

Atomic physics from Effective field theories

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More modest: Hydrogen and muonic hydrogen

With an hadronic bias...

We want to study the strong interactions using light at very low energies:

$q^2 \rightarrow 0$.

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \dots$$

Observables: $ep \rightarrow ep$, $\mu p \rightarrow \mu p$, $\gamma p \rightarrow \gamma p$, atomic physics, muonic atoms,

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Scales (and ratios)

$$m_p \sim \Lambda_\chi$$

$$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p + m_\mu}$$

$$m_r \alpha \sim m_e$$

...

$$Q^2 \rightarrow 0$$

Tool: Effective Field Theories \equiv Factorization

Why?: There is a hierarchy of different scales

EFTs are especially useful in these situations.

- 1) Perturbative calculations much easier and systematic.
- 2) Nonperturbative information is parameterized in a model independent way.
- 3) Power counting.

Effective Field Theory: Non-relativistic protons, photons and (non-relativistic) electron/muons.

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Caswell-Lepage

$$iD_0 = i\partial_0 + Z_p e A^0, \quad i\mathbf{D} = i\nabla - Z_p e \mathbf{A}$$

$$\begin{aligned} \mathcal{L}_{\text{NRQED}} = & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} \\ & + \psi_p^\dagger \left\{ iD_0 + \frac{C_k}{2m_p} \mathbf{D}^2 + \frac{C_4}{8m_p^3} \mathbf{D}^4 + \frac{C_F^{(p)}}{2m_p} \boldsymbol{\sigma} \cdot e\mathbf{B} + \frac{C_D^{(p)}}{8m_p^2} (\mathbf{D} \cdot e\mathbf{E} - e\mathbf{E} \cdot \mathbf{D}) \right. \\ & \left. + i \frac{C_S^{(p)}}{8m_p^2} \boldsymbol{\sigma} \cdot (\mathbf{D} \times e\mathbf{E} - e\mathbf{E} \times \mathbf{D}) + C_{A_1}^{(p)} e^2 \frac{\mathbf{B}^2 - \mathbf{E}^2}{8m_p^3} - C_{A_2}^{(p)} e^2 \frac{\mathbf{E}^2}{8m_p^3} \right\} \psi_p \end{aligned}$$

+(leptons)

$$- \frac{C_3^{(pe)}}{m_p m_e} \psi_p^\dagger \psi_p \psi_e^\dagger \psi_e + \frac{C_4^{(pe)}}{m_p m_e} \psi_p^\dagger \boldsymbol{\sigma} \psi_p \psi_e^\dagger \boldsymbol{\sigma} \psi_e + \dots$$

Dictionary (relation Wilson coefficients with low energy constants):

$C_F^{(p)} \rightarrow \mu_p$ anomalous magnetic moment (low energy constant)

$C_D \rightarrow r_p$ proton radius (quasi low energy constant)

$C_{A_i}^{(p)} \rightarrow \alpha_E, \beta_M$ Proton polarizabilities (quasi low energy constant)

$C_{3/4}^{(pe)} \rightarrow$ Two-photon exchange ...

$Q^2 \rightarrow 0$ INVOLVES COULOMB RESUMMATION \rightarrow ATOMIC PHYSICS

$|\mathbf{q}| \sim m_\mu \alpha \sim m_e \sim 0.5 \text{ MeV}$ (muonic hydrogen)

$|\mathbf{q}| \sim m_e \alpha \sim 5 \cdot 10^{-3} \text{ MeV}$ (hydrogen)

New scales: $m_{\text{lepton}} \alpha$, $m_{\text{lepton}} \alpha^2$

Theoretical setup (muonic hydrogen)

We use an effective field theory, **Potential Non-Relativistic QED**, which describes the muonic hydrogen dynamics and profits from the hierarchy

$$m_\mu \gg m_\mu \alpha \gg m_\mu \alpha^2$$

$$\left. \begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r} \right) \psi(\mathbf{r}) = 0 \\ + \text{corrections to the potential} \\ + \text{interaction with ultrasoft photons} \end{array} \right\} \text{potential NRQED} \quad E \sim mv^2$$

$$NRQED(m_\mu \alpha) \rightarrow pNRQED$$

Theoretical setup (muonic hydrogen)

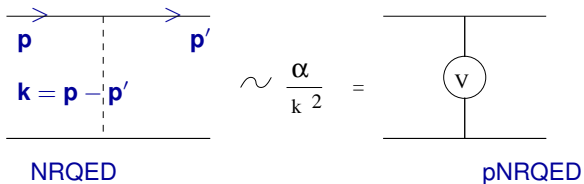
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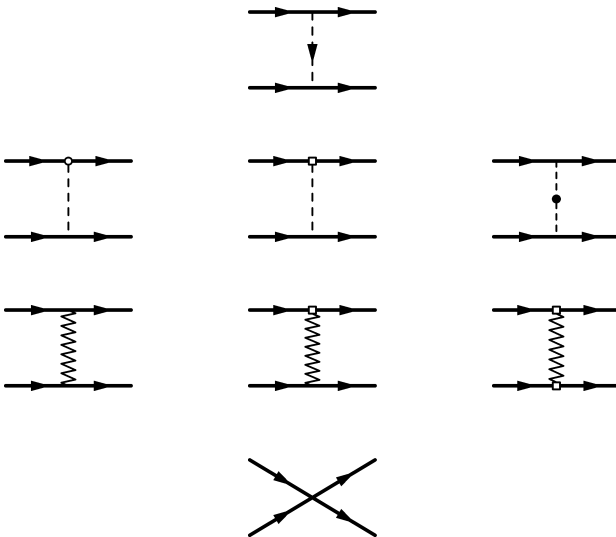
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Matching NRQED to pNRQED



Hydrogen/Positronium/muonium

Tree level



Order $1/m^2$

$$\tilde{V}^{(b)} = \frac{\pi\alpha}{2} \left[Z_p \frac{c_D^{(\mu)}}{m_\mu^2} + Z_\mu \frac{c_D^{(p)}}{m_p^2} \right],$$

$$\tilde{V}^{(c)} = -i2\pi\alpha \frac{(\mathbf{p} \times \mathbf{k})}{\mathbf{k}^2} \cdot \left\{ Z_p \frac{c_S^{(\mu)} \mathbf{s}_1}{m_\mu^2} + Z_\mu \frac{c_S^{(p)} \mathbf{s}_2}{m_p^2} \right\},$$

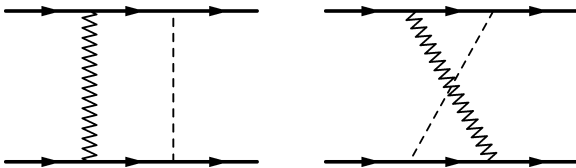
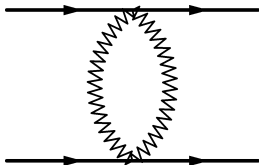
$$\tilde{V}^{(d)} = -Z_\mu Z_p 16\pi\alpha \left(\frac{d_2^{(\mu)}}{m_\mu^2} + \frac{d_2^{(\tau)}}{m_\tau^2} + \frac{d_{2,NR}}{m_p^2} \right),$$

$$\tilde{V}^{(e)} = -Z_\mu Z_p \frac{4\pi\alpha}{m_\mu m_p} \left(\frac{\mathbf{p}^2}{\mathbf{k}^2} - \frac{(\mathbf{p} \cdot \mathbf{k})^2}{\mathbf{k}^4} \right),$$

$$\tilde{V}^{(f)} = -\frac{i4\pi\alpha}{m_\mu m_p} \frac{(\mathbf{p} \times \mathbf{k})}{\mathbf{k}^2} \cdot (Z_p c_F^{(\mu)} \mathbf{s}_1 + Z_\mu c_F^{(p)} \mathbf{s}_2),$$

