Three-particle dynamics and resonances in a finite volume

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Scattering Amplitudes and Resonance Properties from Lattice QCD, MITP 2018

Outline

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- EFT for three-body systems
  - Infinite Volume
  - Finite Volume
- Resonances in a finite volume
- Summary and Outlook

Work in collaboration with:

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P. Klos, S. König, J.E. Lynn, A. Schwenk (TU Darmstadt)

Nuclear Scales and the Unitary Limit

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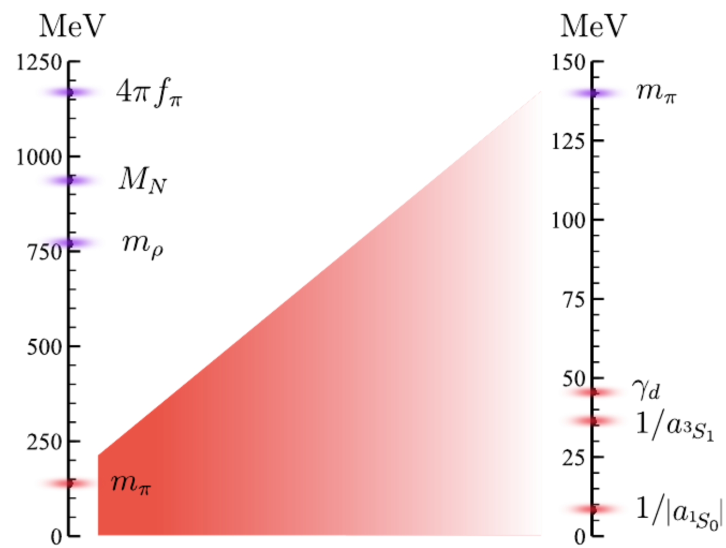
- Chiral EFT: $Q \sim m_\pi$
  $\Rightarrow$ contact interactions and pion exchange between nucleons
- Pionless EFT: $Q \sim \gamma_d \sim 1/a$
  $\Rightarrow$ contact interactions only



- Pionless EFT $\Longrightarrow$ expansion around the unitary limit
- Unitary limit: $a \to \infty$, $R \sim r_e, \ldots \to 0$ (cf. Bertsch problem, 2000)

$$\mathcal{T}_2(k,k) \propto \left[-1/a + r_e k^2/2 + \ldots - ik\right]^{-1} \Longrightarrow i/k$$

  - Scattering amplitude scale invariant, saturates unitarity bound
- Use as basis for EFT for shallow bound/virtual/resonant states

EFT for the Unitary Limit

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- Effective Lagrangian (Kaplan, 1997; Bedaque, HWH, van Kolck, 1999)

$$\mathcal{L}_d = \psi^\dagger \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi + \frac{g_2}{4} d^\dagger d - \frac{g_2}{4} (d^\dagger \psi^2 + (\psi^\dagger)^2 d) - \frac{g_3}{36} d^\dagger d \psi^\dagger \psi + \ldots$$

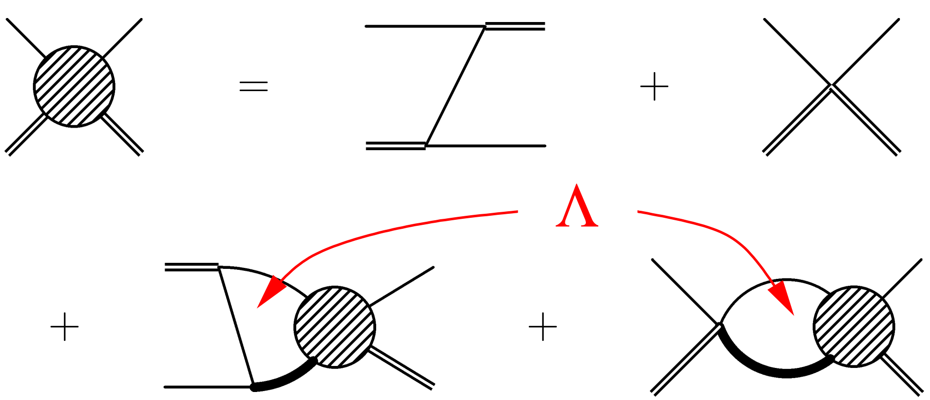
- 2-body amplitude:

- 2-body coupling g_2 near fixed point $(1/a = 0)$

$\Rightarrow$ scale and conformal invariance $\Longleftrightarrow$ unitary limit

(Mehen, Stewart, Wise, 2000; Nishida, Son, 2007; ...)

- 3-body amplitude:



$g_3(\Lambda)$ $\Rightarrow$ limit cycle

$\Rightarrow$ discrete scale inv.

Three-Body Force: Limit Cycle

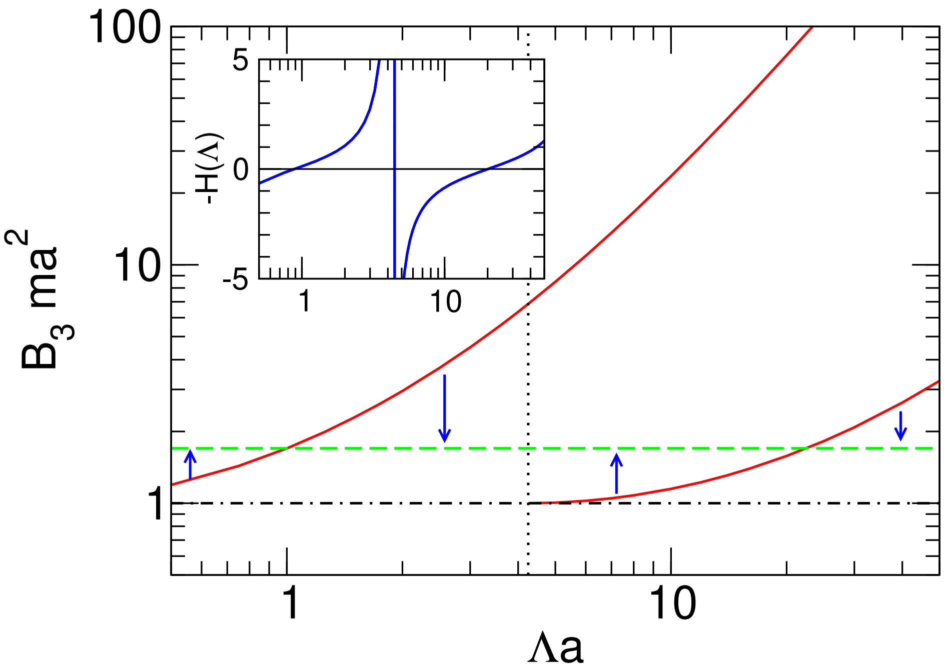
- RG invariance $\Longrightarrow$ running coupling $H(\Lambda) = g_3\Lambda^2/(9g_2^2)$

