SUM RULES FOR LIGHT-LIGHT INTERACTION

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CAUSALITY

$$B(t) = \int dt' \, G(t-t')A(t')$$

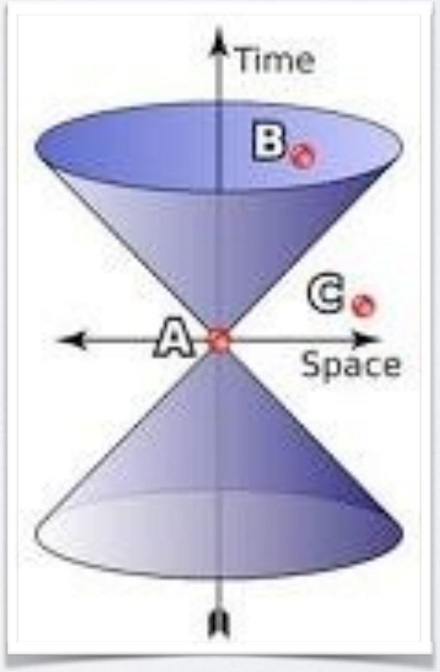
$$G(t-t') = 0, \quad t < t'$$

implies analyticity in Energy

relativistic version:

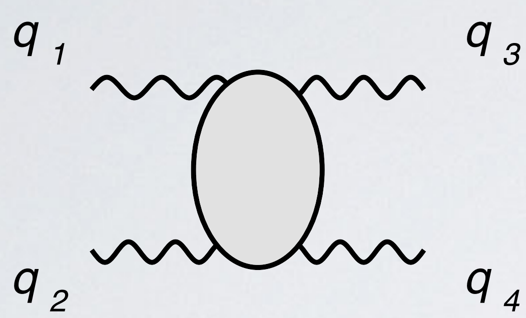
$$B(x) = \int dx' G(x-x')A(x')$$

$$G(x-x') = 0, \quad (x-x')^2 < 0$$



DERIVATION OF SUM RULES FOR LIGHT-BY-LIGHT

[V.P. & VANDERHAEGHEN, PRL 105 (2010)]



$$M_{\lambda_1\lambda_2\lambda_3\lambda_4} = \varepsilon^{*\mu_4}_{\lambda_4}(\vec{q}_4)\,\varepsilon^{*\mu_3}_{\lambda_3}(\vec{q}_3)\,\varepsilon^{\mu_2}_{\lambda_2}(\vec{q}_2)\,\varepsilon^{\mu_1}_{\lambda_1}(\vec{q}_1)\,\mathcal{M}_{\mu_1\mu_2\mu_3\mu_4}$$

Helicity AMPL. Feynman AMPL.

In the forward direction ($t = 0, \quad s = 4\omega^2, \quad u = -s.$):

$$\mathcal{M}_{\mu_1\mu_2\mu_3\mu_4} = A(s)\, g_{\mu_4\mu_2} g_{\mu_3\mu_1} + B(s)\, g_{\mu_4\mu_1} g_{\mu_3\mu_2} + C(s)\, g_{\mu_4\mu_3} g_{\mu_2\mu_1}\,,$$

$$\begin{aligned} M_{++++}(s) &= A(s) + C(s), \\ M_{+-+-}(s) &= A(s) + B(s), \\ M_{++--}(s) &= B(s) + C(s). \end{aligned}$$

1) Crossing symmetry (1 <-> 3, 2 <-> 4):

$$M_{+-+-}(s) = M_{++++}(-s), \quad M_{++--}(s) = M_{++--}(-s)$$

SUM RULES FOR LIGHT-BY-LIGHT (DERIVATION CONTD)

Amplitudes with definite parity under Crossing:

$$f^{(\pm)}(s) = M_{++++}(s) \pm M_{+-+-}(s)$$

$$g(s) = M_{++--}(s)$$

2) Causality => Analyticity => dispersion relations:

$$\mathrm{Re}\left\{ \begin{array}{l} f^{(\pm)}(s) \\ g(s) \end{array} \right\} = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{ds'}{s'-s} \, \mathrm{Im} \left\{ \begin{array}{l} f^{(\pm)}(s') \\ g(s') \end{array} \right\},$$