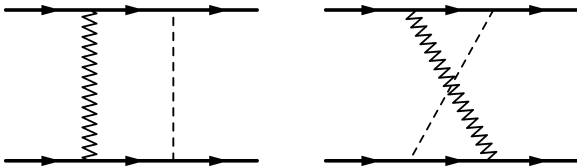
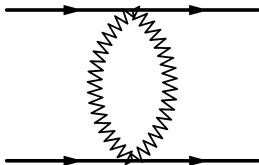
$$\tilde{V}^{(g)} = \frac{4\pi\alpha c_F^{(1)} c_F^{(2)}}{m_\mu m_p} \left(\mathbf{s}_1 \cdot \mathbf{s}_2 - \frac{\mathbf{s}_1 \cdot \mathbf{k} \mathbf{s}_2 \cdot \mathbf{k}}{\mathbf{k}^2} \right),$$

$$\tilde{V}^{(h)} = -\frac{1}{m_p^2} \left\{ (c_3^{pl_i} + 3c_4^{pl_i}) - 2c_4^{pl_i} \mathbf{S}^2 \right\}.$$



$$\tilde{V}_{1loop}^{(a)} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p} \left(\log \frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3} \log 2 + \frac{5}{3} \right),$$

$$\tilde{V}_{1loop}^{(b,c)} = \frac{4Z_\mu^2 Z_p^2 \alpha^2}{3m_\mu m_p} \left(\log \frac{\mathbf{k}^2}{\mu^2} + 2 \log 2 - 1 \right).$$



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Muonic Hydrogen: electron vacuum polarization

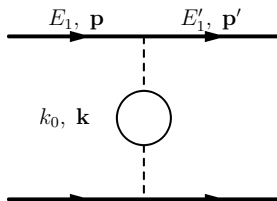


Figure : *Leading correction to the Coulomb potential due to the electron vacuum polarization. $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ and $k_0 = E_1 - E_1'$.*

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_p \alpha_V(k) \frac{1}{\mathbf{k}^2},$$

$$\alpha_{\text{eff}}(k) = \alpha \frac{1}{1 + \Pi(-\mathbf{k}^2)},$$

where

$$\Pi(k^2) = \alpha \Pi^{(1)}(k^2) + \alpha^2 \Pi^{(2)}(k^2) + \alpha^3 \Pi^{(3)}(k^2) + \dots$$

$$\alpha_V(k) = \alpha_{\text{eff}}(k) + \sum_{\substack{n, m=0 \\ n+m=\text{even}>0}} Z_\mu^n Z_p^m \alpha_{\text{eff}}^{(n, m)}(k) = \alpha_{\text{eff}}(k) + \delta\alpha(k), \quad \delta\alpha(k) = O(\alpha^4)$$

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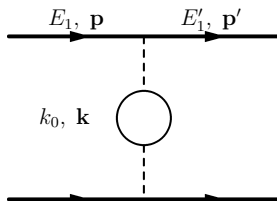


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Order $1/m^2$

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Order $1/m^2$ from energy-dependent terms

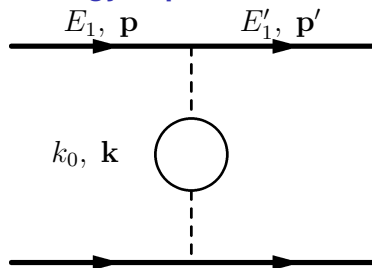


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$$\delta \tilde{V}_E = -\frac{Z_\mu Z_p e^2}{4m_\mu m_p} \frac{(\mathbf{p}^2 - \mathbf{p}'^2)^2}{\mathbf{k}^2} \frac{\alpha}{\pi} m_e^2 \int_4^\infty d(q^2) \frac{1}{(m_e^2 q^2 + \mathbf{k}^2)^2} u(q^2).$$

$$u(q^2) = \frac{1}{3} \sqrt{1 - \frac{4}{q^2}} \left(1 + \frac{2}{q^2}\right).$$

Theoretical setup.

Muonic hydrogen Lamb shift: $\Delta E_L \equiv E(2P_{3/2}) - E(2S_{1/2})$ and

hyperfine splitting: $\Delta E_{HF} \equiv E(nS_{3/2}) - E(nS_{1/2})$

$$L_{pNRQED} = \int d^3\mathbf{r} d^3\mathbf{R} dt S^\dagger(\mathbf{r}, \mathbf{R}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} \right. \\ \left. - V(\mathbf{r}, \mathbf{p}, \sigma_1, \sigma_2) + e\mathbf{r} \cdot \mathbf{E}(\mathbf{R}, t) \right\} S(\mathbf{r}, \mathbf{R}, t) - \int d^3\mathbf{r} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

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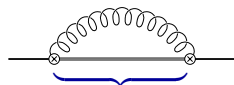
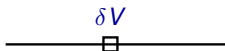
Observable: Spectrum or decays

Corrections to the Green Function ($h_s^{(0)} = \mathbf{p}^2/m + V^{(0)}$)

$$G_s(E) = P_s \frac{1}{h_s^{(0)} - H_I - E} P_s = G_s^{(0)} + \delta G_s \quad G_s^{(0)}(E) = \frac{1}{h_s^{(0)} - E}$$

A) Ultrasoft loops (lamb shift-like): $\mathbf{x} \cdot \mathbf{E} \leftarrow$

B) Quantum mechanics perturbation theory \leftarrow



$$1/(E - V^{(0)} - \mathbf{p}^2/m)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^3)$

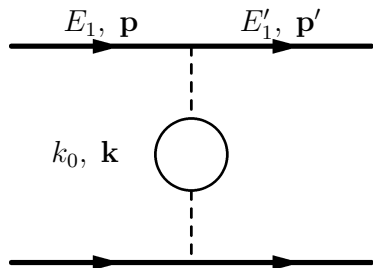


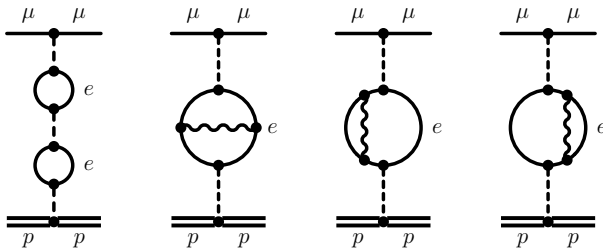
Figure : *Leading correction to the Coulomb potential due to the electron vacuum polarization. $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ and $k_0 = E_1 - E_1'$.*

1-loop static potential

$$E_{LO} = \langle n | \delta V | n \rangle = 205.0074 \text{ meV} = \mathcal{O}(m_r\alpha^3)$$

$E_L \propto \beta_0$. Measure of (Non)-Asymptotically free theory!

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



Pachuki/Borie

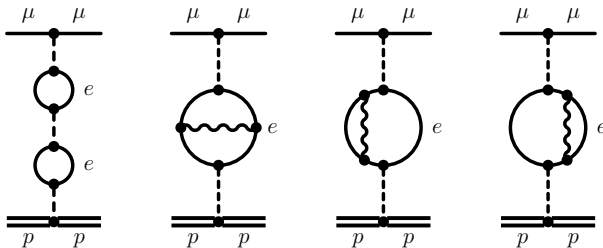
2-loop static potential is the same as two-loop vacuum polarization iterations
(*two loop vacuum polarization*)

$$\delta E = \langle n | \delta V | n \rangle = 1.5079 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Quantum mechanics perturbation theory (*iteration one-loop*)

$$\delta E \sim \langle n | \delta V \frac{1}{H_C - E_n} \delta V | n \rangle = 0.151 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



Pachuki/Borie

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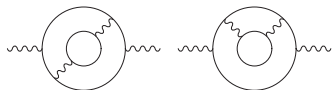
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Vacuum polarization effects: $\mathcal{O}(m_r\alpha^5)$

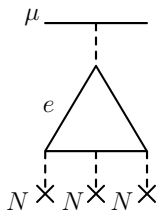
3-loop static potential (three loop vacuum polarization, Kinoshita-Nio)

$$0.00752 \text{ meV} = \mathcal{O}(m_r\alpha^5)$$

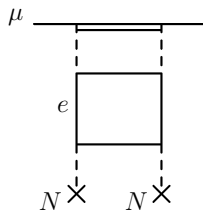
Slightly corrected by Ivanov et al.