- $H(\Lambda)$ periodic: limit cycle

$$\Lambda \to \Lambda\, e^{n\pi/s_0} \approx \Lambda(22.7)^n$$

(cf. Wilson, 1971)

- Anomaly: scale invariance broken to discrete subgroup



$$\boxed{H(\Lambda) \approx \frac{\cos(s_0\ln(\Lambda/\Lambda_*)+\arctan(s_0))}{\cos(s_0\ln(\Lambda/\Lambda_*)-\arctan(s_0))}\,, \qquad s_0 \approx 1.00624}$$

(Bedaque, HWH, van Kolck, 1999)

- Limit cycle $\Longleftrightarrow$ Discrete scale invariance $\Longleftrightarrow$ Efimov physics

Limit Cycle: Efimov Physics

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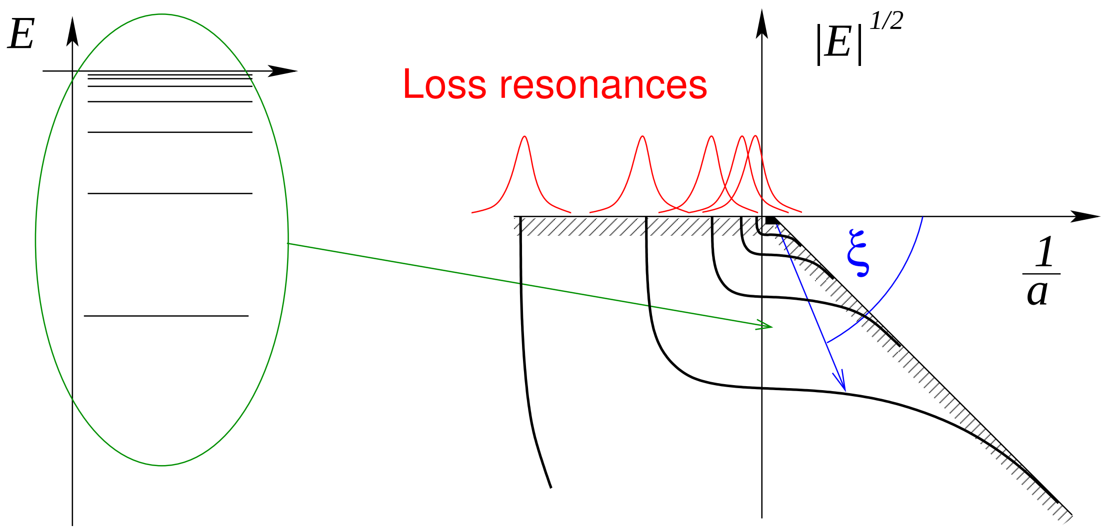
- Universal spectrum of three-body states (Efimov, 1970)

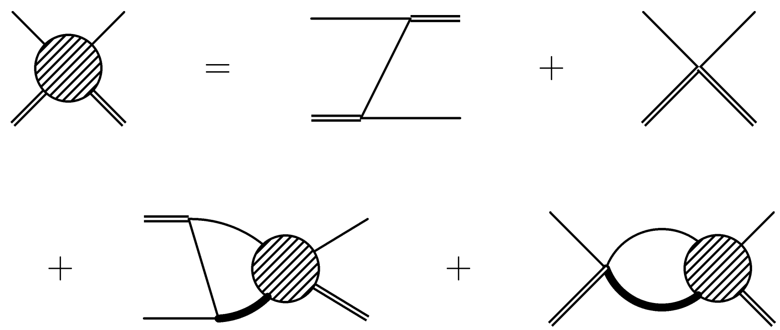


  - Discrete scale invariance for fixed angle ξ
  - Geometrical spectrum for $1/a \to 0$
- Ultracold atoms $\Longrightarrow$ variable scattering length $\Longrightarrow$ loss resonances
- Hadrons & Nuclei $\Longrightarrow$ universal correlations and scaling relations
- Works also for natural interactions: $a \sim r_e \sim R$

Infinite Volume Equation

- Use dimer picture NREFT to bridge infinite and finite volume

  HWH, Pang, Rusetsky, JHEP 1709 (2017) 109, JHEP 1710 (2017) 115



$$\mathcal{M}(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + \int_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{k};E)\tau(\mathbf{k};E)\mathcal{M}(\mathbf{k},\mathbf{q};E)$$

$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{pq}-mE} + \frac{H_0}{\Lambda^2} + \frac{H_2}{\Lambda^2}(\mathbf{p}^2+\mathbf{q}^2) + \cdots$$

$$\tau^{-1}(\mathbf{k};E) = k^*\cot\delta(k^*) + k^*\,, \quad k^* = \sqrt{\frac{3}{4}\mathbf{k}^2 - mE}$$

Finite-Volume 3-Body Problem

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- Kreuzer, HWH, PLB 673 (2009) 260, EPJA 43 (2010) 229, PLB 694 (2011) 424; Kreuzer, Grießhammer, EPJA 48 (2012) 93

  Dimer formalism, numerical solution of 3-body equation

- Polejaeva, Rusetsky, EPJA 48 (2012) 67

  Finite volume energy levels determined solely by the S-matrix

- Briceno, Davoudi, PRD 87 (2013) 094507

  Dimer formalism, quantization condition

- Hansen, Sharpe, PRD 90 (2014) 116003; PRD 92 (2015) 114509

  Quantization condition

- Guo, PRD 95 (2017) 054508

  Quantization condition in the 1+1-dimensional case

- Mai, Döring, EPJA 53 (2017) 240

  Three-body unitarity + analyticity

- . . .