3) Optical theorem (unitarity):

$$\mathrm{Im}\, f^{(\pm)}(s) = -\frac{s}{8}\,[\,\sigma_0(s) \pm \sigma_2(s)\,],$$

$$\mathrm{Im}\, g(s) = -\frac{s}{8}\,[\,\sigma_{||}(s) - \sigma_{\perp}(s)\,].$$

$\sigma_{0,2}(\sigma_{||,\perp})$ Are circularly (linearly) polarized Photon–Photon Fusion cross-sections

SUM RULES FOR LIGHT-BY-LIGHT (DERIVATION CONTD)

Sum rules:

$$\mathrm{Re}\, f^{(+)}(s) = -\frac{1}{2\pi} \fint_0^\infty ds'\, s'^2\, \frac{\sigma(s')}{s'^2 - s^2}\,, \qquad \sigma = (\sigma_0 + \sigma_2)/2 = (\sigma_{||} + \sigma_\perp)/2$$

$$\mathrm{Re}\, f^{(-)}(s) = -\frac{s}{4\pi} \fint_0^\infty ds'\, \frac{s'\,\Delta\sigma(s')}{s'^2 - s^2}\,, \qquad \Delta\sigma = \sigma_2 - \sigma_0$$

$$\mathrm{Re}\, g(s) = -\frac{1}{4\pi} \fint_0^\infty ds'\, s'^2\, \frac{\sigma_{||}(s') - \sigma_\perp(s')}{s'^2 - s^2}\,,$$

4) "Low-energy Theorem": $\mathcal{L}_{\mathrm{EH}} = c_1 (F_{\mu\nu} F^{\mu\nu})^2 + c_2 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2,$

Low-energy expansion

$$f^{(+)}(s) = -2(c_1 + c_2) s^2 + O(s^4)$$

$$f^{(-)}(s) = O(s^5)$$

$$g(s) = -2(c_1 - c_2) s^2 + O(s^4)$$

SUM RULES FOR LIGHT-BY-LIGHT

$O(s^1)$:

$$0 = \int_0^\infty \frac{\mathrm{d}s}{s} \Big[\sigma_2(s) - \sigma_0(s)\Big]$$

Gerasimov & Moulin (1976)
Brodsky & Schmidt (1995)

$O(s^2)$:

$$c_1 = \frac{1}{8\pi} \int_0^\infty \frac{\mathrm{d}s}{s^2}\, \sigma_{||}(s)\,,$$

$$c_2 = \frac{1}{8\pi} \int_0^\infty \frac{\mathrm{d}s}{s^2}\, \sigma_{\perp}(s)$$

LECs are positive
Photons attract!

GENERALIZATION TO VIRTUAL PHOTONS

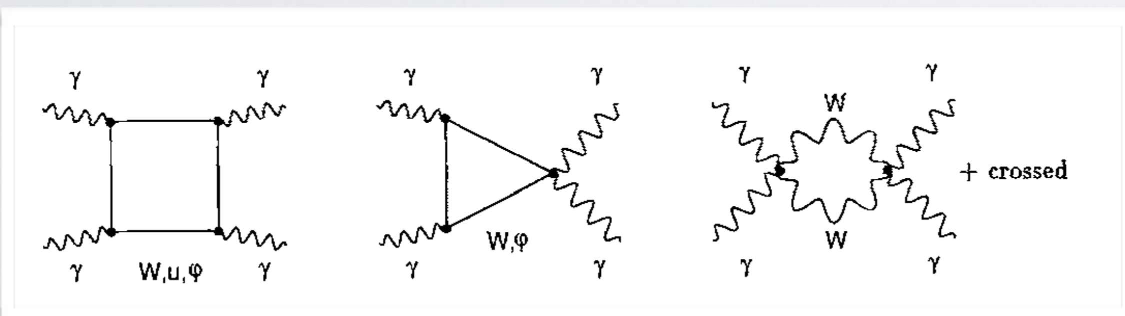
[V.P., PAUK & VANDERHAEGHEN, PRD (2012)]

$$
\begin{aligned}
0 &= \int_{s_0}^{\infty} ds \frac{1}{(s+Q_1^2)} \left[\sigma_0 - \sigma_2\right]_{Q_2^2=0}, \\
0 &= \int_{s_0}^{\infty} ds \, \frac{1}{(s+Q_1^2)^2} \left[\sigma_{\parallel} + \sigma_{LT} + \frac{(s+Q_1^2)}{Q_1 Q_2} \tau_{TL}^a \right]_{Q_2^2=0}, \\
0 &= \int_{s_0}^{\infty} ds \, \left[\frac{\tau_{TL}}{Q_1 Q_2}\right]_{Q_2^2=0}.
\end{aligned}
$$