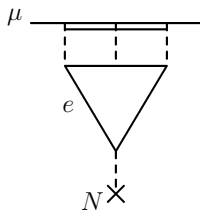
Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



(1:3)



(2:2)



(3:1)

Light-by-light (Wichmann-Kroll and Delbrück) contribution very small
(Karshenboim et al.)

$$\Delta E \simeq -0.0009 \text{ meV} = \mathcal{O}(m_r\alpha^5)$$

Earlier work by Borie

$1/m$ potential

$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(1)}(r)}{m_\mu} \rightarrow \mathcal{O}(m_r \alpha^6)$$

relativistic corrections+vacuum polarization

$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \rightarrow \mathcal{O}(m_r \alpha^4, \alpha^5)$$

Lamb shift

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

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$\mathcal{O}(m\alpha^5)$ 0.0169 (Pachucki and Veitia; Borie)

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Lamb shift

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.018759 (Jentschura; Karshenboim&Ivanov&Korzinin; Peset&AP)

relativistic corrections+vacuum polarization

$$V(\mathbf{x}, \mathbf{p}, \sigma_1, \sigma_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \rightarrow \mathcal{O}(m_r \alpha^4, \alpha^5)$$

Hyperfine

$$\frac{m}{m_p} \times \mathcal{O}(m \alpha^4)$$

$$\frac{m}{m_p} \times \mathcal{O}(m \alpha^5) \text{ Martynenko; Indelicato; Pachucki; Borie; Peset-Pineda; ...}$$

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relativistic corrections+vacuum polarization

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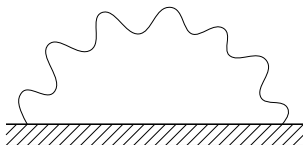
$$\frac{V^{(2)}(r)}{m_\mu^2} \rightarrow \mathcal{O}(m_r \alpha^4, \alpha^5)$$

Hyperfine

$$\frac{m}{m_p} \times \mathcal{O}(m\alpha^4)$$

$$\frac{m}{m_p} \times \mathcal{O}(m\alpha^5) \text{ Martynenko; Indelicato; Pachucki; Borie; Peset-Pineda; ...}$$

Ultrasoft effects: $\mathcal{O}(m\alpha^5)$



$$\Delta E = -0.6677 \text{ meV}$$

$$\mathcal{O}(m\alpha^5 \frac{m_\mu}{m_p}) : \quad \Delta E = -0.045 \text{ meV}$$

All (soft+ultrasoft):

$$\Delta E = -0.71896 \text{ meV.}$$

Start the overlap with hadronic effects.

Hadronic corrections

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.} \delta^3(\mathbf{r}) \rightarrow \Delta E \sim \frac{1}{m_p^2} D_d^{had.} (m_r \alpha)^3$$

$$D_d^{(p\mu)} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D^{(p)}$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_s^{had.} (\mathbf{S}_1 + \mathbf{S}_2)^2 \delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

$c_3, c_4, d_2, c_D^{(p)}, \dots$ matching coefficients of NRQED.

$$\delta \mathcal{L} = \dots + \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} - e \frac{c_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p - \frac{c_3}{m_p^2} N_p^\dagger N_{p\mu} \mu + \frac{c_4}{m_p^2} N_p^\dagger \sigma N_{p\mu} \sigma \mu$$

Lamb Shift

	$\mathcal{O}(m_r \alpha^3)$	$V_{VP}^{(0)}$	205. 00737
	$\mathcal{O}(m_r \alpha^4)$	$V_{VP}^{(0)}$	1. 50795
	$\mathcal{O}(m_r \alpha^4)$	$V_{VP}^{(0)}$	0. 15090
	$\mathcal{O}(m_r \alpha^5)$	$V_{VP}^{(0)}$	0. 00752
	$\mathcal{O}(m_r \alpha^5)$	$V_{LbL}^{(0)}$	-0. 00089(2)
	$\mathcal{O}(m_r \alpha^4 \times \frac{m_\mu^2}{m_p^2})$	$V^{(2,1)} + V^{(3,0)}$	0. 05747
	$\mathcal{O}(m_r \alpha^5)$	$V_{VP}^{(2,2)} + V^{(2,1)} \times V_{VP}^{(0,2)}$	0. 01876
	$\mathcal{O}(m_r \alpha^5)$	$V_{no-VP}^{(2,2)} + \text{ultrasoft}$	-0. 71896
	$\mathcal{O}(m_r \alpha^6 \times \ln(\frac{m_\mu}{m_e}))$	$V^{(2,3)}; c_D^{(\mu)}$	-0. 00127
	$\mathcal{O}(m_r \alpha^6 \times \ln \alpha)$	$V_{VP}^{(2,3)}; c_D^{(\mu)}$	-0. 00454
	$\mathcal{O}(m_r \alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}; c_D^{(p)}$	-5. 1975 $\frac{r_p^2}{\text{fm}^2}$
	$\mathcal{O}(m_r \alpha^5 \times m_r^2 r_p^2)$	$V_{VP}^{(2,2)} + V^{(2,1)} \times V_{VP}^{(0,2)}; c_D^{(p)}$	-0. 0282 $\frac{r_p^2}{\text{fm}^2}$
	$\mathcal{O}(m_r \alpha^6 \ln \alpha \times m_r^2 r_p^2)$	$V^{(2,3)}; c_D^{(p)}$	-0. 0014 $\frac{r_p^2}{\text{fm}^2}$
	$\mathcal{O}(m_r \alpha^5 \times \frac{m_r^2}{m_p^2})$	$V_{VP\text{had}}^{(2)}; a_2^{\text{had}}$	0. 0111(2)
	$\mathcal{O}(m_r \alpha^5 \times \frac{m_r^2 m_\mu}{m_p^2 m_\pi})$	$\delta E^{\text{TPE}}; V^{(2)}; c_3^{\text{had}}$	0. 0344(125)

Hyperfine muonic hydrogen 1S

i)	$\mathcal{O}(m_r \alpha^4)$	δE_{Fermi}	182.65604
ii)	$\mathcal{O}(m_r \alpha^5)$	$V^{(2,1)} \times V_{\text{VP}}^{(0,2)}$	0.73449
iii)	$\mathcal{O}(m_r \alpha^5)$	$V_{\text{VP}}^{(2,2)}$	0.37465
iv)	$\mathcal{O}(m_r \alpha^5 \frac{m_\mu}{m_p})$	δE^{TPE}	-1.161(20)
v)	$\mathcal{O}(m_r \alpha^6)$	$V^{(2,1)} \times V^{(2,1)}$	0.01457
vi)	$\mathcal{O}(m_r \alpha^6)$	$V^{(2,3)}; \delta c_4^{(-1)}$	-0.01755
vii)	$\mathcal{O}(m_r \alpha^6)$	$V^{(2,1)} \times V_{\text{VP}}^{(0,3)}$	0.00556
viii)	$\mathcal{O}(m_r \alpha^6)$	$V_{\text{VP}}^{(2,3)}$	0.00292
ix)	$\mathcal{O}(m_r \alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,1)} \times V^{(2,1)}$	0.01752
x)	$\mathcal{O}(m_r \alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,2)} \times V_{\text{VP}}^{(0,2)}; c_{4,\text{TPE}}$	-0.00466
	Total sum		182.623(27)