Finite Volume Equation

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$$\mathbf{k} = \frac{2\pi}{L}\,\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3\,, \qquad\qquad \int_{\mathbf{k}}^{\Lambda} \to \frac{1}{L^3}\sum_{\mathbf{k}}^{\Lambda}$$

$$\mathcal{M}_L(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + \frac{8\pi}{L^3}\sum_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{q};E)\tau_L(\mathbf{k};E)\mathcal{M}_L(\mathbf{k},\mathbf{q};E)$$

$$\tau_L^{-1}(\mathbf{k};E) = k^*\cot\delta(k^*) - \frac{4\pi}{L^3}\sum_{\mathbf{l}} \frac{1}{\mathbf{k}^2 + \mathbf{l}^2 + \mathbf{kl} - mE}$$

- Poles in the amplitude $\Rightarrow$ finite-volume energy spectrum
- $k^*\cot\delta(k^*)$ fitted in the two-particle sector
- $H_0, H_2, \ldots$ fitted to three-particle energies
- S-matrix in the infinite volume $\Rightarrow$ equation with $H_0, H_2, \ldots$

Quantization Condition

- Particle-dimer scattering amplitude

$$\mathcal{M}_L = Z + Z\tau_L\mathcal{M}_L$$

- Three-particle scattering amplitude

$$\tau_L + \tau_L\mathcal{M}_L\tau_L = (\tau_L^{-1} - Z)^{-1}$$

- Quantization condition – poles in the three-particle amplitude

$$\det(\tau_L^{-1} - Z) = 0$$

- Spectrum is determined by on-shell input
- In agreement with:

  Polejaeva & Rusetsky, Hansen & Sharpe, Briceno & Davoudi, Mai & Döring

Reduction of QC: Symmetries

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Döring, HWH, Mai, Pang, Rusetsky, Wu, Phys. Rev. D **97** (2018) 114508

- Symmetry in a finite box: octahedral group O_h, including inversions (rest frame), little groups (moving frames)
- Reduction: analog of the partial-wave expansion in a finite volume
- Analog of a sphere $|\mathbf{k}| =$ const.: shells

$$s = \left\{ \mathbf{k} \;:\quad \mathbf{k} = g\mathbf{k}_0\,,\quad g \in O_h \right\}$$

- Each shell s characterized by reference momentum $\mathbf{k}_0$
- Shells are counted with increasing $|\mathbf{k}|$
- Momenta with $|\mathbf{k}| = |\mathbf{k}'|$ but unrelated by O_h belong to different shell

Expansion in Basis of Irreps

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- Expansion of arbitrary function of $\mathbf{p}$, belonging to shell s:

$$f(\mathbf{p}) = f(g\mathbf{p}_0) = \sum_{\Gamma} \sum_{ij} T_{ij}^{(\Gamma)}(g) f_{ji}^{(\Gamma)}(\mathbf{p}_0) \,, \qquad \Gamma = A_1^{\pm}, A_2^{\pm}, E^{\pm}, T_1^{\pm}, T_2^{\pm}$$

- Projecting out the components

$$\frac{G}{s_\Gamma} f_{ji}^{(\Gamma)}(\mathbf{p}_0) = \sum_{g \in O_h} (T_{ij}^{(\Gamma)}(g))^* f(g\mathbf{p}_0) \,, \qquad G = \text{dim}(O_h) = 48$$

- Kernel invariant under O_h: $Z(g\mathbf{p}, g\mathbf{q}) = Z(\mathbf{p}, \mathbf{q})$

$$Z_{nm}^{(\Gamma\Gamma', ij)}(r, s) \quad = \quad \frac{G}{s_\Gamma} \delta_{\Gamma\Gamma'} \delta_{ij} Z_{nm}^{(\Gamma)}(r, s)$$

Quantization condition in new basis partially diagonalizes

Reduction of the Equation

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- Equation for energy spectrum

$$f(\mathbf{p}) = \frac{8\pi}{L^3} \sum_s \sum_{g \in O_h} \frac{\vartheta(s)}{G} Z(\mathbf{p}, g\mathbf{k}_0(s))\tau(s) f(g\mathbf{k}_0(s))$$

where $\vartheta(s)$: multiplicity of shell s

- Projection on a given irrep Γ:

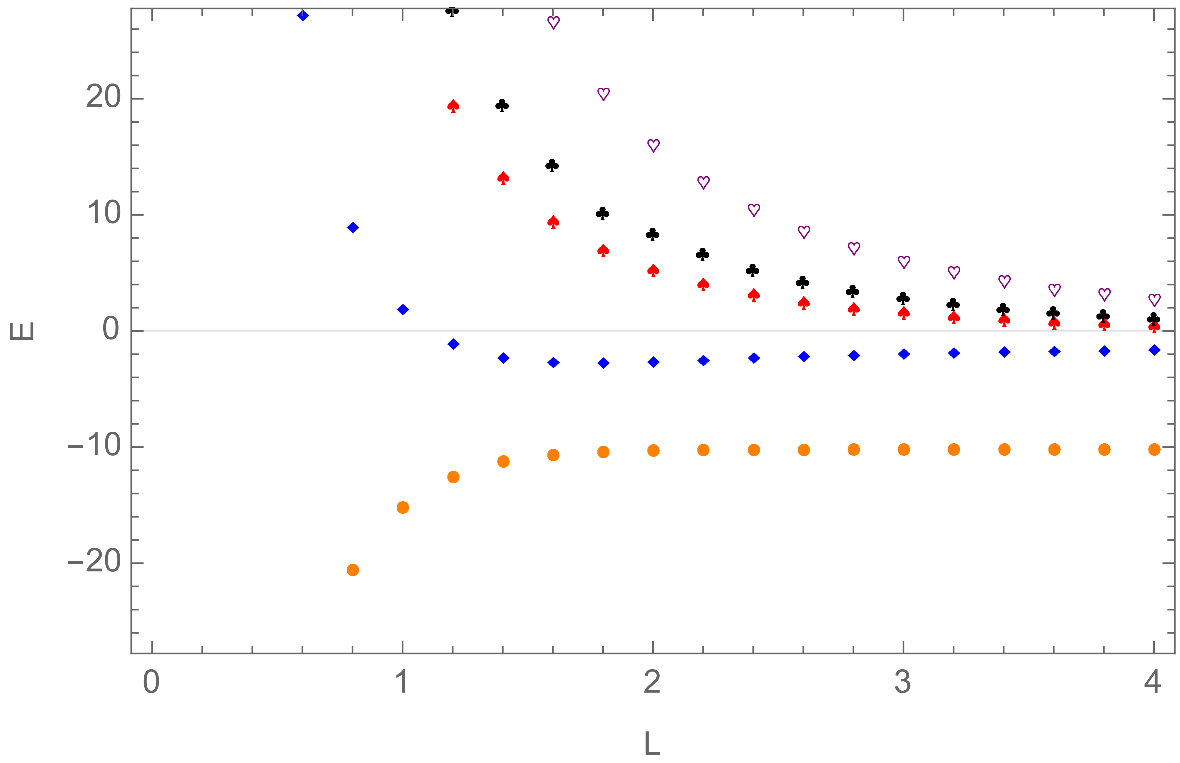
$$f_i^{(\Gamma)}(r) = \frac{8\pi}{L^3} \sum_s \frac{\vartheta(s)\tau(s)}{G} \sum_j Z_{ij}^{(\Gamma)}(r,s) f_j^{(\Gamma)}(s)\,.$$