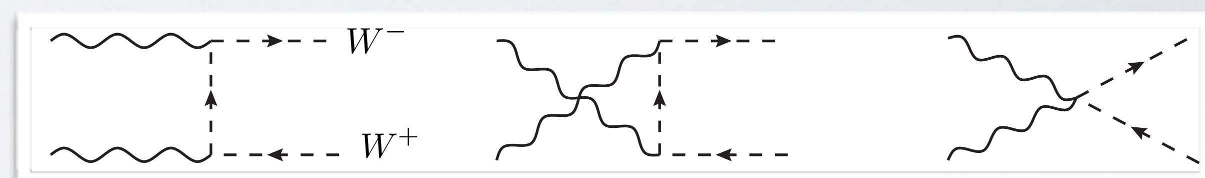
PERTURBATIVE VERIFICATION OF SUM RULES FOR LIGHT-BY-LIGHT

$$c_1 = \frac{1}{8\pi} \int_0^\infty \frac{\mathrm{d}s}{s^2}\, \sigma_{||}(s)\,,$$

$$c_2 = \frac{1}{8\pi} \int_0^\infty \frac{\mathrm{d}s}{s^2}\, \sigma_{\perp}(s)$$



Böhm & Schuster, EPJC (1994)



V.Pauk (2011)

$$\sigma_{||} = \frac{2\pi\alpha^2}{M^2 s^3}\left(s(18M^4 + 3M^2 s + 4s^2)\sqrt{1-\frac{4M^2}{s}} - 24M^4(s-3M^2)\mathrm{ArcTanh}\left(\sqrt{1-\frac{4M^2}{s}}\right)\right)$$

$$\sigma_{\perp} = \frac{2\pi\alpha^2}{M^2 s^3}\left(s(6M^4 + 3M^2 s + 4s^2)\sqrt{1-\frac{4M^2}{s}} - 24M^4(s-M^2)\mathrm{ArcTanh}\left(\sqrt{1-\frac{4M^2}{s}}\right)\right)$$

