Dispersive approach to hadronic light-by-light scattering: Reconstructing $\gamma^*\gamma^* \to \pi\pi$

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based in parts on arXiv:1402.7081 and arXiv:1309.6877

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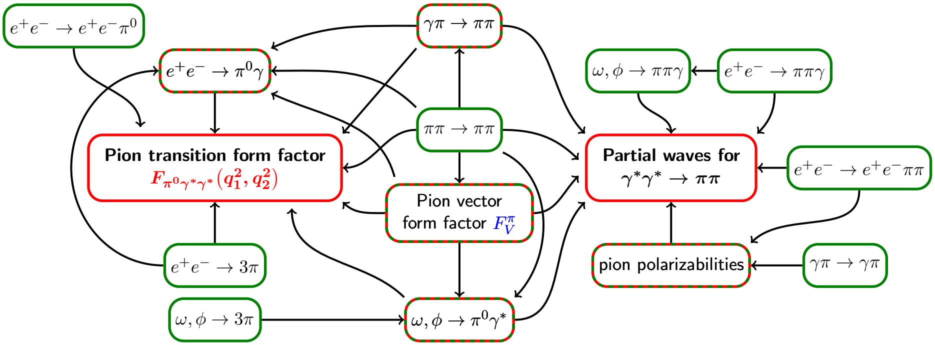
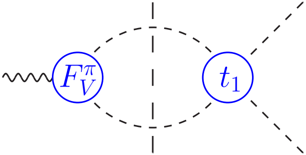


- Pion transition form factor: Talk by B. Kubis
- Ideal world: measure space-like, doubly-virtual $\gamma^*\gamma^* \to \pi\pi$ for arbitrary virtualities

  $\hookrightarrow$ partial-wave analysis
- This talk: how to realistically **reconstruct** $\gamma^*\gamma^* \to \pi\pi$

- **Dispersive approach**: resum $\pi\pi$ rescattering F_V^π as example
- **Unitarity** for **pion vector form factor**

$$\mathrm{Im}\,F_V^\pi(s) = \theta\big(s-4M_\pi^2\big)F_V^\pi(s)e^{-i\delta_1(s)}\sin\delta_1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

Pion vector form factor

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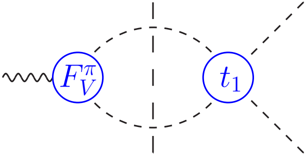


$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

- Solution in terms of **Omnès function** Omnès 1958

$$F_V^\pi(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1(s')}{s'(s'-s)}\right\}$$

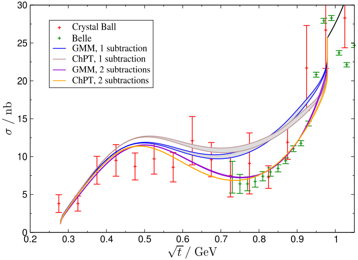
- Asymptotics + normalization $\Rightarrow P(s) = 1$

$\gamma^*\gamma^* \to \pi\pi$ partial waves

Roy(–Steiner) equations $=$ Dispersion relations $+$ partial-wave expansion

$+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- **On-shell case $\gamma\gamma \to \pi\pi$** Moussallam 2010, MH, Phillips, Schat 2011 $\hookrightarrow$ precision determination of $\sigma \to \gamma\gamma$ coupling
- **Singly-virtual $\gamma^*\gamma \to \pi\pi$** Moussallam 2013
- **Doubly-virtual $\gamma^*\gamma^* \to \pi\pi$: anomalous thresholds**
  Colangelo, MH, Procura, Stoffer arXiv:1309.6877



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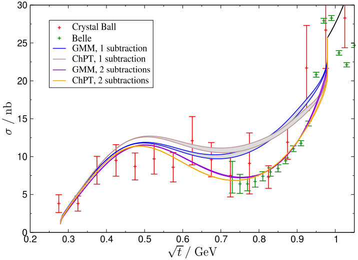
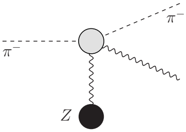
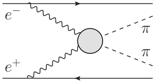
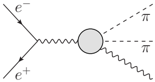
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds** Colangelo, MH, Procura, Stoffer arXiv:1309.6877
- Constraints
  - **Low energies**: pion polarizabilities, ChPT
  - **Primakoff**: $\gamma\pi \to \gamma\pi$ (COMPASS), $\gamma\gamma \to \pi\pi$ (JLab)
  - **Scattering**: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - **(Transition) Form factors**: F_V^π, $\omega, \phi \to \pi^0\gamma^*$