- Quantization condition partially diagonalizes

$$\det\left(\tau(s)^{-1}\vartheta(s)^{-1}\delta_{rs}\delta_{ij} - \frac{8\pi}{L^3}\,\frac{1}{G}\,Z_{ij}^{(\Gamma)}(r,s)\right) = 0\,.$$

Finite-Volume Spectrum in A_1

- Three-body spectrum below and above three-particle threshold (CM frame)



- Parameters: $m = a = 1$, $\Lambda = 225$, $H_0(\Lambda) = 0.192$

Bound-State Spectrum

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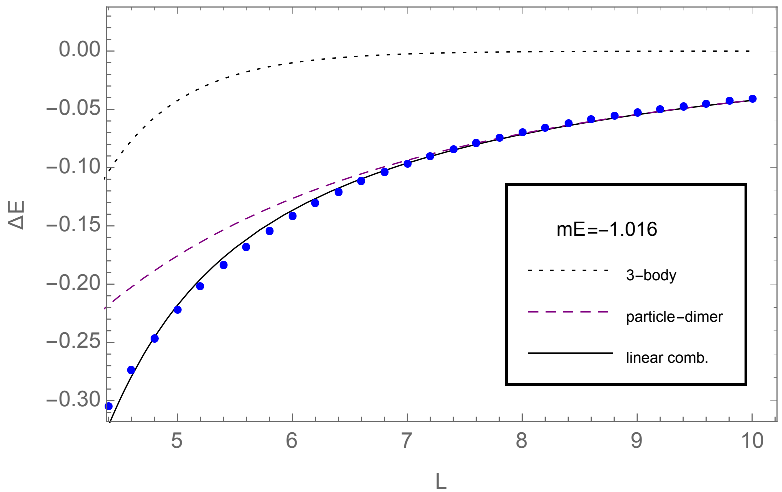
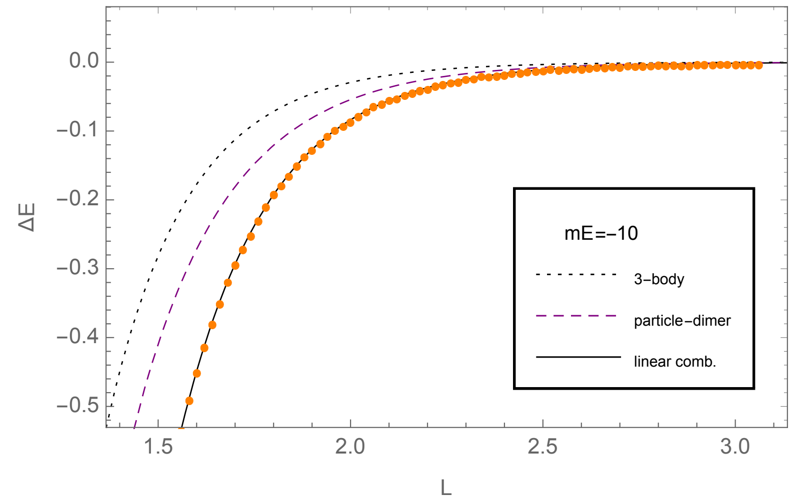
- Bound-State Spectrum: $E = -1.016$ and $E = -10$





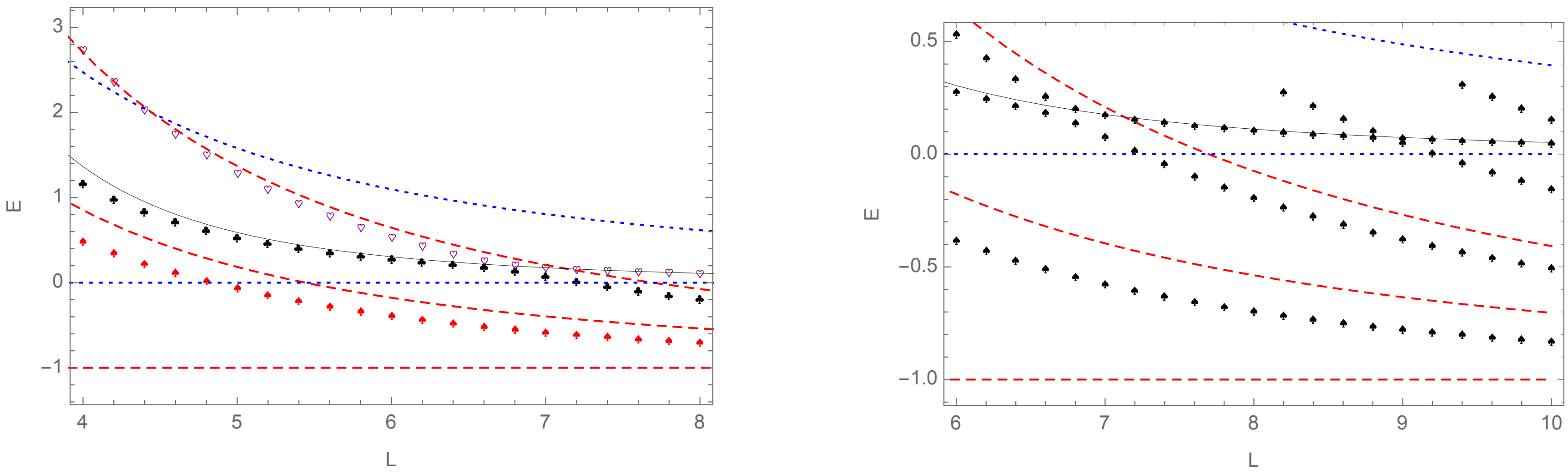
- Three-particle comp.: $\dfrac{C}{L^{3/2}} \exp\left(-\dfrac{2}{\sqrt{3}}\,\kappa L\right)$ (Meißner, Ríos and Rusetsky, '16)