Low-energy Expansion

Integrate

$$c_1 = \frac{29\alpha^2}{160 M^4}$$

$$c_2 = \frac{27\alpha^2}{160 M^4}$$

Ghosts, Higgs Sector

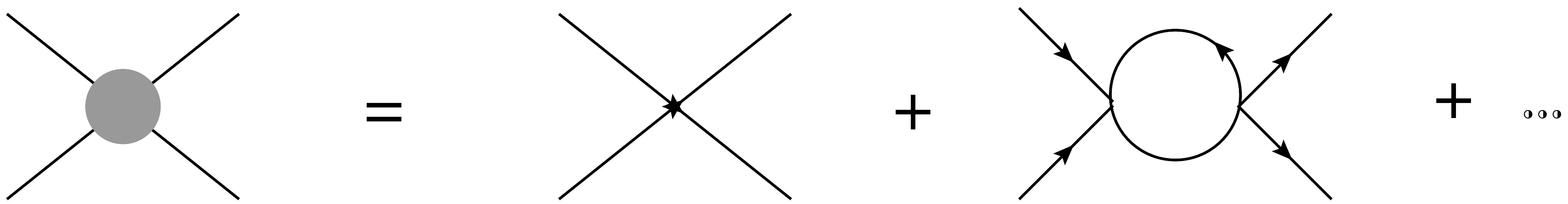
Unitarity, Causality

Relativistic 0-range Scattering

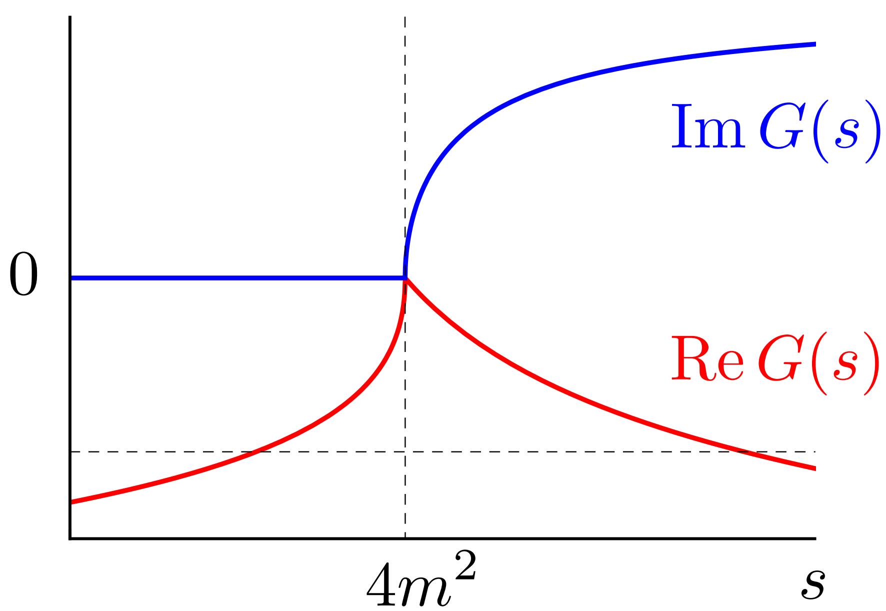
$$T = V + V\,G\,T$$
$$V = \lambda$$

Bubble-chain sum:

$$T(s) = \frac{1}{\lambda^{-1} - G(s)}$$



$$G(s) = -i \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[(p+\ell)^2 - m^2]\,(\ell^2 - m^2)}$$ with $p^2 = s$.



