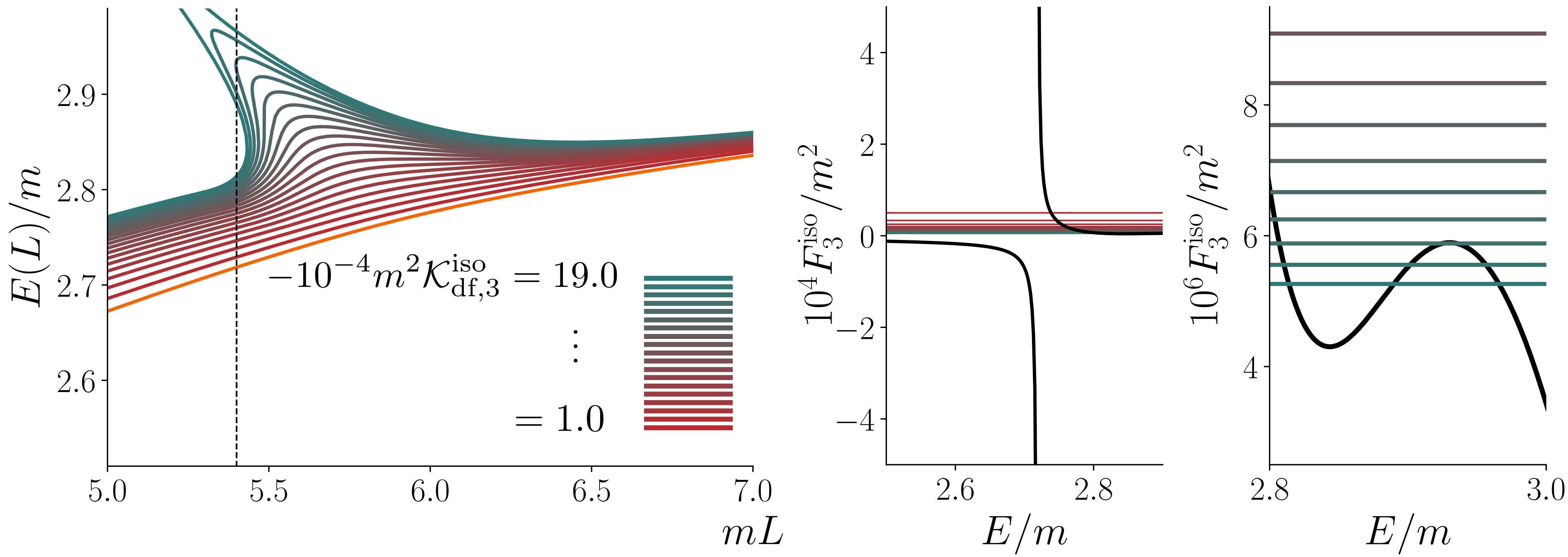
Unphysical solutions

$ma = -10$ (strongly attractive interaction)



Unphysical solutions

$$1/\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E^*) = -F_3^{\mathrm{iso}}[E, \vec{P}, L, \mathcal{M}_2^s]$$



- 2 extra solutions appear as L is varied, due to non-monotonicity in F_3^{iso}
  - Unphysical because leads to poles in correlator with wrong sign
  - Occur for larger magnitudes of $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}$ and smaller mL
- Possible sources: unphysical parameter choices or enhanced exp(-mL) effects