- Particle-dimer comp.: $\dfrac{C'}{L} \exp\left(-\dfrac{2}{\sqrt{3}}\sqrt{\kappa^2 - a^{-2}}L\right)$ (Lüscher, '86)

- …or a linear combination (HWH, Pang and Rusetsky, '17)

Scattering States

- Structure of scattering states
  - Avoided level crossing between three-particle and particle-dimer states



- Resonances (two-body) $\Rightarrow$ avoided level crossings

  Wiese, Nucl. Phys. B (Proc. Suppl.) **9** (1989) 609

- Signature of few-body resonances? e.g. 3 and 4 neutrons

  Klos, König, HWH, Lynn, Schwenk, arXiv:1805.02029, Phys. Rev. C (in press)

Discrete Variable Representation

Klos, König, HWH, Lynn, Schwenk, arXiv:1805.02029, Phys. Rev. C (in press)

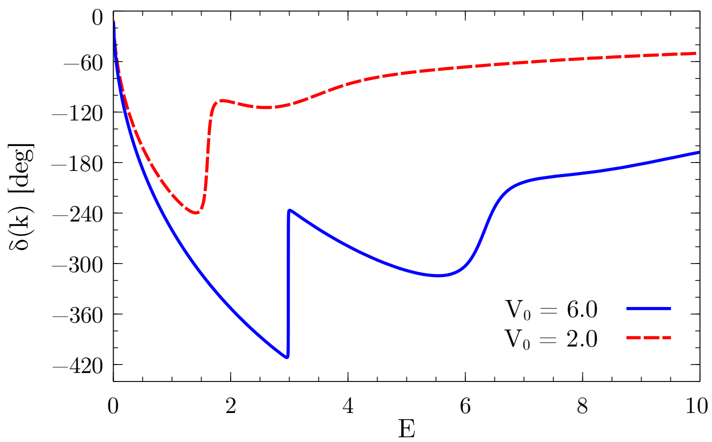
- Solve few-body problem in a box and investigate resonance signatures
- Numerical method: Discrete Variable Representation (DVR)
  (cf. Bulgac, Forbes, Phys. Rev. C **87** (2013) 051301)
- Start with some initial basis, here: $\phi_j(x) = \frac{1}{\sqrt{L}} \exp(i \frac{2\pi j}{L} x)$
- Consider (x_k, w_k) with $\sum_k w_k \, \phi_i^*(x_k) \phi_j(x_k) = \delta_{ij}$
- Unitary Transformation: $\psi_k(x) = \sum_i \underbrace{\left(\sqrt{w_k} \phi_i(x_k)\right)^*}_{\mathcal{U}_{ki}^*} \phi_i(x)$

  $\Rightarrow$ localized at x_k $\quad \Longrightarrow \quad$ local interaction becomes diagonal
- Convenient basis for box calculations
- Test method for n-body resonances ($n = 2, 3, 4$) using simple potentials

Two-Body Resonances

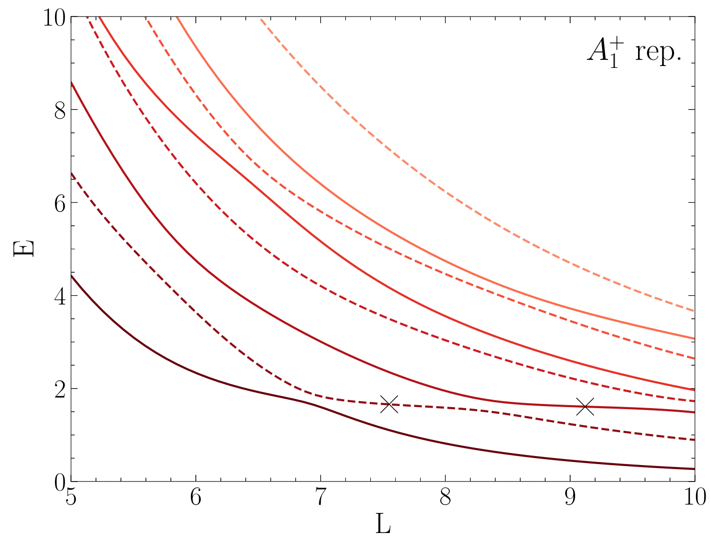
- Use barrier to produce S-wave resonance (no bound states)

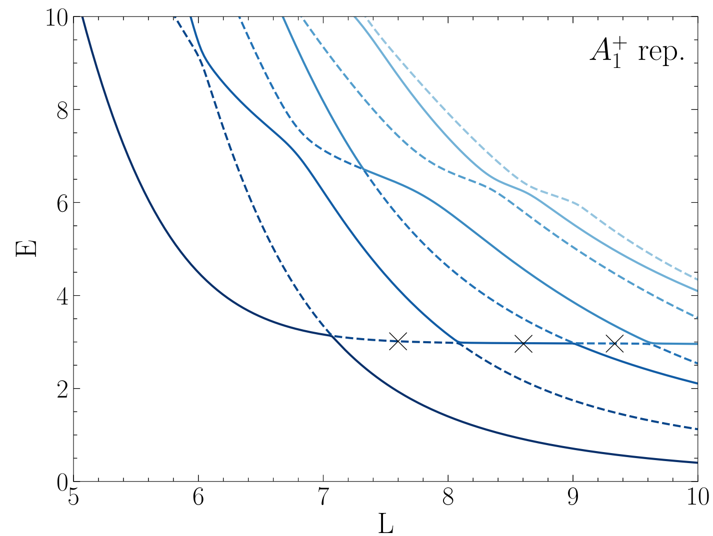
$$V(r) = V_0 \exp\left(-\left(\frac{r-a}{R_0}\right)^2\right)$$