$\lambda > 0$: no poles

$\lambda < 0$: one pole and one K-matrix pole

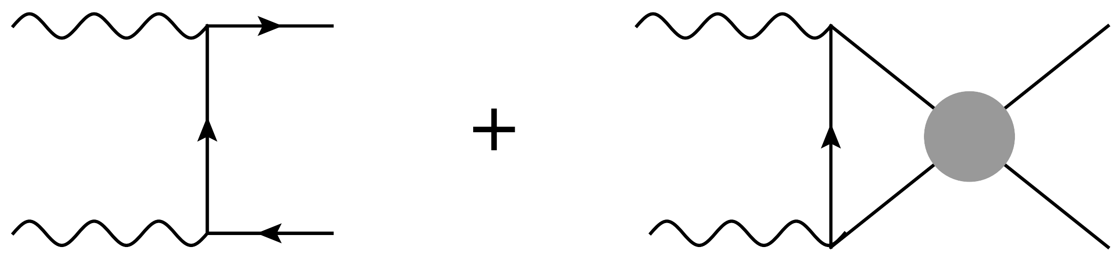
$$T(s) = \frac{1}{K^{-1}(s) - i}$$

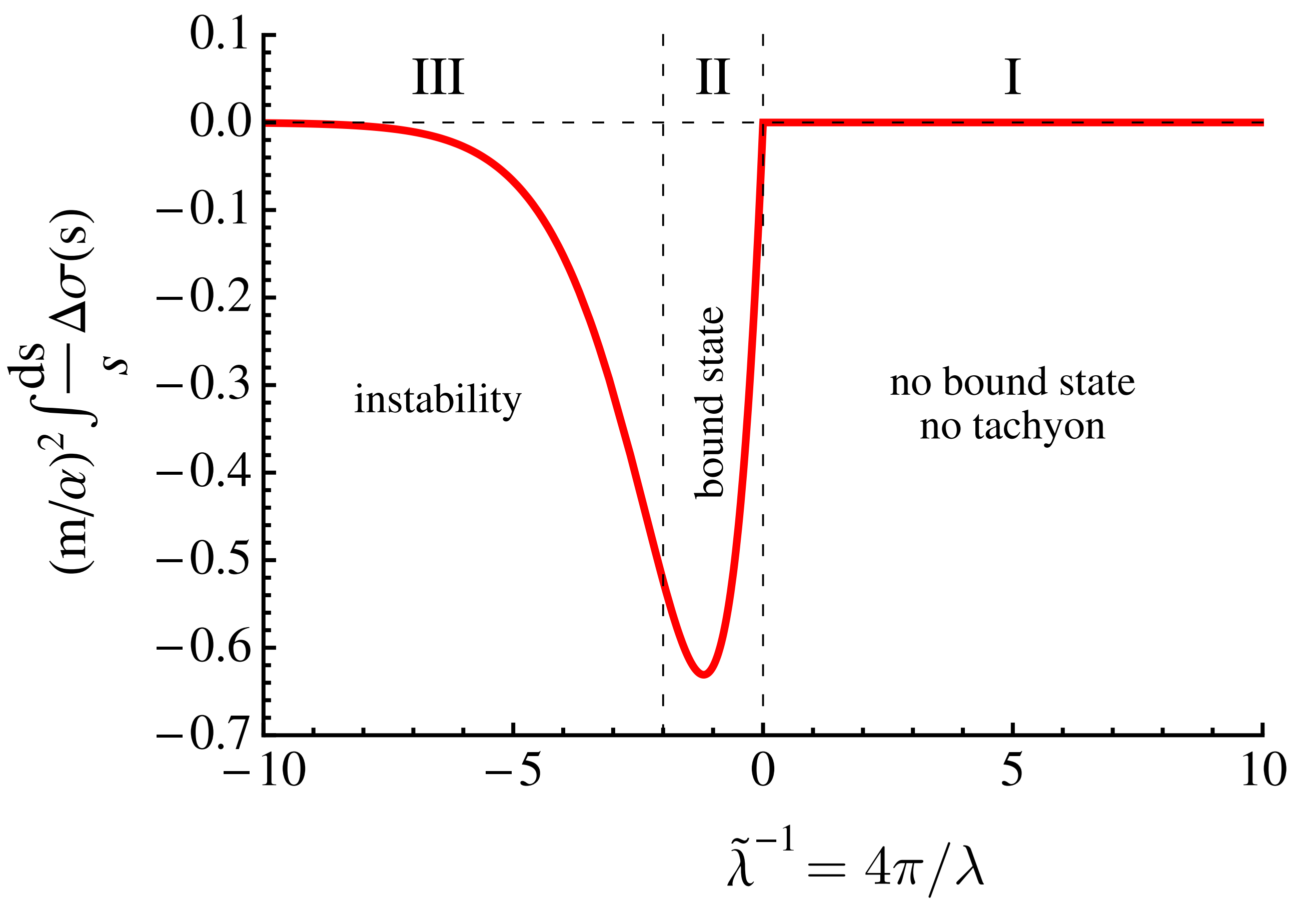
Causality criterion

PAUK, V.P. & VANDERHAEGHEN, PLB (2013)

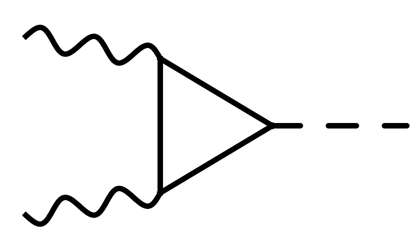
$$\int_{s_0}^{\infty} \mathrm{d}s\, \frac{\Delta\sigma(s)}{s} = 0.$$

$$\mathcal{L} = (D^\mu \phi)^* D_\mu \phi - m^2 \phi^* \phi + \frac{\lambda}{4}(\phi^* \phi)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$





To cancel the integral one need to introduce the bound state as the asymptotic state i.e., new channel:



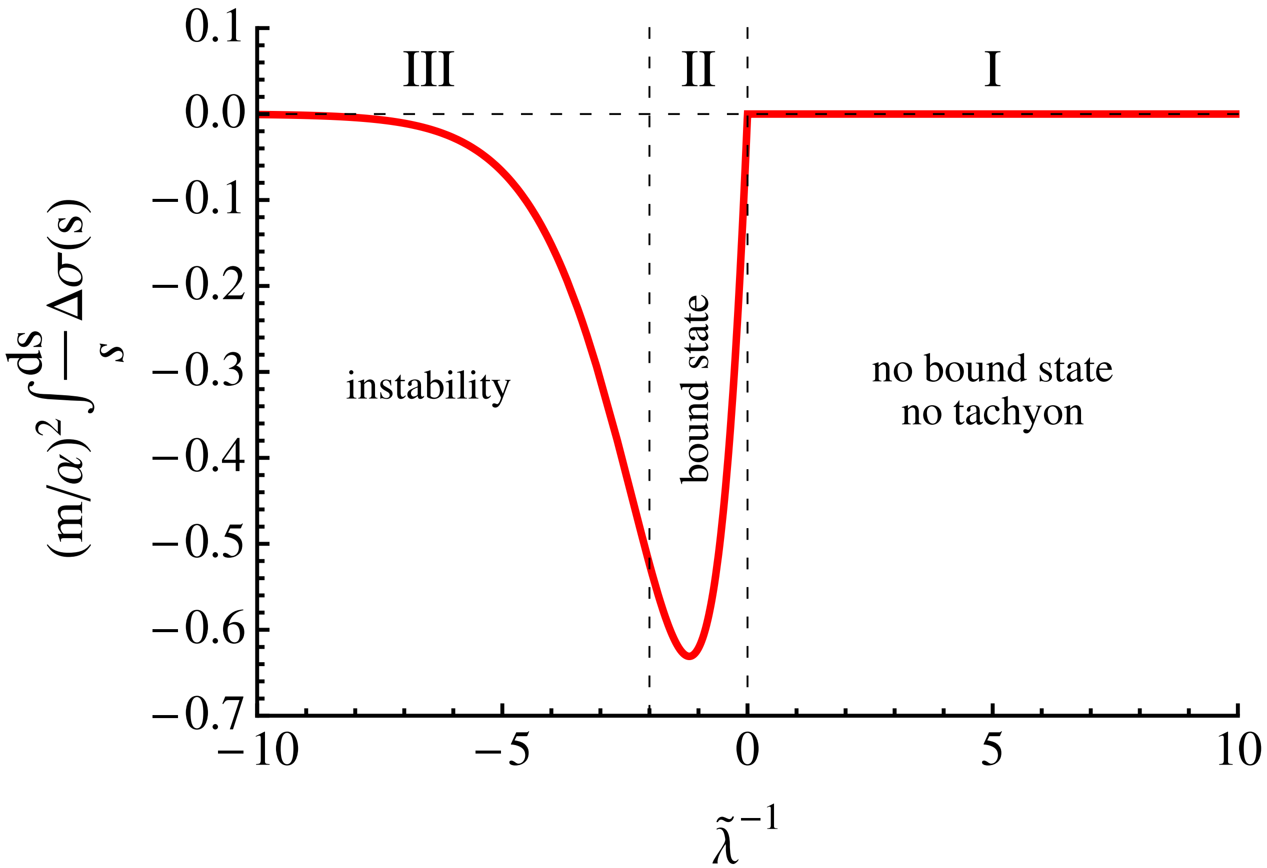
Surpassing Wigner's causality bound for NR scattering

$$|\mathbf{k}| \cot \delta(s) = -\frac{1}{a} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} r_n |\mathbf{k}|^{2n}$$

Wigner's bound:
effective range is non-positive!

WIGNER, PHYS REV (1955)

PHILLIPS & COHEN, PLB (1997),
HAMMER & DEAN LEE, ANN PHYS (2010), ...



In the tachyon (acausal) regime at least one of the effective range parameters is negative.

Therefore, causality yields:

$$r_n \geq 0$$

VP (2014)

LO phase shift, crosses 90 degrees!

