Three-photon exchange nuclear structure contribution to the Lamb sfift

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QED theory of compound particles

- What after proton radius puzzle disappears ?
- Comparison of the Lamb shift in eX and μX, with X=He, Li, Be, B
- Nontrivial QED expansion the for finite size particles (nuclear self-energy)
- Nuclear polarizability is the issue
- Little progress in three-photon exchange nuclear structure effects for nuclei
- Cancellation between elastic and inelastic contributions

Measurements of atomic properties

Measurement of transition frequencies can be very accurate [Garching, 2013]

- $\nu(1S-2S)_{\mathrm{H}} = 2466\,061\,413\,187\,018(11)\ \mathrm{Hz}$
- sensitive to the nuclear size and to the nuclear polarizability
- from $\nu(1S-2S)_{\mathrm{H-D}}$: $r_d^2 - r_p^2 = 3.820\,07(65)\ \mathrm{fm}^2$
- from $\nu(2S-2p)_{\mu\mathrm{H}-\mu\mathrm{D}}$: $r_d^2 - r_p^2 = 3.811\,20(339)\ \mathrm{fm}^2$
- Can one determine the nuclear charge radius for other elements?

$2^3S - 2^3P$ transition in ^{4}He in MHz

	$(m/M)^0$	$(m/M)^1$	$(m/M)^2$	Sum
α^2	−276 775 637.536	102 903.459	−4.781	−276 672 738.857
α^4	−69 066.189	−6.769	−0.003	−69 072.961
α^5	5 234.163	−0.186	—	5 233.978
α^6	87.067	−0.029	—	87.039
α^7	−8.0 (1.0)	—	—	−8.0(1.0)
FNS	3.427	—	—	3.427
NPOL	−0.002	—	—	−0.002
Theory				−276 736 495.41 (1.00)
Exp.	[Florence.2004]			−276 736 495.649 5 (21)
Exp.	[Zheng.2017]			−276 736 495.600 0 (14)

The nuclear finite size effect

- $\delta_{\rm fs} E = \frac{2\,\pi\,Z\,\alpha}{3}\,\phi^2(0)\,\langle r_{ch}^2\rangle = C\,\langle r_{ch}^2\rangle$
- this formula is universal, valid for an arbitrary atomic system
- higher order $O(Z\,\alpha\,r_{\rm ch}/\lambda\!\!\!^{-})$, small for electronic atoms
- determination of mean square nuclear charge radius from:

$$\langle r_{ch}^2\rangle = \frac{E_{\rm exp} - E_{\rm the}}{C}$$

α charge radius from He $2^3S - 2^3P$

- $E(2^3S - 2^3P, {}^4\mathrm{He})_{\mathrm{centroid}} = 276\,736\,495\,600.0(1.4)$ kHz, Zheng, 2017
- finite size effect: $E_{\mathrm{fs}} = 3\,427$ kHz
- since E_{fs} is proportional to r^2

$$\frac{\Delta r}{r} = \frac{1}{2}\,\frac{\delta E_{\mathrm{fs}}}{E_{\mathrm{fs}}} \approx \frac{1}{2}\,\frac{10}{3\,427} = 1.5 \cdot 10^{-3}$$

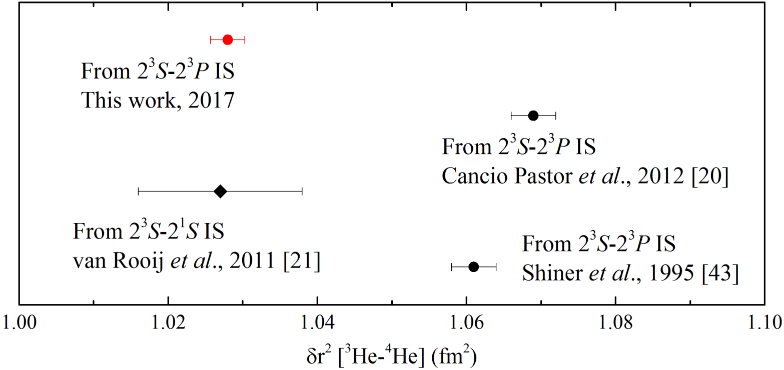
- electron scattering gives $r_{\mathrm{He}} = 1.681(4)$ fm, what corresponds to about $2.5 \cdot 10^{-3}$ relative accuracy
- ~ 10 kHz accuracy requires calculation of $m\,\alpha^7$ correction

^{3}He - ^{4}He isotope shift of $2^3S - 2^3P$ in kHz

$E(^3\mathrm{He}, 2^3P - 2S)$ (centroid)	$276\,702\,827\,204.8\,(2.4)$
$-E(^4\mathrm{He}, 2^3P - 2S)$ (centroid)	$-276\,736\,495\,600.0\,(1.4)$
$-\delta E_{\mathrm{iso}}(2^3P - 2^3S)$ (point nucleus)	$33\,667\,149.3\,(0.9)$
δE	$-1\,245.9\,(2.9)$
C	$-1212.2\,(1)\ \mathrm{kHz/fm}^2$
δR^2	$1.028\,(2)\ \mathrm{fm}^2$