- Identify energy via phase shift
- Extract resonance energies from avoided level crossings

(fit inflection points)





$$E_R = 1.606(1) \;\Leftrightarrow\; 1.63(3)$$

$$E_R = 2.9821(3) \;\Leftrightarrow\; 2.98(3)$$

Three-Body Resonances

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- Check with established three-body resonance from literature

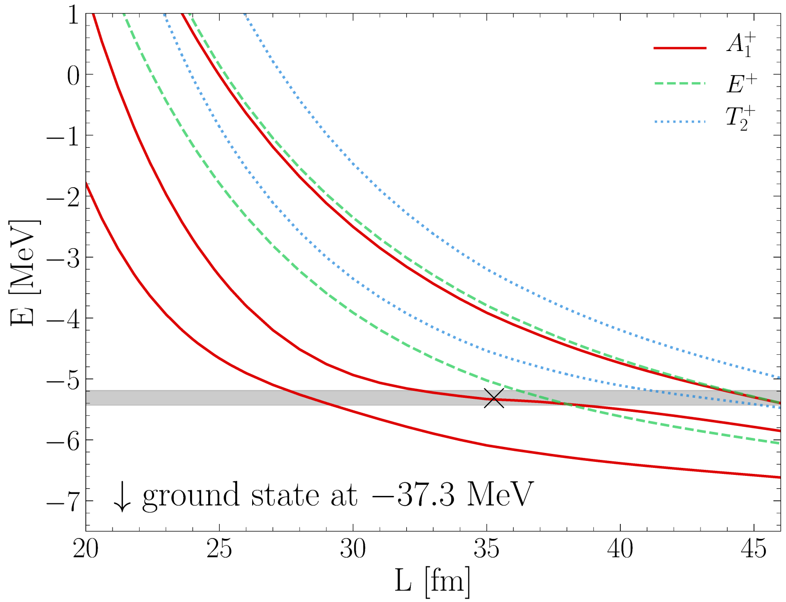
  Blandon et al., Phys. Rev. A **75** (2007) 042508

$$V(r) = V_0\, e^{-\left(\frac{r}{R_0}\right)^2} + V_1\, e^{-\left(\frac{r-a}{R_1}\right)^2}$$

$V_0 = -55$ MeV, $V_1 = 1.5$ MeV

$R_0 = \sqrt{5}$ fm, $R_1 = 10$ fm

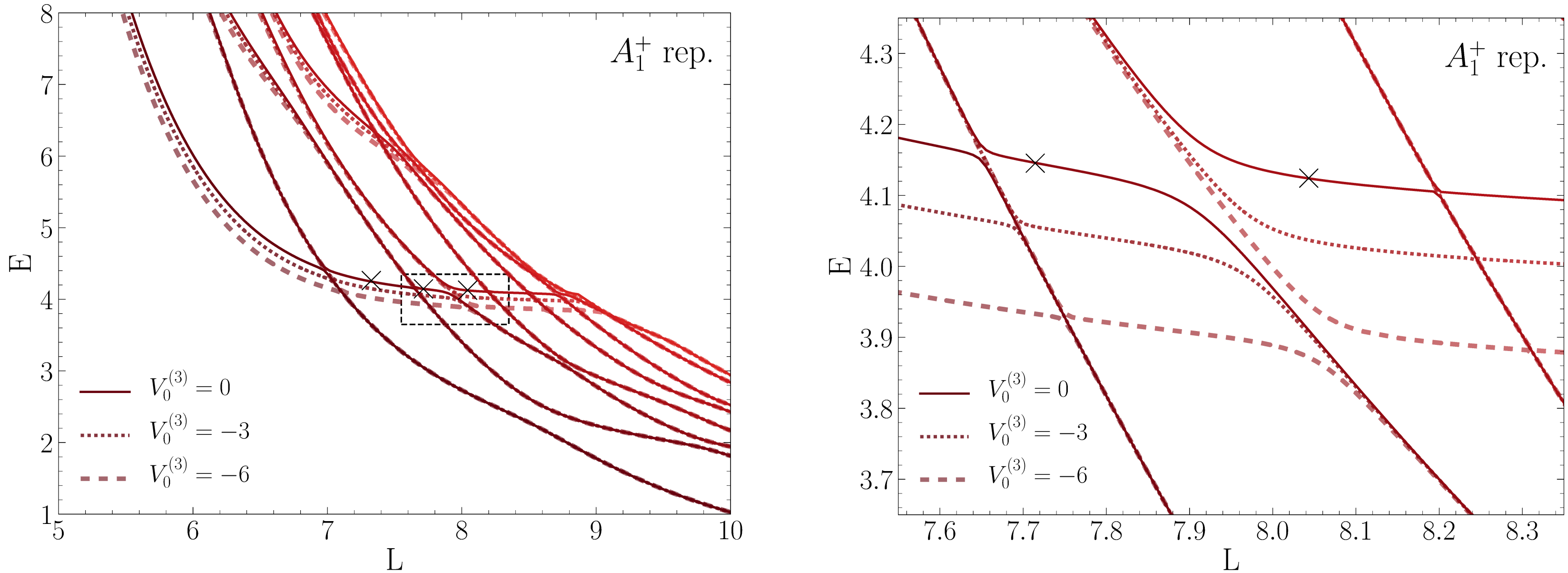
$m = 939$ MeV



- Two- and three-body bound states at $E = -6.76$ MeV and $E = -37.35$ MeV
- Three-body resonance at $E_R = -5.31$ MeV, $\Gamma_R = 0.24$ MeV
- Fit inflection points to extract energy: $E_R = -5.32(1)$ MeV
- Width not extracted