Table : *The different contributions to the hyperfine splitting for the 1S in muonic hydrogen in meV units. The value for the iv) and x) entries is obtained from the hydrogen hyperfine measurement.*

Hyperfine muonic hydrogen 2S

i)	$\mathcal{O}(m_r \alpha^4)$	δE_{Fermi}	22.832005
ii)	$\mathcal{O}(m_r \alpha^5)$	$V^{(2,1)} \times V_{\text{VP}}^{(0,2)}$	0.074474
iii)	$\mathcal{O}(m_r \alpha^5)$	$V_{\text{VP}}^{(2,2)}$	0.048275
iv)	$\mathcal{O}(m_r \alpha^5 \frac{m_\mu}{m_p})$	δE^{TPE}	-0.1451(25)
v)	$\mathcal{O}(m_r \alpha^6)$	$V^{(2,1)} \times V^{(2,1)}$	0.002581
vi)	$\mathcal{O}(m_r \alpha^6)$	$V^{(2,3)}; \delta c_4^{(-1)}$	-0.002194
vii)	$\mathcal{O}(m_r \alpha^6)$	$V^{(2,1)} \times V_{\text{VP}}^{(0,3)}$	0.000375
viii)	$\mathcal{O}(m_r \alpha^6)$	$V_{\text{VP}}^{(2,3)}$	0.000563
ix)	$\mathcal{O}(m_r \alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,1)} \times V^{(2,1)}$	0.001846
x)	$\mathcal{O}(m_r \alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,2)} \times V_{\text{VP}}^{(0,2)}; c_{4,\text{TPE}}$	-0.000473
Total sum			22.8123(33)

Table : *The different contributions to the hyperfine splitting for the 2S in muonic hydrogen in meV units. The value for the iv) and x) entries is obtained from the hydrogen hyperfine measurement.*

HOW to determine the Two-Photon Exchange correction?

- ▶ Dispersion relations
- ▶ lattice (not yet)
- ▶ Chiral perturbation theory (\rightarrow (non-analytic) m_q dependence, N_c dependence)
- ▶

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \dots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\text{Tr}[D_\mu U D^\mu U] + \dots \quad U = u^2 = e^{i\frac{\mathbf{p}}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger (iv^\mu \nabla_\mu + g_A u_\mu S^\mu) N + \dots + (\Delta) + \dots - e\frac{C_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \quad \nabla_\mu = \partial_\mu + \Gamma_\mu \quad u_\mu = iu^\dagger (\nabla_\mu U) u$$

$$\Gamma_\mu = \frac{1}{2} \left\{ u^\dagger (\partial_\mu + ieQA_\mu) u + u (\partial_\mu + ieQA_\mu) u^\dagger \right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2} \sum_i c_{3,R}^{pl_i} \bar{N}_p \gamma^0 N_p \bar{l}_i \gamma^0 l_i + \frac{1}{m_p^2} \sum_i c_{4,R}^{pl_i} \bar{N}_p \gamma^j N_p \bar{l}_i \gamma_j l_i$$

$$\delta\mathcal{L} = \dots + \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} - e\frac{C_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p - \frac{C_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu + \frac{C_4}{m_p^2} N_p^\dagger \sigma N_p \mu^\dagger \sigma \mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

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$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\text{Tr}[D_\mu U D^\mu U] + \dots \quad U = u^2 = e^{i\frac{\mathbf{n}}{F_\pi}}$$

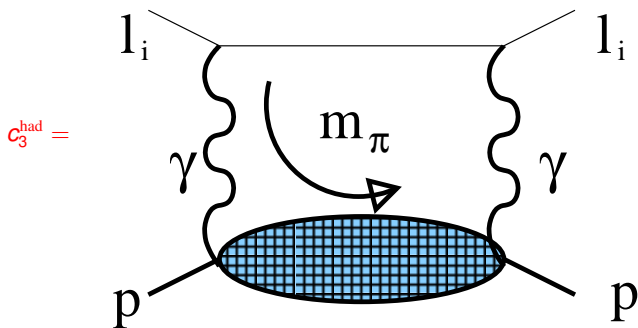
$$\mathcal{L}_N = N^\dagger (iv^\mu \nabla_\mu + g_A u_\mu S^\mu) N + \dots + (\Delta) + \dots - e\frac{C_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \quad \nabla_\mu = \partial_\mu + \Gamma_\mu \quad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2} \left\{ u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger \right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2} \sum_i c_{3,R}^{pl_i} \bar{N}_p \gamma^0 N_p \bar{l}_i \gamma^0 l_i + \frac{1}{m_p^2} \sum_i c_{4,R}^{pl_i} \bar{N}_p \gamma^j N_p \bar{l}_i \gamma_j l_i$$

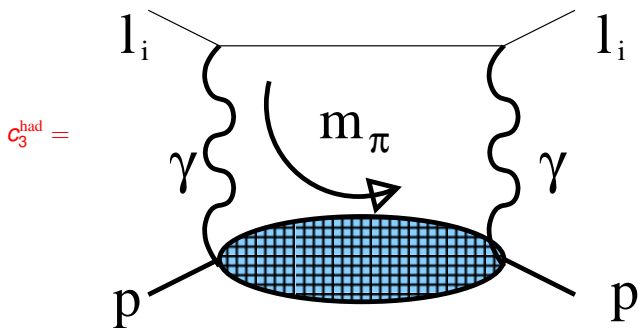
$$\delta\mathcal{L} = \dots + \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} - e\frac{C_D}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p - \frac{C_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu + \frac{C_4}{m_p^2} N_p^\dagger \sigma N_p \mu^\dagger \sigma \mu$$



$$T^{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle p, s | T J^\mu(x) J^\nu(0) | p, s \rangle,$$

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) + \frac{1}{m_p^2} \left(p^\mu - \frac{m_p \rho}{q^2} q^\mu \right) \left(p^\nu - \frac{m_p \rho}{q^2} q^\nu \right) S_2(\rho, q^2)$$

$$S_1 = ?? \quad S_2 = ??$$

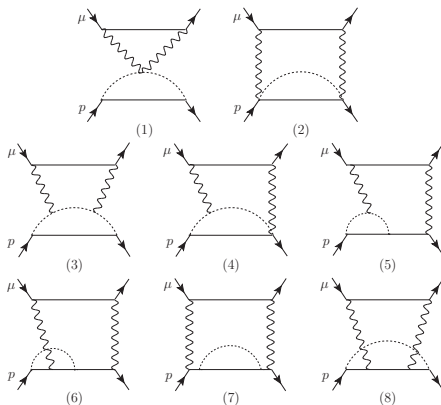


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$$S_1 = ?? \quad S_2 = ??$$

TWO-PHOTON EXCHANGE correction



m_μ extra suppression+ χ PT (Model independent)

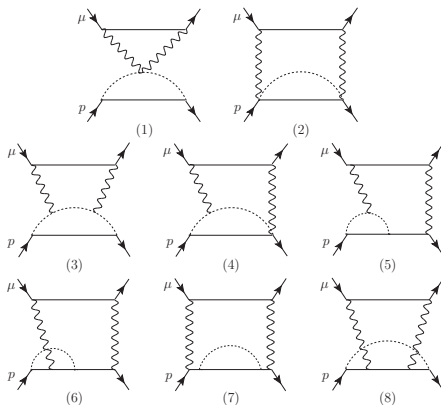
Power-like chiral enhanced ($\rightarrow \chi$ PT can predict the leading order!)