^{3}He - ^{4}He charge radii difference

$$\begin{aligned}
\delta r^2(\mathrm{Inguscio\ 2012}, 2^3P - 2^3S) &= 1.069\,(3)\ \mathrm{fm}^2\,,\\
\delta r^2(\mathrm{Shiner\ 1995}, 2^3P - 2^3S) &= 1.061\,(3)\ \mathrm{fm}^2\,,\\
\delta r^2(\mathrm{Vassen\ 2011}, 2^1S - 2^3S) &= 1.027\,(11)\ \mathrm{fm}^2\\
\delta r^2(\mathrm{Zheng\ 2017}, 2^3P - 2^3S) &= 1.028\,(2)\ \mathrm{fm}^2
\end{aligned}$$


From 2³S-2³P IS
This work, 2017
From 2³S-2³P IS
Cancio Pastor et al., 2012 [20]
From 2³S-2¹S IS
van Rooij et al., 2011 [21]
From 2³S-2³P IS
Shiner et al., 1995 [43]
1.00
1.02
1.04
1.06
1.08
1.10
δr² [³He-⁴He] (fm²)

Three-photon exchange

- significant nuclear structure effects in muonic atoms $\sim \alpha$, $\sim \alpha^2$
- $\sim \alpha$ aka two-photon exchange is a subject of many works
- $\sim \alpha^2$ aka three-photon exchange, almost unknown
- Friar (1979): the elastic correction
- Friar (1977): the Coulomb distortion correction to the nuclear polarizability

Elastic three-photon exchange

$$E^{(6)}_{\rm fns}(nS) = -(Z\alpha)^6 m^3 r_C^2 \frac{2}{3n^3}\left[\frac{9}{4n^2} - 3 - \frac{1}{n} + 2\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m\, r_{C2}\, Z\alpha)\right]$$
$$+ (Z\alpha)^6 m^5 r_C^4 \frac{4}{9n^3}\left[-\frac{1}{n} + 2 + 2\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m\, r_{C1}\, Z\alpha)\right]$$
$$+ (Z\alpha)^6 m^5 r_{CC}^4 \frac{1}{15n^5},$$

$$E^{(6)}_{\rm fns}(nP_{1/2}) = (Z\alpha)^6 m\left(\frac{m^2 r_C^2}{6} + \frac{m^4 r_{CC}^4}{45}\right)\frac{1}{n^3}\left(1 - \frac{1}{n^2}\right),$$
$$E^{(6)}_{\rm fns}(nP_{3/2}) = (Z\alpha)^6 m^5 r_{CC}^4 \frac{1}{45n^3}\left(1 - \frac{1}{n^2}\right),$$
$$E^{(6)}_{\rm fns}(nL_J) = 0 \text{ for } L > 1,$$

Inelastic three-photon exchange $\sim r^2$

$$E_2^{(6)}(nS) = E_C(nS) - \alpha^6\, m^3\, \frac{2}{3\,n^3} \left[r_d^2 \left(\frac{9}{4n^2} - 3 - \frac{1}{n} + 2\,\gamma - \ln\frac{n}{2} + \Psi(n) + \ln\alpha \right) + r_s^2 \left(\frac{47}{12} - \frac{13}{2}\ln 2 - \frac{5}{2}\ln\frac{\langle E\rangle_2}{m} - \gamma \right) + r_p^2\, \ln(r_{p2}\, m) \right].$$

where

$$\ln\frac{\langle E\rangle_2}{m} = \frac{1}{r_s^2} \left\langle \phi \middle| \vec{R}\, \ln\frac{(H_N - E_N)}{m}\, \vec{R} \middle| \phi \right\rangle.$$

Inelastic three-photon exchange $\sim r^4$

$$
\begin{aligned}
E_1^{(6)}(nS) \;=\; & \alpha^6\, m^5\, \frac{4}{9\, n^3} \left[r_d^4 \left(-\frac{1}{n} + \gamma - \ln\frac{n}{2} + \Psi(n) + \ln\alpha \right) + r_{dd}^4\, \frac{3}{20\, n^2} \right. \\
& - r_{ss}^4 \left(\frac{3}{5}\ln 2 - \frac{9}{10} \right) + r_s^4 \left(\frac{1}{3} - \frac{3}{2}\ln 2 - \frac{1}{2}\ln\frac{\langle E\rangle_1}{m} + 2\,\beta \right) \\
& \left. + r_p^4 \Big(2 + \gamma + \ln(r_{p1}\, m) \Big) + r_s^2\, r_p^2 \left(\frac{10}{3} - 5\ln 2 - \ln\frac{\langle E\rangle_2}{m} \right) \right].
\end{aligned}
$$

where

$$
\begin{aligned}
\ln\frac{\langle E\rangle_1}{m} \;=\; & -\frac{1}{\langle R^2\rangle^2} \left[\langle 0|R^2 \ln\frac{(H-E)'}{m} R^2|0\rangle - \frac{6}{5}\langle 0|R^i \ln\frac{(H-E)'}{m} R^2\, R^i|0\rangle \right. \\
& \left. + \frac{3}{10}\langle 0|(R^i\, R^j - \delta^{ij}\, R^2/3) \ln\frac{(H-E)'}{m} (R^i\, R^j - \delta^{ij}\, R^2/3)|0\rangle \right].
\end{aligned}
$$