Three-Body Resonances

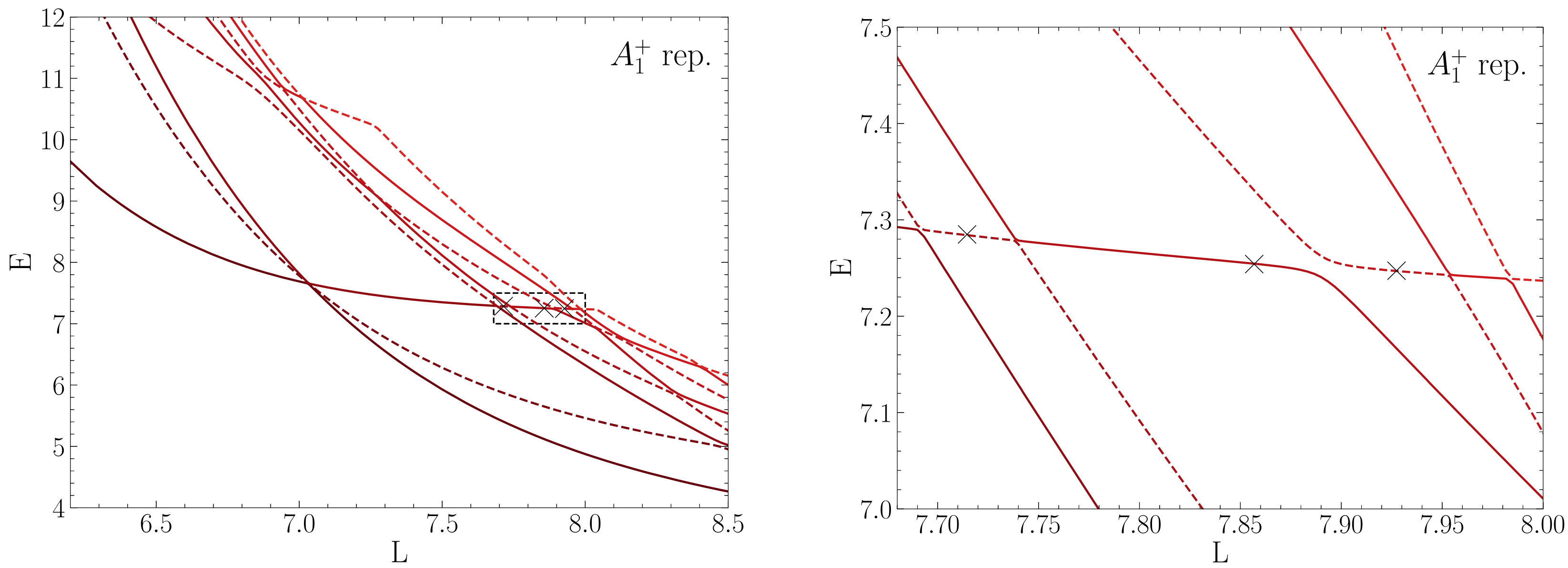
- Go back to shifted Gaussian potential with $V_0 = 2$ and investigate three-body resonances for bosons (no bound states)



- Add three-body force $V_0^{(3)} = 0, -3, -6$ to distinguish three-body resonances from two-body resonances embedded in three-body continuum
- Three-body resonance extracted at $E_R = 4.17(8)$

Four-Body Resonances

- Investigate four-body resonances for same potential (no bound states)



- Four-boson resonance extracted at $E_R = 7.27(2)$
- Avoided level crossings appear also for multi-body resonances
- Rigorous justification?

Summary and Outlook

- Effective Field Theory to analyze three-body spectrum in a finite volume
- Strategy: use EFT to bridge infinite and finite volume worlds
  - Fit coupling constants from finite volume spectra then predict infinite volume observables using these couplings
- Quantization condition can be reduced to different irreps of O_h
- Nature of states from L dependence (cf. Akaki's talk)
- Future: higher partial waves, derivative couplings, relativistic kinematics, inelastic reactions, ...
- Few-body resonances from finite volume spectra
  - Discrete Variable Representation provides efficient method
  - Extracted resonances of up to 4 particles (no asymptotic two-body channel required)
- Apply to 3 and 4 neutron systems using chiral interactions

Additional Slides

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Expansion of the Kernel

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- Kernel invariant under O_h: $Z(g\mathbf{p}, g\mathbf{q}) = Z(\mathbf{p}, \mathbf{q})$

$$
\begin{aligned}
Z_{nm}^{(\Gamma\Gamma',ij)}(r,s) &= \sum_{g,g'\in O_h} (T_{in}^{(\Gamma)}(g'))^* Z(g'\mathbf{p}_0(r), g\mathbf{k}_0(s)) T_{jm}^{(\Gamma')}(g) \\
&= \sum_{g,g'\in O_h} (T_{in}^{(\Gamma)}(g'))^* Z(\underbrace{g^{-1}g'}_{=g''}\mathbf{p}_0(r), \mathbf{k}_0(s)) T_{jm}^{(\Gamma')}(g) \\
&= \sum_{g,g''\in O_h} \sum_k (T_{ik}^{(\Gamma)}(g))^* (T_{kn}^{(\Gamma)}(g''))^* Z(g''\mathbf{p}_0(r), \mathbf{k}_0(s)) T_{jm}^{(\Gamma')}(g) \\
&= \sum_{g''\in O_h} \sum_k \frac{G}{s_\Gamma} \delta_{\Gamma\Gamma'}\delta_{ij}\delta_{km} (T_{kn}^{(\Gamma)}(g''))^* Z(g''\mathbf{p}_0(r), \mathbf{k}_0(s)) \\
&= \frac{G}{s_\Gamma}\delta_{\Gamma\Gamma'}\delta_{ij} \sum_{g\in O_h} (T_{mn}^{(\Gamma)}(g))^* Z(g\mathbf{p}_0(r), \mathbf{k}_0(s)) \\
&= \frac{G}{s_\Gamma}\delta_{\Gamma\Gamma'}\delta_{ij} Z_{nm}^{(\Gamma)}(r,s)
\end{aligned}
$$

Weinberg’s Folk Theorem

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Quantum Field Theory has no content besides unitarity, analyticity, cluster decomposition and symmetries