$$c_3^{\text{had}} \sim \alpha^2 \frac{m_\mu}{m_\pi} + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) \quad \delta E \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$$

Error ($\Delta = M_\Delta - M_p \sim 300$ MeV): $\text{LO} \times \frac{m_\pi}{\Delta} \simeq \text{LO} \times \frac{1}{2}$

$$\rightarrow c_3^{\text{had}} = \alpha^2 \frac{m_\mu}{m_\pi} 47.2(23.6)$$

TWO-PHOTON EXCHANGE correction



m_μ extra suppression + χ PT (Model independent)

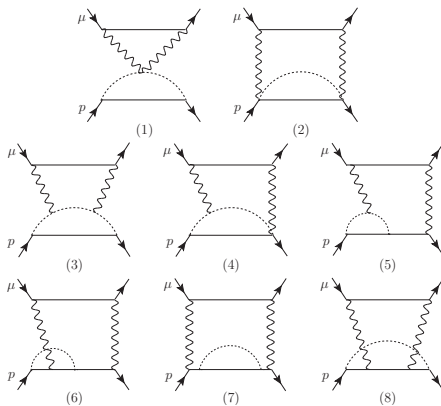
Power-like chiral enhanced ($\rightarrow \chi$ PT can **predict** the leading order!)

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TWO-PHOTON EXCHANGE correction



m_μ extra suppression+ χ PT (Model independent)

Power-like chiral enhanced ($\rightarrow \chi$ PT can **predict** the leading order!)

$$c_3^{\text{had}} \sim \alpha^2 \frac{m_\mu}{m_\pi} + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) \quad \delta E \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi})$$

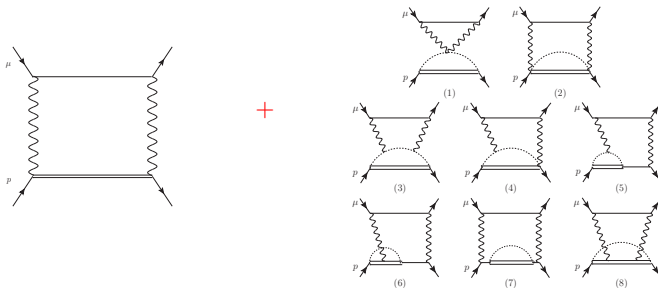
Error ($\Delta = M_\Delta - M_p \sim 300$ MeV): $\text{LO} \times \frac{m_\pi}{\Delta} \simeq \text{LO} \times \frac{1}{2}$

$$\rightarrow c_3^{\text{had}} = \alpha^2 \frac{m_\mu}{m_\pi} 47.2(23.6)$$

Large N_c . Including the Δ particle

Error:

$$\frac{m_\mu}{\Delta} \sim N_c \frac{m_\mu}{\Lambda_{QCD}} \rightarrow N_c \frac{m_\mu}{\Lambda_{QCD}} \sim \frac{1}{3}$$



$$c_3^{\text{had}} \sim \alpha^2 \frac{m_\mu}{m_\pi} \left[1 + \# \frac{m_\pi}{\Delta} + \dots \right] + \mathcal{O} \left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}} \right) = \alpha^2 \frac{m_\mu}{m_\pi} \begin{cases} 47.2(23.6) & (\pi), \\ 56.7(20.6) & (\pi + \Delta), \end{cases}$$

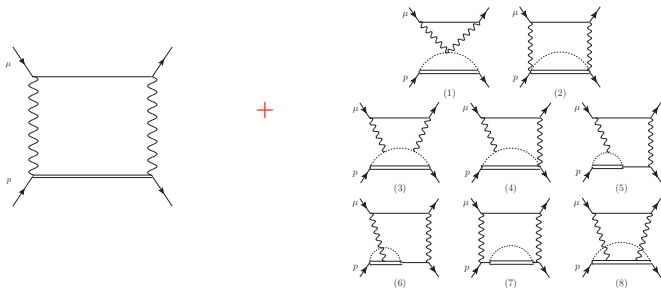
$$\Delta E_{\text{TPE}} = 28.59(\pi) + 5.86(\pi \& \Delta) = 34.4(12.5) \mu\text{eV} \quad (\text{Peset\&AP}).$$

(Model dependent+DR: $\Delta E_{\text{TPE}} = 33(2) \mu\text{eV}$ (Birse-McGovern))

Large N_c . Including the Δ particle

Error:

$$\frac{m_\mu}{\Delta} \sim N_c \frac{m_\mu}{\Lambda_{QCD}} \rightarrow N_c \frac{m_\mu}{\Lambda_{QCD}} \sim \frac{1}{3}$$



$$c_3^{\text{had}} \sim \alpha^2 \frac{m_\mu}{m_\pi} \left[1 + \# \frac{m_\pi}{\Delta} + \dots \right] + \mathcal{O} \left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}} \right) = \alpha^2 \frac{m_\mu}{m_\pi} \begin{cases} 47.2(23.6) & (\pi), \\ 56.7(20.6) & (\pi + \Delta), \end{cases}$$

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(Model dependent+DR: $\Delta E_{\text{TPE}} = 33(2) \mu\text{eV}$ (Birse-McGovern))

$$\Delta E_{\text{TPE}} \sim m_\mu \alpha^5 \times \frac{m_\mu^2}{(4\pi F_\pi)^2} \times \frac{m_\mu}{m_\pi} \sum_{n=0}^{\infty} c_n (N_c \sqrt{m_q})^n$$

$$\frac{\#}{\sqrt{m_q}} + ? + ? \sqrt{m_q} + \dots$$

plus large N_c

$$\frac{\#}{\sqrt{m_q}} + \left[\# N_c + ? + \frac{?}{N_c} + \dots \right] + \left[\# N_c^2 + ? N_c + ? + \dots \right] \sqrt{m_q} + \dots$$

? → Size of the counterterm in HBET

μeV	DR	<i>Pachucki</i>	<i>Carlson et al</i>	HBET(π)	(π & Δ)
ΔE_{Born}		23.2(1.0)	24.7(1.6)	10.1(5.1)	8.3(4.3)

Table : Predictions for the Born contribution to the $n = 2$ Lamb shift. The first two entries correspond to dispersion relations. The last two entries are the predictions of HBET (AP and Peset&AP).

$$\Delta E_{\text{Born}} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3} = 0.010 \frac{48}{\pi} \int \frac{d^3 k}{4\pi} \frac{1}{\mathbf{k}^6} \left(G_E^2 - 1 + \frac{1}{3} \langle r^2 \rangle \mathbf{k}^2 \right)$$

(μeV)	[1]	[2]	[3]	[4]	$B_{\chi\text{PT}}(\pi)$	HBET(π)	(π & Δ)
ΔE_{pol}	12(2)	11.5	7.4(2.4)	15.3(5.6)	$8.2^{(+1.2)}_{(-2.5)}$	18.5(9.3)	26.2(10.0)

Table : Polarizability contribution to the $n = 2$ Lamb shift. The first four entries use dispersion relations for the inelastic term and different modeling functions for the subtraction term. [1] Pachucki, [2] Martynenko, [3] Carlson&Vanderhaeghen, [4] Gorchtein et al.. The 5th entry is the prediction obtained using $B_{\chi\text{PT}}$ (Alarcon et al.). The last two entries are the predictions of HBET (Nevado&AP and Peset&AP).