$\hookrightarrow$ discuss these constraints in the following

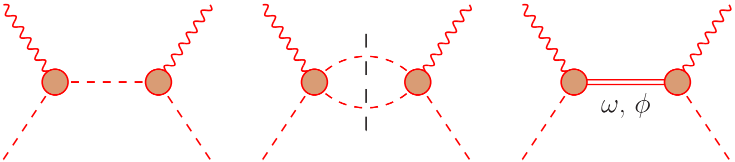








Left-hand cut



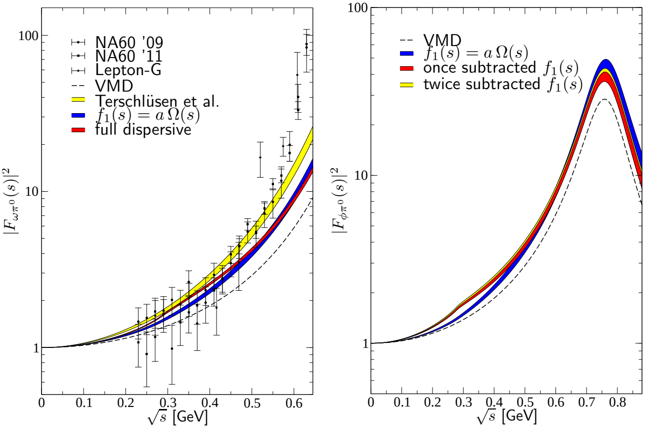
- **Pion pole**: coupling determined by F_V^π as before
- **Multi-pion intermediate states**: approximate in terms of **resonances**
  - $2\pi \sim \rho$: can even be done **exactly** using $\gamma^* \to 3\pi$ amplitude

    $\hookrightarrow$ see pion transition form factor
  - $3\pi \sim \omega, \phi$: narrow-width approximation

    $\hookrightarrow$ **transition form factors** for $\omega, \phi \to \pi^0\gamma^*$
  - Higher intermediate states also potentially relevant: **axials**, **tensors**

    $\hookrightarrow$ **sum rules** to constrain their transition form factors Pauk, Vanderhaeghen 2014

$\omega, \phi \to \pi^0\gamma^*$ transition form factor



Schneider, Kubis, Niecknig 2012

- Puzzle of steep rise in $F_{\omega\pi^0}$

  $\hookrightarrow$ Measurement of $F_{\phi\pi^0}$ would be extremely valuable

- Clarification important for pion transition form factor, but also $\gamma^*\gamma^* \to \pi\pi$

Subtraction functions

Omnès representation for S-wave

$$h_{0,++}(s)=\Delta_{0,++}(s)+\Omega_0(s)\bigg[\frac{1}{2}(s-s_+)a_+(q_1^2,q_2^2)+\frac{1}{2}(s-s_-)a_-(q_1^2,q_2^2)+q_1^2q_2^2b(q_1^2,q_2^2)$$

$$+\frac{s(s-s_+)}{2\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_+)(s'-s)|\Omega_0(s')|}+\frac{s(s-s_-)}{2\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_-)(s'-s)|\Omega_0(s')|}$$

$$+\frac{2q_1^2q_2^2s}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\sin\delta_0(s')\Delta_{0,00}(s')}{s'(s'-s_+)(s'-s_-)|\Omega_0(s')|}\bigg] \qquad s_\pm=q_1^2+q_2^2\pm 2\sqrt{q_1^2q_2^2}$$

- Inhomogeneities $\Delta_{0,++}(s),\Delta_{0,00}(s)$ include left-hand cut
- **Subtraction functions**
  - $b(q_1^2,q_2^2)$ and $a_+(q_1^2,q_2^2)-a_-(q_1^2,q_2^2)$ multiply $q_1^2q_2^2$ and $\sqrt{q_1^2q_2^2}$

    $\hookrightarrow$ inherently doubly-virtual observables $\Rightarrow$ need ChPT (or lattice)
  - However: $a(q_1^2,q_2^2)=(a_+(q_1^2,q_2^2)+a_-(q_1^2,q_2^2))/2$ fixed by singly-virtual measurements

    $\hookrightarrow$ compare with chiral prediction, uncertainty estimates for the other functions

Subtraction functions: chiral constraints

- 1-loop result for arbitrary q_i^2, e.g.