$\Rightarrow$ to calculate the most general S-matrix for any theory below some scale simply use the most general effective Lagrangian $\mathcal{L}_{\text{eff}}$ consistent with these principles in terms of the appropriate asymptotic states

(Weinberg, 1979)

Nuclear forces from chiral EFT

$$e^{iZ[J]} = \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{D}G\, e^{i\int d^4x\,\mathcal{L}_{QCD}[q,\bar{q},G;J]} \Leftrightarrow \int \mathcal{D}\pi\,\mathcal{D}N e^{i\int d^4x\,\mathcal{L}_{eff}[\pi,N;J]}$$

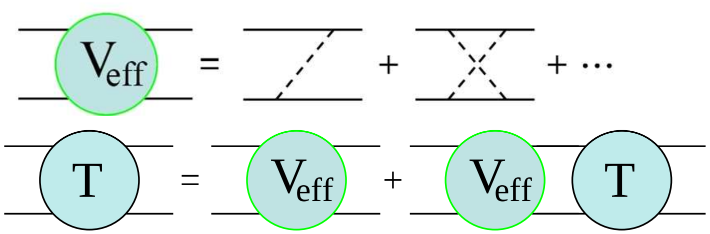
- Write down most general $\mathcal{L}_{\text{eff}}$ consistent with chiral symmetry

$$\mathcal{L}_{\text{QCD}} \to \mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \mathcal{L}_{NN} + \ldots$$

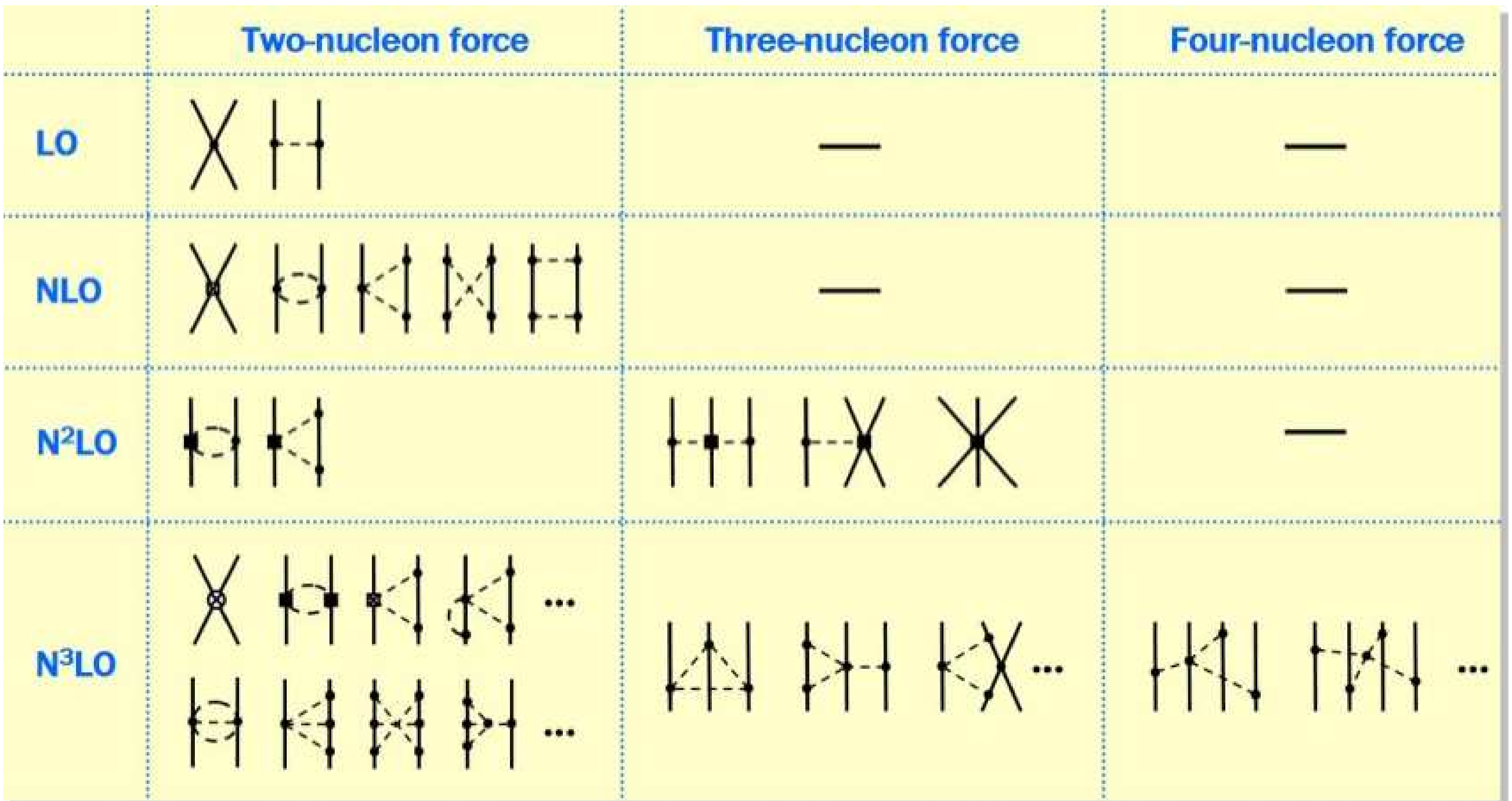
- Compute S-Matrix elements in perturbation theory (chiral exp.)
- Fix low-energy constants (LECs) & make predictions...

Complication for $\geq 2N$: large scattering length ($\Rightarrow$ ^{2}H, nuclei)

- Weinberg's approach ('90):
  - Use $\mathcal{L}_{eff}$ to compute V_{eff}
    (syst. expansion in Q/Λ_χ and m_π/Λ_χ)
  - Solve LS equation with V_{eff}



Nuclear forces from chiral EFT



- Hierarchy of nuclear forces: $V_{2N} \gg V_{3N} \gg V_{4N} \gg \ldots$

(cf. E. Epelbaum, HWH, U.-G. Meißner, Rev. Mod. Phys. **81** (2009) 1773; R. Machleidt, D.R. Entem, Phys. Rep. **503** (2011) 1)