Polarizability=Inelastic+subtraction

$$c_{3,\text{sub}}^{pl_i} = -e^4 M_\rho m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} (3k_{0,E}^2 + \mathbf{k}^2) S_1(0, -k_E^2)$$

$$c_{3,\text{inel}}^{pl_i} = -e^4 M_\rho m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \\ \times \left\{ (3k_{0,E}^2 + \mathbf{k}^2) (S_1(ik_{0,E}, -k_E^2) - S_1(0, -k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\}$$

$$\Delta E^{(\text{sub})}(\pi\text{-loop}) = -1.62 \mu\text{eV} ; \quad \Delta E^{(\text{sub})}(\pi\Delta\text{-loop}) = -1.23 \mu\text{eV}.$$

$$\delta c_{3,\text{sub}}^{pl_i} \sim -\delta c_{3,\text{inel}}^{pl_i} \simeq -\frac{4}{3} \alpha^2 \frac{m_{l_i}}{\Delta} b_{1,F}^2 \ln(\nu/m_{l_i}) \rightarrow \Delta E^{(\text{sub})}|_{\nu=m_\rho} \sim -11.4 \mu\text{eV}$$

Also seen in Relativistic χ PT (Alarcon et al)

Polarizability=Inelastic+subtraction

$$\mathcal{C}_{3,\text{sub}}^{pl_i} = -e^4 M_\rho m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} (3k_{0,E}^2 + \mathbf{k}^2) S_1(0, -k_E^2)$$

$$\begin{aligned} \mathcal{C}_{3,\text{inel}}^{pl_i} &= -e^4 M_\rho m_{l_i} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \\ &\times \left\{ (3k_{0,E}^2 + \mathbf{k}^2) (S_1(ik_{0,E}, -k_E^2) - S_1(0, -k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\} \end{aligned}$$

$$\Delta E^{(\text{sub})}(\pi\text{-loop}) = -1.62 \mu\text{eV}; \quad \Delta E^{(\text{sub})}(\pi\Delta\text{-loop}) = -1.23 \mu\text{eV}.$$

$$\delta\mathcal{C}_{3,\text{sub}}^{pl_i} \sim -\delta\mathcal{C}_{3,\text{inel}}^{pl_i} \simeq -\frac{4}{3}\alpha^2 \frac{m_{l_i}}{\Delta} b_{1,F}^2 \ln(\nu/m_{l_i}) \rightarrow \Delta E^{(\text{sub})}|_{\nu=m_\rho} \sim -11.4 \mu\text{eV}$$

Also seen in Relativistic χ PT (Alarcon et al)

Hyperfine: Hydrogen and muonic hydrogen

Experiment:

$$E_{\text{hyd, HF}}^{\text{exp}}(1S) = 1420.405751768(1) \text{ MHz},$$

$$E_{\mu p, \text{HF}}^{\text{exp}}(2S) = 22.8089(51) \text{ meV}.$$

Theory:

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{\text{had.}} (\mathbf{S}_1 + \mathbf{S}_2)^2 \delta^3(\mathbf{r})$$

$$D_s^{\text{had.}} = 2c_4$$

c_4 , matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \dots - \frac{C_4}{m_p^2} N_p^\dagger \sigma N_p \mu^\dagger \sigma \mu$$

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c_4 , Spin-dependent effects

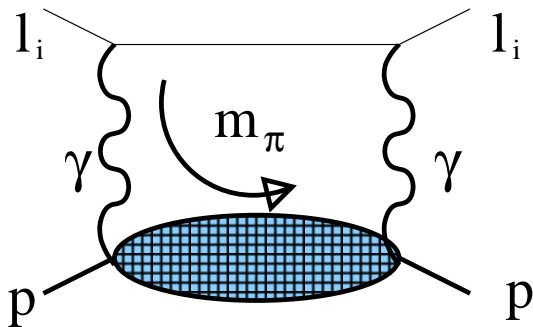


Figure : Symbolic representation (plus permutations) of the spin-dependent correction.

$$c_4^{pl} = -\frac{ig^4}{3} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{1}{k^4 - 4m_l^2 k_0^2} \left\{ A_1(k_0, k^2)(k_0^2 + 2k^2) + 3k^2 \frac{k_0}{m_p} A_2(k_0, k^2) \right\}$$

Drell-Sullivan(67)

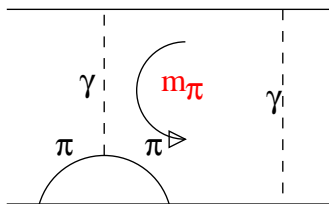
$$T^{\mu\nu} = i \int d^4 x e^{iq \cdot x} \langle p, s | T J^\mu(x) J^\nu(0) | p, s \rangle,$$

which has the following structure ($\rho = q \cdot p/m$):

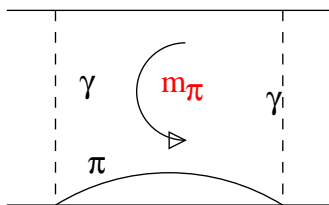
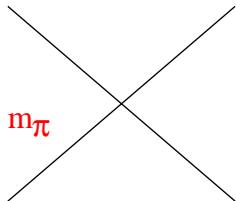
$$\begin{aligned} T^{\mu\nu} &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) \\ &+ \frac{1}{m_p^2} \left(p^\mu - \frac{m_{p\rho}}{q^2} q^\mu \right) \left(p^\nu - \frac{m_{p\rho}}{q^2} q^\nu \right) S_2(\rho, q^2) \\ &- \frac{i}{m_p} \epsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\rho, q^2) \\ &- \frac{i}{m_p^3} \epsilon^{\mu\nu\rho\sigma} q_\rho ((m_{p\rho}) s_\sigma - (q \cdot s) p_\sigma) A_2(\rho, q^2) \end{aligned}$$

A_1, A_2 (χ PT): Ji-Osborne; Peset-Pineda

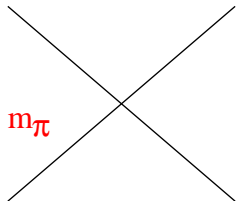
Leading chiral logs to the hyperfine splitting



$$\sim \frac{1}{f_\pi^2} \ln m_\pi$$



$$\sim \frac{1}{f_\pi^2} \ln m_\pi$$



$$\delta V = 2 \frac{C_4}{m_p^2} \mathbf{S}^2 \delta^{(3)}(\mathbf{r}).$$

$$\delta E_{HF} \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$$

The leading chiral logs can be determined for **Hydrogen** and **muonic hydrogen** hyperfine splitting (AP).

$$\begin{aligned} c_4^{pl_i} &\simeq \left(1 - \frac{\mu_p^2}{4}\right) \alpha^2 \ln \frac{m_l^2}{\nu^2} + \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2} \\ &+ \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \frac{2}{3} \left(\frac{2}{3} + \frac{7}{2\pi^2}\right) \pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2} \\ &+ \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \frac{8}{27} \left(\frac{5}{3} - \frac{7}{\pi^2}\right) \pi^2 g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \\ &\stackrel{(N_c \rightarrow \infty)}{\simeq} \alpha^2 \ln \frac{m_l^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2} \alpha^2 \pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2}. \end{aligned}$$

$$E_{HF} = 4 \frac{c_4^{pl_i}}{m_p^2} \frac{1}{\pi} (\mu_{lp} \alpha)^3 \sim m_l \alpha^5 \frac{m_l^2}{m_p^2} \times (\ln m_q, \ln \Delta, \ln m_l).$$

$$c_4^{pl_i} = c_{4,R}^{pl_i} + c_{4,\text{point-like}}^{pl_i} + c_{4,\text{Born}}^{pl_i} + c_{4,\text{pol}}^{pl_i} + \mathcal{O}(\alpha^3).$$