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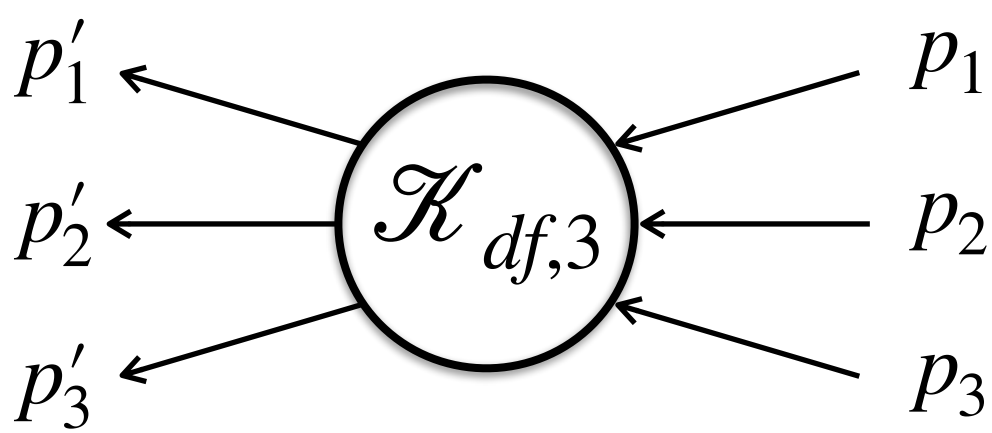
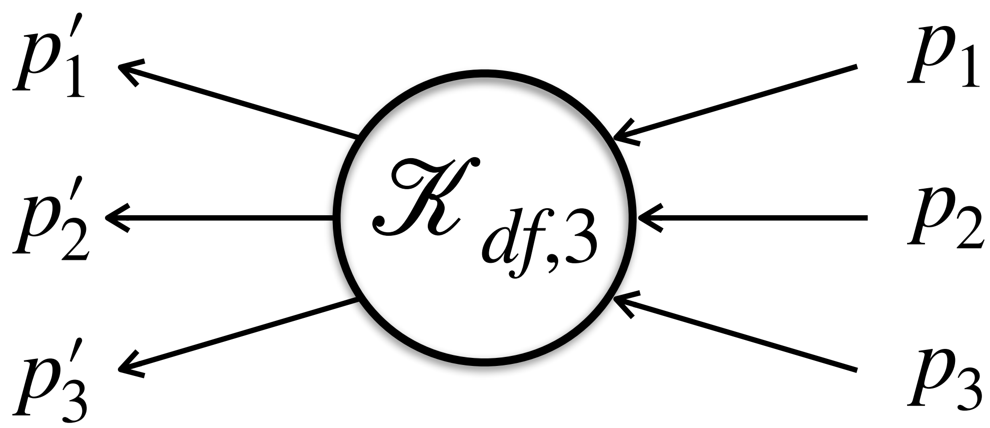
Beyond the isotropic approximation

[Tyler Blanton, Fernando Romero-Lopez & SRS, in progress]

- In 2-particle case, assume s-wave dominance at low energies, then systematically add in higher waves (suppressed by q^{2l})
- We are implementing the same general approach for $\mathcal{K}_{df,3}$, making use of the facts that it is relativistically invariant and completely symmetric under initial- & final-state permutations, and expanding about threshold
- We work in the G-parity invariant theory with 3 identical scalars, so the first channel beyond s-wave has l=2 (d-wave)



Beyond the isotropic approximation



$$\Delta = s - 9m^2$$

$$\Delta_1 = (p_2 + p_3)^2 - 4m^2 \text{ etc .}$$

$$\Delta_1' = (p_2' + p_3')^2 - 4m^2 \text{ etc .}$$

$$t_{ij} = (p_i - p_j')^2$$

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E) + c_A \mathcal{K}_{3A} + c_B \mathcal{K}_{3B} + \mathcal{O}(\Delta^3)$$

$$\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3} = c_0 + c_1 \Delta + c_2 \Delta^2$$

c_0 is the leading term—
only term kept in isotropic approx

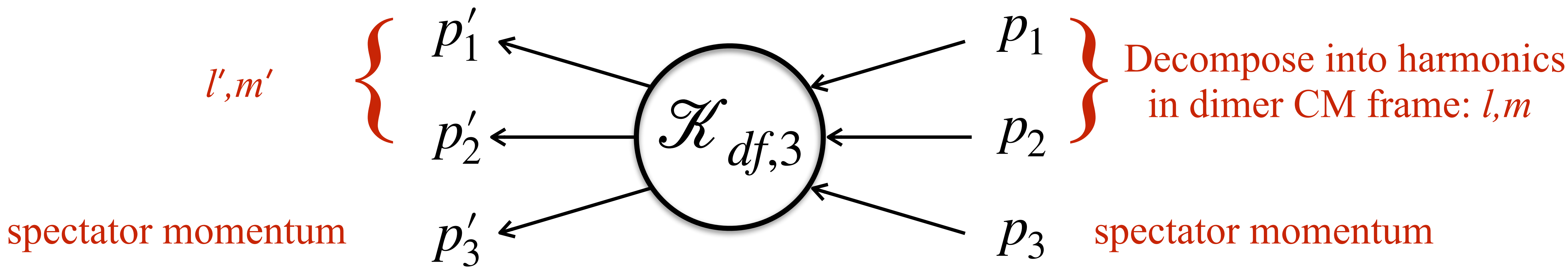
$$\mathcal{K}_{3A} = \sum_{i=1}^{3} \left(\Delta_i^2 + \Delta_i'^2\right)$$

c_1 is coefficient of the only linear term

$$\mathcal{K}_{3B} = \sum_{i,j=1}^{3} t_{ij}^2$$

Only three coefficients needed at quadratic order:
c_2 , c_A & c_B
Many fewer than the 7 angular variables + s dependence present at arbitrary energy!