Levinson's theorem: $\delta(0) = \pi N_{\text{bound states}}$

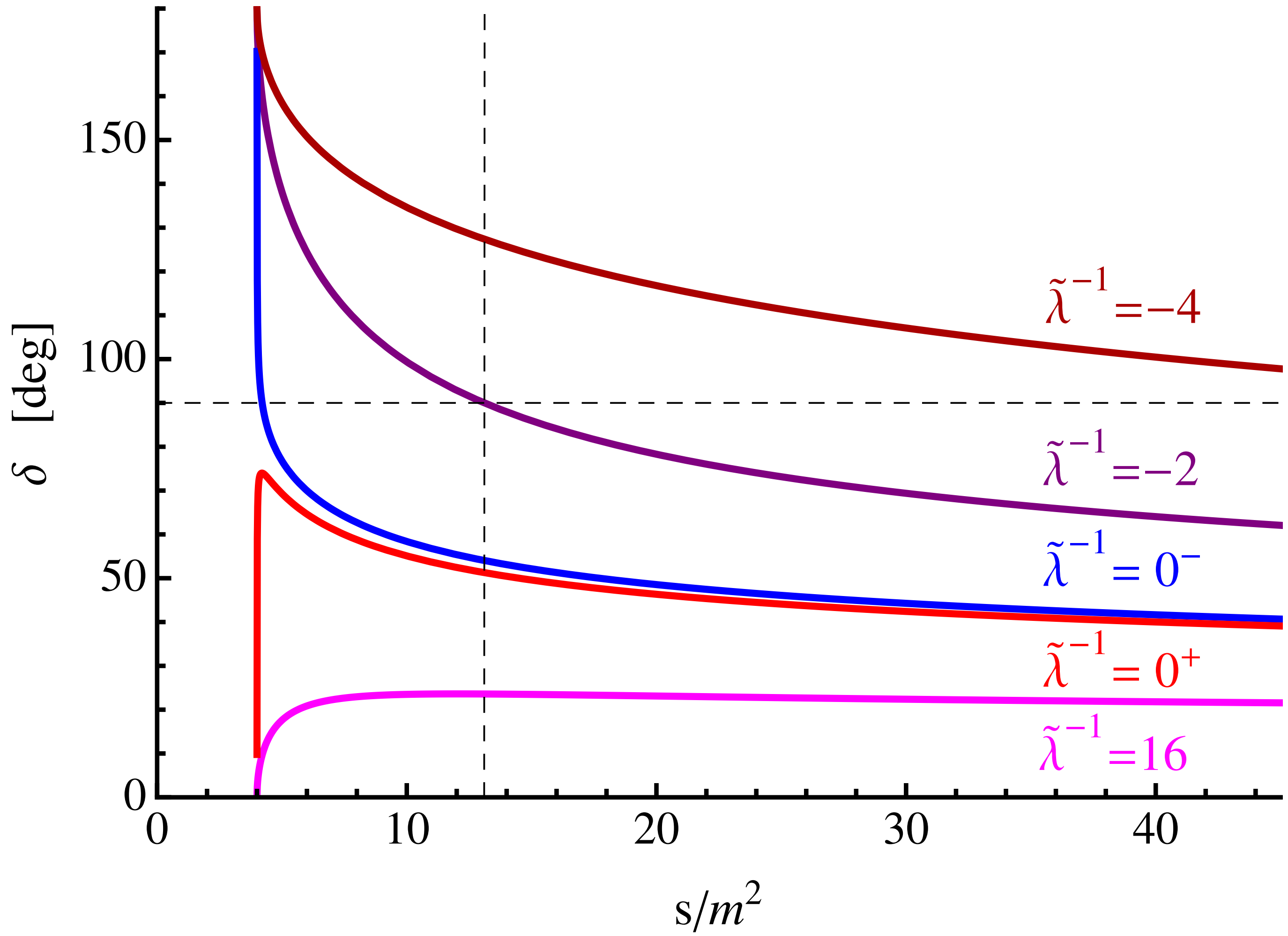
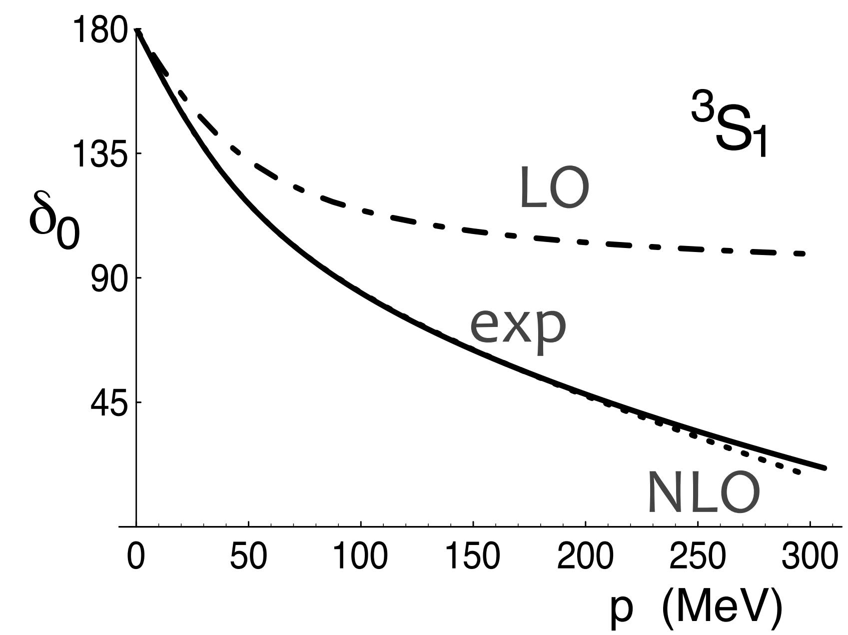
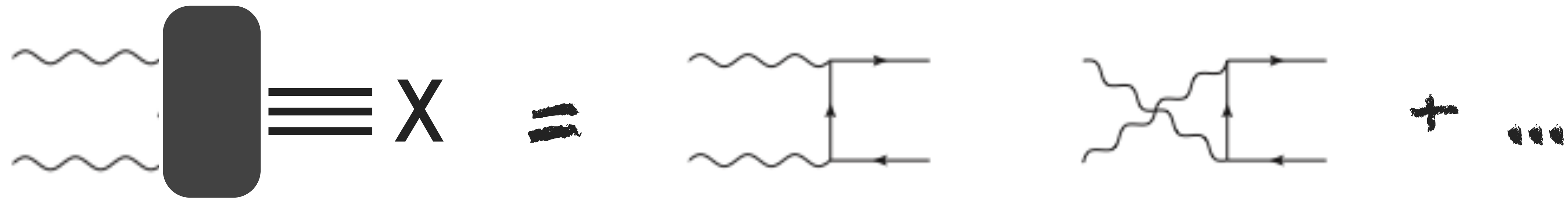