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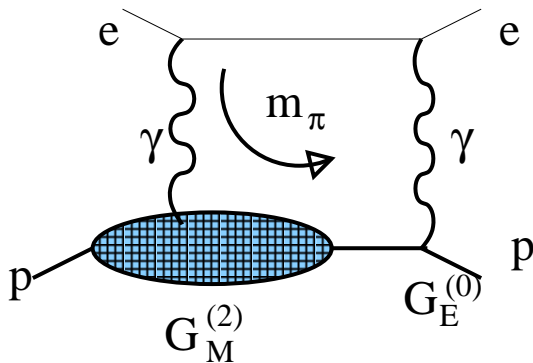


Figure : Symbolic representation (plus permutations) of the Zemach correction.

$$\delta C_{4,Zemach}^{pl} = (4\pi\alpha)^2 m_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{k^4} G_E^{(0)} G_M^{(2)}.$$

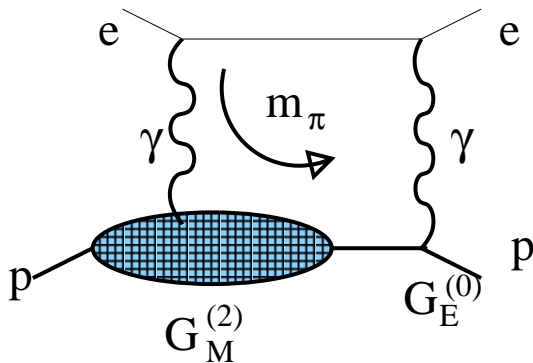


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$c_{4,\text{Born}}$, Zemach magnetic radius

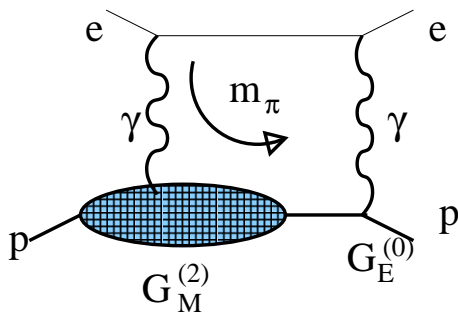


Figure : Symbolic representation (plus permutations) of the Born correction.

Chiral logs can be determined and constitute the leading contribution!

$$\langle r_Z \rangle = -\frac{4}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left[G_E(Q^2) G_M(Q^2) - 1 \right] = -\frac{4}{\pi} \int_0^\infty q^{D-4} \frac{dQ}{Q^2} G_E^{(0)}(Q^2) G_M^{(2)}(Q^2).$$

$$\langle r_Z \rangle = -\frac{3}{4\pi} \frac{1}{\alpha^2 M_p} c_{4,\text{Born}}^{p_i} \simeq -\frac{\pi}{2} \frac{M_p}{(4\pi F_0)^2} \left[g_A^2 \ln \frac{m_\pi^2}{\nu^2} + \frac{4}{9} g_{\pi N \Delta}^2 \ln \frac{\Delta^2}{\nu^2} \right] \stackrel{(\nu=m_\rho)}{=} 1.35 \text{ fm}.$$

"Experiment" $\sim 1.04 - 1.08 \text{ fm}$.

$c_{4,\text{pol}}$, Polarizability

Determined from dispersion relations. No subtractions needed.

Chiral computation can still give relevant information.

$$c_{4,\text{point-like}}^{pl_j} = \left(1 - \frac{\kappa_p^2}{4}\right) \alpha^2 \ln \frac{m_{l_j}^2}{\nu^2},$$

$$c_{4,\text{pol}}^{pl_j} = \frac{M_p^2}{(4\pi F_0)^2} \frac{\alpha^2}{\pi} \frac{8}{3} \left(\frac{7\pi}{8} - \frac{\pi^3}{12}\right) \left[g_A^2 \ln \frac{m_\pi^2}{\nu^2} - \frac{8}{9} g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \right] + \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2}.$$

$$c_{4,\text{point-like}}^{pl_j} + c_{4,\text{pol}}^{pl_j} \stackrel{(N_c \rightarrow \infty)}{\simeq} \alpha^2 \ln \frac{m_{l_j}^2}{\nu^2}$$

Polarizability contribution small and vanishes in the large N_c limit!!

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$$+ \frac{b_{1,F}^2}{18} \alpha^2 \ln \frac{\Delta^2}{\nu^2}.$$

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Polarizability contribution small and vanishes in the large N_c limit!!

Fixing c_4^{pe} . Hydrogen

Hydrogen. By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\text{HF,logarithms}}(m_\rho) = -0.031 \text{ MHz},$$

which accounts for approximately 2/3 of the difference between theory (pure QED) and experiment.

$$E_{\text{HF}}(\text{QED}) - E_{\text{HF}}(\text{exp}) = -0.046 \text{ MHz}.$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c_{4,NR}^{pe} = -48.69(3)\alpha^2$ and $c_{4,R}^{pe}(m_\rho) \simeq c_{4,R}^p(m_\rho) \simeq -16\alpha^2$.

$$C_{4,\text{TPE}}^{p\mu} = C_{4,\text{TPE}}^{pe} + [C_{4,\text{TPE}}^{p\mu} - C_{4,\text{TPE}}^{pe}](\chi\text{PT}) + \mathcal{O}(\alpha).$$

$$\begin{aligned} C_{4,\text{point-like}}^{p\mu} - C_{4,\text{point-like}}^{pe} &= \left(1 - \frac{\kappa_p^2}{4}\right) \ln \frac{m_\mu^2}{m_e^2} + \frac{m_\mu^2}{m_p^2} \left(1 + \frac{\kappa_p}{2} \left(1 - \frac{\kappa_p}{6}\right)\right) \ln \frac{m_\mu^2}{\nu_{\text{pion}}^2} \\ &\simeq 2.09 - 0.09 = 2.00(9), \end{aligned}$$

$$C_{4,\text{pol}}^{p\mu} - C_{4,\text{pol}}^{pe} = \begin{cases} 0.17(9) & (\pi), \\ 0.25(10) & (\pi \& \Delta), \end{cases}$$

DR (Carlson et al) $\sim -0.3(1.4)$. Relativistic χPT 0.08(27)/0.11(55)(Hagelstein et al)

$$\begin{aligned} C_{4,\text{Born}}^{p\mu} - C_{4,\text{Born}}^{pe} &= - \int_0^\infty dp \frac{1}{3p} G_M^{(1)}(-p^2) \\ &\times \left[\left(\frac{p^2 \kappa_p}{m_\mu^2} + \frac{32m_\mu^4 - 8m_\mu^2 p^2 (\kappa_p + 2) - 2p^4 \kappa_p}{m_\mu^2 p (\sqrt{4m_\mu^2 + p^2} + p)} + 8 \right) - (m_\mu \rightarrow m_e) \right], \end{aligned}$$

$G_M^{(1)}(\chi\text{PT})$: Gasser et al.; Bernard et al.

$$C_{4,\text{Born}}^{p\mu} - C_{4,\text{Born}}^{pe} = \begin{cases} 0 + 1.11(55) & (\pi), \\ 0 + 1.42(53) & (\pi \& \Delta). \end{cases}$$

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Overall, combining the three contributions, we obtain

$$[c_{4,\text{TPE}}^{p\mu} - c_{4,\text{TPE}}^{pe}](\chi PT) = 3.68(72)$$

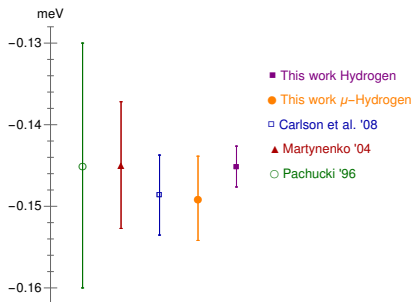


Figure : Two-photon exchange contribution to the hyperfine splitting of the 2S muonic hydrogen. Peset-Pineda

Variation of this idea has later been applied using DR (Tomalak). Error $\sim 1/2$.