Extended polarizability β

$$\beta = -\sum_{\Lambda_1,\Lambda_2} \Big[\langle 0|R^i\, R^j + 3\,\delta^{ij}\, R^2|\Lambda_1\rangle\, \langle\Lambda_1|R^i|\Lambda_2\rangle\, \langle\Lambda_2|R^j|0\rangle + \langle 0|R^i|\Lambda_1\rangle\, \langle\Lambda_1|\delta^{ij}\, R^2 - 3\, R^i\, R^j|\Lambda_2\rangle\, \langle\Lambda_2|R^j|0\rangle + \langle 0|R^i|\Lambda_1\rangle\, \langle\Lambda_1|R^j|\Lambda_2\rangle\, \langle\Lambda_2|\delta^{ij}\, R^2 - 3\, R^i\, R^j|0\rangle \Big] \frac{3}{10\,\langle R^2\rangle^2}\, f\!\left(\frac{\Lambda_1}{\Lambda_2}\right)$$

with

$$f(x) = x\,\ln\!\left(1+\frac{1}{\sqrt{x}}\right) - \sqrt{x} - \ln(1+\sqrt{x})\,.$$

Results

The nuclear charge radius difference determined from the measurement of the H-D isotope shift of the $1S$–$2S$ transition,

$$\delta r^2[\text{electronic}] \equiv r_d^2 - r_p^2 = 3.820\,70\,(31)\ \text{fm}^2\,,$$

which agrees with but is twice as accurate as the previous value of 3.820 07 (65) fm^2.

For muonic hydrogen and deuterium, our result for the inelastic three-photon exchange nuclear structure contribution to the $2P_{1/2}$–$2S$ transition energy of 0.008 75 (88) meV shifts the deuteron-proton charge radius difference by 0.0014 fm^2, with the result

$$\delta r^2[\text{muonic}] \equiv r_d^2 - r_p^2 = 3.8126\,(34)\ \text{fm}^2\,.$$

The results derived from the electronic and muonic atoms disagree by about $2\,\sigma$

Results for the nuclear structure three photon-exchange

transition	units	Elastic	Inelastic	Sum	Elastic by others
$E^{(6)}(2S-1S,\ e\mathrm{H})$	Hz	-584	$-344\,(344)$	$-928\,(344)$	$-587\,(2)^{a}$
$E^{(6)}(2S-1S,\ e\mathrm{D}-e\mathrm{H})$	Hz	$-2\,846$	$817\,(41)$	$-2\,029\,(41)$	$-2\,834\,(13)^{a}$
$E^{(6)}(2P_{1/2}-2S,\ \mu\mathrm{H})$	meV	$-0.001\,27$	$\pm 0.000\,27$	$-0.001\,27\,(27)$	$-0.001\,34^{b}$
$E^{(6)}(2P_{1/2}-2S,\ \mu\mathrm{D})$	meV	$-0.006\,56$	$0.008\,75\,(88)(27)^{\dagger}$	$0.002\,19\,(88)(27)^{\dagger}$	$-0.006\,50\,(60)^{c}$
$E^{(6)}(2P_{1/2}-2S,\ \mu^{3}\mathrm{He}^{+})$	meV	$-0.384\,7$	unknown		$-0.378\,6\,(60)^{d}$
$E^{(6)}(2P_{1/2}-2S,\ \mu^{4}\mathrm{He}^{+})$	meV	$-0.304\,8$	unknown		$-0.311\,5\,(140)^{e}$

[a] CODATA (2014)

[b] Antognini (2013) the difference of entries “Our choice” and “Non-rel. finite-size” in Table 2 of that work, $-0.0019\ r_p^2$.

[c] Krauth (2016), the sum of entries r_3 and r_3' in Table 2 of that work, $-0.002\,124\,(4)\ r_d^2 + 0.003\,10\,(60)\ \mathrm{meV}$.

[d] Franke (2017), the sum of entries r_3 and r_3' in Table 2 of that work, $-0.1288\,(13)\ r_h^2 + 0.1177\,(33)\ \mathrm{meV}$.

[e] Diepold (2018), the sum of entries r_3 and r_3' in Table 4 of that work, $-0.1340\,(30)\ r_\alpha^2 + 0.0672\,(112)$ meV.

† the second uncertainty comes from the interaction with individual nucleons and cancels in the μD-μH isotope shift.

Collaborators on this project

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- V. Patkóš, Charles University, Praga
- M. Kalinowski, student, University of Warsaw