Figure 7: Phase shift for different values of $\tilde{\lambda}$.



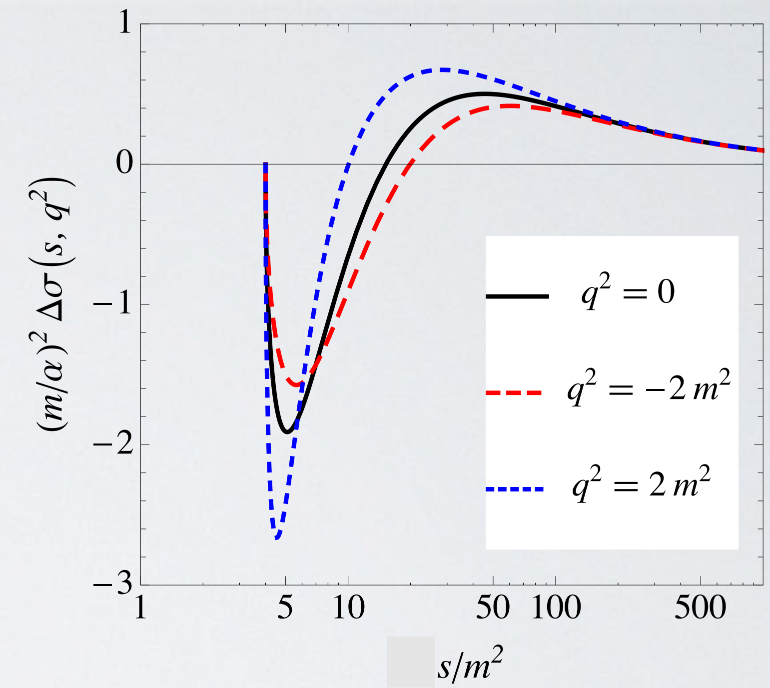
Kaplan, Savage & Wise
PLB (1998)

Bound state is accompanied by the 90 degree crossing, i.e. a K-matrix pole, which does not correspond here to any S-matrix pole



$$0 = \int_0^\infty \mathrm{d}s \, \frac{\sigma_2(s) - \sigma_0(s)}{s},$$

PT



non-PT

cancellation of (pseudo)scalar and tensor meson contributions

	m_M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	$\int ds\, \Delta\sigma/s$ [nb]
π^0	134.98	$(7.8 \pm 0.6) \times 10^{-3}$	-195.0 ± 15.0
η	547.85	0.51 ± 0.03	-190.7 ± 11.2
η'	957.66	4.30 ± 0.15	-301.0 ± 10.5
Sum η, η'			-492 ± 22

	m_M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	$\int ds\, \Delta\sigma/s$ narrow res. [nb]	$\int ds\, \Delta\sigma/s$ Breit-Wigner [nb]
$a_2(1320)$	1318.3	1.00 ± 0.06	134 ± 8	137 ± 8
$f_2(1270)$	1275.1	3.03 ± 0.35	448 ± 52	479 ± 56
$f_2'(1525)$	1525	0.081 ± 0.009	7 ± 1	7 ± 1
Sum f_2, f_2'			455 ± 53	486 ± 57