Δ , (ppm)	Δ_Z	Δ_R^p	$\Delta_Z + \Delta_R^p$	Δ_0^{pol}	Δ_{HFS}
this work, $\mu\text{H } r_E, r_M^W$	-7415(84)	844(7)	-6571(87)	364(89)	-6207(127)
this work, electron r_E, r_M^W	-7487(95)	844(7)	-6643(98)	364(89)	-6279(135)
this work, $\mu\text{H } r_E, r_M^e$	-7333(48)	846(6)	-6486(49)	364(89)	-6122(105)
this work, electron r_E, r_M^e	-7406(56)	847(6)	-6559(57)	364(89)	-6195(109)
Hagelstein et al. [59]				-61_{-52}^{+70}	
Peset et al. [29]					-6247(109)
Carlson et al. [28, 39]	-7587	835	-6752(180)	351(114)	-6401(213)
Martynenko et al. [38]	-7180		-6656	410(80)	-6246(342)
Pachucki [7]	-8024		-6358	0(658)	-6358(658)

Figure : From Tomalak, 2017

CONCLUSIONS

Effective Field Theories provide with a **model independent, efficient and systematic (Power counting)** approach to the dynamics of **NR** systems and a unified framework to determine the nonperturbative effects.

Rigorous connection between Quantum Field Theories (**Wilson coefficients**) and a **NR** Quantum-mechanical formulation of the **NR** systems (**potentials**). For instance. The proton radius is a **Wilson coefficient of the effective theory**. In general it is an scheme/scale dependent object.

Unlike dispersion relations, no assumption on the high energy behavior.

The spin-independent TPE energy shift (and the associated error) is (and can only be) computed in a model independent way with χ PT. Overall number consistent with determinations from a combined use of dispersion relations and models, but individual contributions are quite different.

χ PT predicts the chiral logs of the hyperfine splitting **and the difference between hydrogen and muonic hydrogen**.

Analytic understanding of the QCD dynamics: m_q and N_c dependence.

$$\Delta E_L^{\text{th}} = \left[206.0243(30) - 5.2270(7) \frac{r_p^2}{\text{fm}^2} + 0.0455(125) \right] \text{ meV} .$$

$$E_{\text{HF}}(1S) = 182.623(27) \text{ meV}, \quad E_{\text{HF}}(2S) = 22.8123(33) \text{ meV}$$

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$$E_{\text{HF}}(1S) = 182.623(27) \text{ meV}, \quad E_{\text{HF}}(2S) = 22.8123(33) \text{ meV}$$

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Rigorous connection between **Quantum Field Theories (Wilson coefficients)** and a **NR Quantum-mechanical** formulation of the **NR** systems (**potentials**). For instance. The proton radius is a **Wilson coefficient of the effective theory**. In general it is an scheme/scale dependent object.

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BACK UP SLIDES

Definition of the proton radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + i F_2(q^2) \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p),$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \dots$$

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$$r_p^2 = 6 \frac{d}{dq^2} G_{E,p}(q^2) |_{q^2=0}$$

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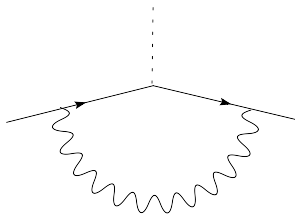
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Infrared divergent! → Wilson coefficient



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c_3^{had} : Two-Photon-Exchange contribution= Born+polarizability

Born:

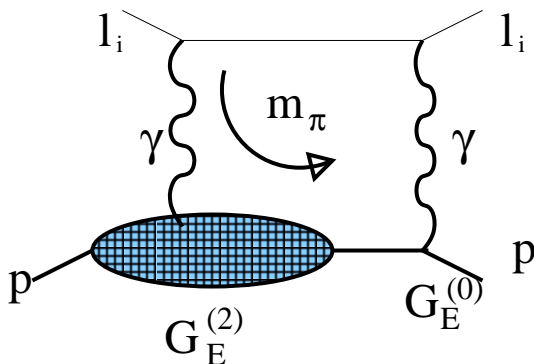


Figure : Symbolic representation (plus permutations) of the Born (r^3) correction.

$$\Delta E_{\text{Born}} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\text{fm}^3} = \frac{48}{\pi} \int \frac{d^3 k}{4\pi} \frac{1}{\mathbf{k}^6} \left(G_E^2 - 1 + \frac{1}{3} \langle r^2 \rangle \mathbf{k}^2 \right) = \frac{96}{\pi} \int \frac{d^{D-1} k}{4\pi} \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\begin{aligned} \delta C_{3,\text{Born}}^{p_l} &= \frac{\pi}{3} \alpha^2 m_p^2 m_\mu \langle r^3 \rangle_{(2)} = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0} \right)^2 \frac{m_l}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} \right. \\ &\quad \left. + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta} \right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta} \right)^{2r} \right\}, \end{aligned}$$

where $(\Delta = M_\Delta - M_p \sim 300 \text{ MeV})$

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1) \Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \quad r \geq 0,$$

$$B_n \equiv \int_0^1 dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln \left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1} \right]$$

$$H_n \equiv \frac{n!(2n-1)!! \Gamma[-3/2]}{2(2n)!! \Gamma[1/2+n]}.$$

Including Pions and Δ particles

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Including Pions and Δ particles

	$\langle r^3 \rangle_E$	$\langle r^5 \rangle_E$	$\langle r^6 \rangle_E$	$\langle r^3 \rangle_{(2)}$
π	0.4980	1.619	5.4(2.7)	0.9960
$\pi \& \Delta$	0.4071	1.522	5.0(2.0)	0.8142
<i>Dipole</i>	0.7706	1.775	3.325	2.023
<i>Kelly</i>	0.9838	3.209	7.440	2.526
<i>Distler et al.</i>	1.16(4)	8.0(1.2)(1.0)	29.8(7.6)(12.6)	2.85(8)

Table : The first two rows give the prediction from the effective theory (Peset&AP). The third row corresponds to the standard dipole fit with $\langle r^2 \rangle = 0.6581 \text{ fm}^3$. The fourth and fifth rows correspond to different parameterizations of experimental data. For completeness, we also quote $\langle r^3 \rangle_{(2)} = 2.71 \text{ fm}^3$ from Friar.

	$\langle r^4 \rangle_E$	$\langle r^8 \rangle_E$	$\langle r^2 \rangle_M$	$\langle r^4 \rangle_M$	$\langle r^6 \rangle_M$
π	0.71(36)	104(52)	0.35(18)	0.71(35)	6.3(3.2)
$\pi \& \Delta$	0.60(29)	99(37)	0.44(16)	0.79(28)	6.9(2.4)

Table : Values of $\langle r^n \rangle_E$ and $\langle r^n \rangle_M$ in units of fm from the effective theory. The first row is for the effective theory with only pions and the second is for the theory with pions and Deltas. Uncertainties in the last two digits are shown in parentheses.

The proton radius in ep scattering from χ PT

Hessels, Horbatsch, AP

$$G_E(Q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} Q^{2n} \langle r^{2n} \rangle$$

- ▶ **Extrapolation** from $|\mathbf{q}| \sim 100$ MeV to $|\mathbf{q}| = 0$
- ▶ dependence on the fitting functions: normalization factors, full data set ...

Higher moments diverge in the chiral limit

$$\langle r^{2k} \rangle \sim m_\pi^{2-2k}$$

Extrapolation controlled by χ PT (at low Q^2): $r_p \sim 0.855$.

Bigger values for the moments produce larger values of r_p .