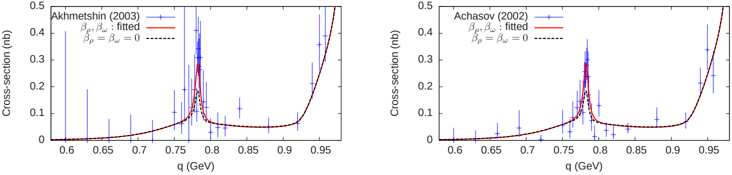
$$a^{\pi^0}(q_1^2,q_2^2) = -\frac{M_\pi^2}{8\pi^2 F_\pi^2 (q_1^2-q_2^2)^2}\bigg\{q_1^2+q_2^2+2\Big(M_\pi^2(q_1^2+q_2^2)+q_1^2q_2^2\Big)C_0(q_1^2,q_2^2)$$
$$+q_1^2\bigg(1+\frac{6q_2^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_1^2)+q_2^2\bigg(1-\frac{6q_1^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_2^2)\bigg\}$$

- Special case: $q_1^2=q_2^2=0$

$$a^{\pi^\pm}(0,0) = \frac{\bar{l}_6-\bar{l}_5}{48\pi^2F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^\pm} \qquad b^{\pi^\pm}(0,0)=0$$
$$a^{\pi^0}(0,0) = -\frac{1}{96\pi^2F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^0} \qquad b^{\pi^0}(0,0) = -\frac{1}{1440\pi^2F_\pi^2M_\pi^2}+\cdots$$

$\hookrightarrow$ resum higher chiral orders into **pion polarizabilities**

Subtraction functions: dispersive representation



Moussallam 2013

- Singly-virtual case: phenomenological representation with chiral constraints

  $\hookrightarrow$ parameters fixed from $e^+e^- \to \pi^0\pi^0\gamma$ (CMD2 and SND) Moussallam 2013

- **Dispersive representation**: imaginary part from $2\pi, 3\pi, \ldots$

  $\hookrightarrow$ analytic continuation from time-like to space-like kinematics

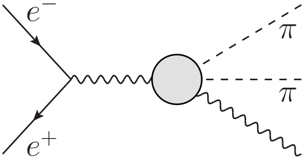
- Example: $I = 2 \Rightarrow$ isovector photons $\Rightarrow 2\pi \sim \rho$

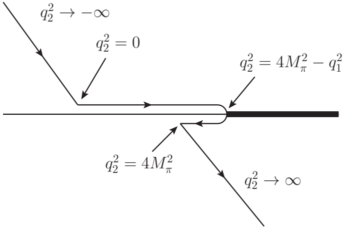
$$a^2(q_1^2,q_2^2) = \alpha_0\Big[\alpha^2 + \alpha\Big(q_1^2\mathscr{F}^\rho(q_1^2) + q_2^2\mathscr{F}^\rho(q_2^2)\Big) + q_1^2 q_2^2 \mathscr{F}^\rho(q_1^2)\mathscr{F}^\rho(q_2^2)\Big]$$

$$\mathscr{F}^\rho(q^2) = \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds\,\frac{q_{\pi\pi}^3(s)\big(F_\pi^V(s)\big)^*\Omega_1(s)}{s^{3/2}(s-q^2)} \qquad q_{\pi\pi}(s) = \sqrt{\frac{s}{4} - M_\pi^2}$$

$\hookrightarrow$ α_0 and α can be determined from $a^2(q^2,0)$ alone!

Anomalous thresholds





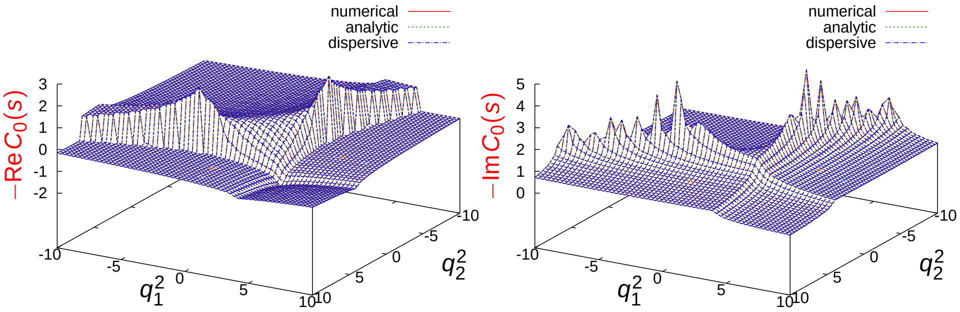
- **Analytic continuation** in q_i^2 in time-like region non-trivial in doubly-virtual case
- Singularities from second sheet move onto first one