Decomposing into spectator/dimer basis



$$\mathcal{K}_{3A},\ \mathcal{K}_{3B} \quad \Rightarrow \quad l'=0,2\ \&\ l=0,2$$

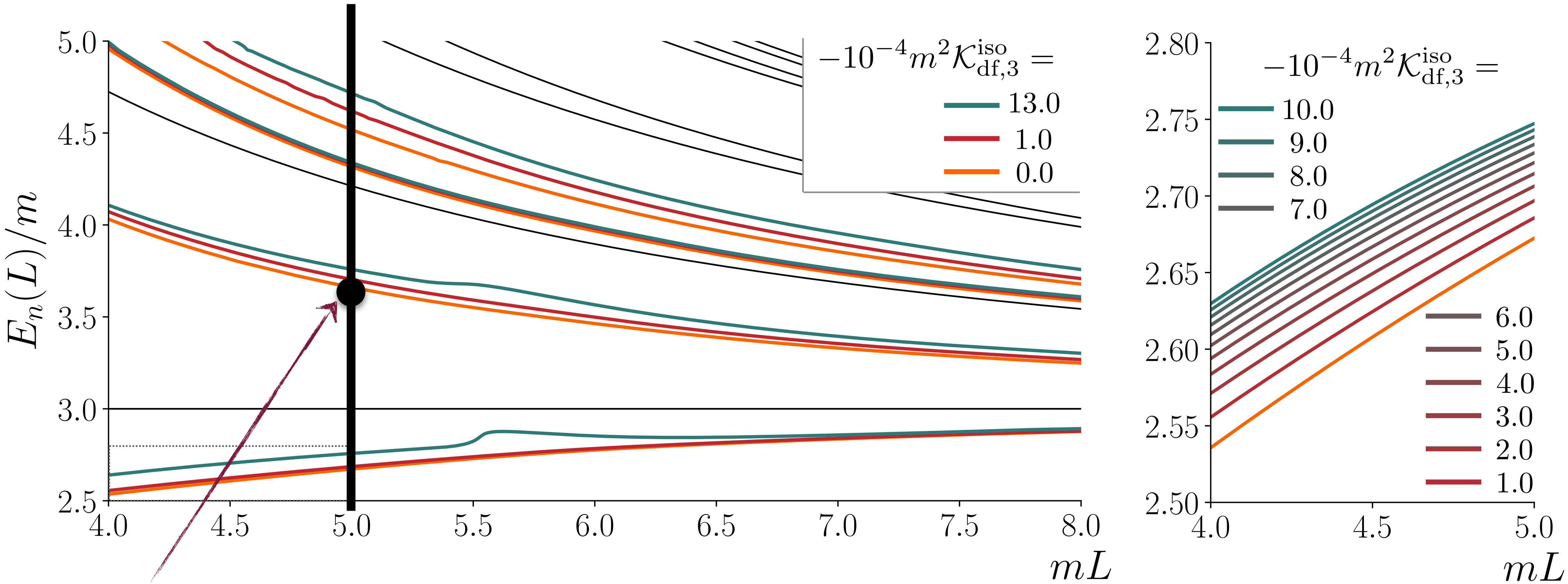
For consistency, need $\mathcal{K}_2^{(0)} \sim 1+q^2+q^4$ & $\mathcal{K}_2^{(2)} \sim q^4$

$$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2}\left[\frac{1}{a_0} + r_0\frac{q^2}{2} + P_0 r_0^3 q^4\right] \qquad \frac{1}{\mathcal{K}_2^{(2)}} = \frac{1}{16\pi E_2}\frac{1}{q^4}\frac{1}{a_2^5}$$

Implemented quantization condition through quadratic order, for **P**=0, including projection onto overall cubic group irreps

First results including l=2

$$\mathscr{K}_{\mathrm{df},3} = 0\,,\ a_0 = -10\,,\ r_0 = 0.5\,,\ P_0 = 0.5\,,\ -1.5 \le a_2 \le 0.1$$



What happens to
this level as
a_2 is turned on?

First results including l=2

$$\mathscr{K}_{\mathrm{df},3} = 0\,,\ a_0 = -10\,,\ r_0 = 0.5\,,\ P_0 = 0.5\,,\ -1.5 \le a_2 \le 0.1$$

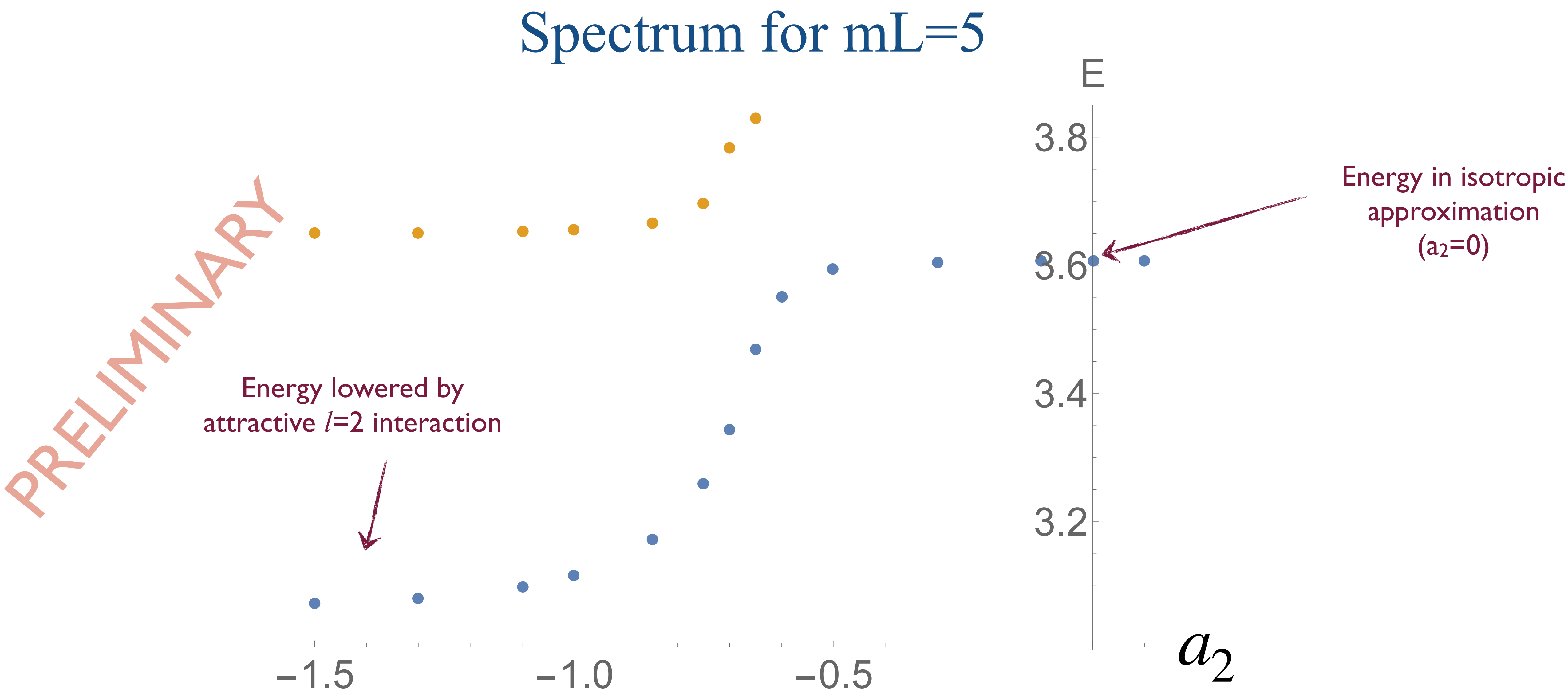
First results including $l=2$

$$\mathscr{K}_{\rm df,3} = 0\,,\; a_0 = -10\,,\; r_0 = 0.5\,,\; P_0 = 0.5\,,\; -1.5 \le a_2 \le 0.1$$

More in progress!

First results including l=2

$$\mathcal{K}_{\mathrm{df},3} = 0\,,\ a_0 = -10\,,\ r_0 = 0.5\,,\ P_0 = 0.5\,,\ -1.5 \le a_2 \le 0.1$$



More in progress!

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Outlook & open issues

- Complete formalism to-do list: nondegenerate particles of arbitrary spins
- Understand relation between different approaches
- Extend numerical experiments to include K-matrix poles ($ma > 1$ in isotropic approximation)
- Understand unphysical solutions
- Determine generalization of Lellouch-Lüscher factor to allow study of three-particle decays such as $K \to 3\pi$
- Use formalism to analyze results from simulations: simplest case is 3 pions
    - Need more understanding of appropriate parametrizations of $\mathcal{K}_{\mathrm{df},3}$
- …

Backup slides

Evidence for exponentially suppressed finite-volume effects

Use threshold expansion to determine 3-particle interaction at threshold

$$R_6(L) \equiv -L^6 \left\{ E(L) - 3 - \frac{c_3}{L^3} - \frac{c_4}{L^4} - \frac{c_5}{L^5} - \frac{\tilde{c}_6}{L^6} \right\}$$
$$= \frac{\mathcal{M}_{3,\mathrm{thr}}}{48} + \mathcal{O}(1/L).$$

Result depends on choice of regularization of Lüscher zeta function, an exponentially suppressed effect