  $\hookrightarrow$ need to **deform** the **integration contour**
- Problem already occurs for a simple triangle loop function $C_0(s)$

  $\hookrightarrow$ extra factor $t_\ell(s)/\Omega_\ell(s)$ is well defined in the whole complex plane

  $\hookrightarrow$ remedy in case of $C_0(s)$ can be taken over to full Omnès solution
- Becomes relevant for $e^+e^+ \to e^+e^-\pi\pi$ in time-like kinematics

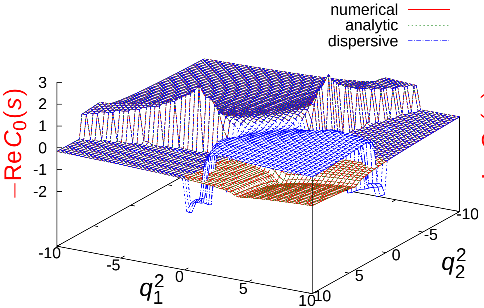
Numerical check of anomalous thresholds

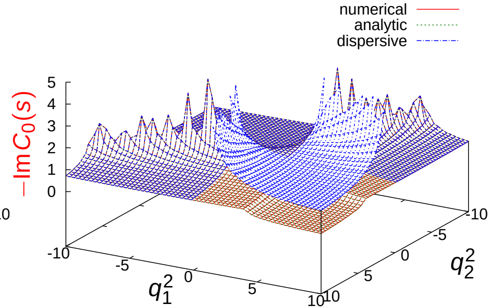


- Comparison for $s = 5$, $M_\pi = 1$

  $\hookrightarrow$ **Dispersive reconstruction** of $C_0(s)$ works!

Numerical check of anomalous thresholds

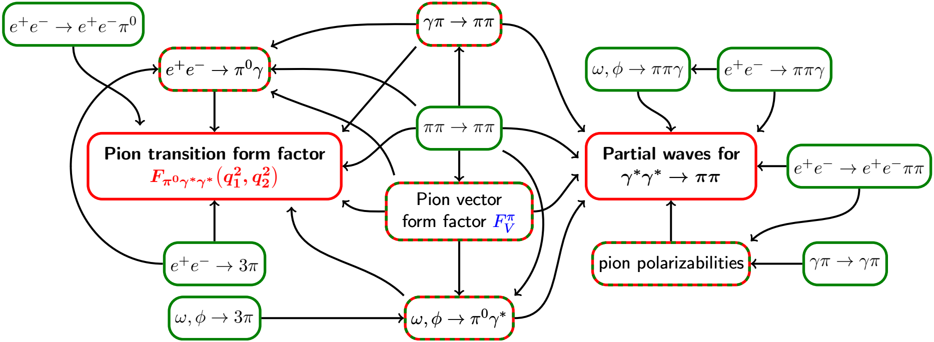




- Ignore anomalous piece

$\hookrightarrow$ Substantial deviations for **large virtualities**!

Experimental input: summary



- (Transition) Form factors to fix **left-hand cut**
- $e^+e^- \to e^+e^-\pi\pi$ **singly-virtual** / $e^+e^- \to \pi\pi\gamma$ + ChPT + pion polarizabilities to fix **subtraction constants**

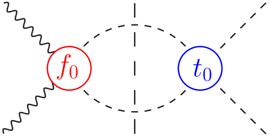
Summary

- **Road map** towards a representation for HLbL scattering **connected to data** as closely as possible
  - **Pion pole**: pion transition form factor
  - $\pi\pi$ **intermediate states**: helicity partial waves for $\gamma^*\gamma^* \to \pi\pi$
- **Roy–Steiner equations** for $\gamma^*\gamma^* \to \pi\pi$
  - **Left-hand cut**: (transition) form factors, $\gamma^* \to 3\pi$
  - **Subtraction constants**: chiral constraints, pion polarizabilities, **singly-virtual data** already help a lot!
  - **Anomalous thresholds** for **time-like** $e^+e^- \to e^+e^-\pi\pi$

- **Left-hand cut** approximated by **pion pole** + **resonances**
- **Unitarity** for $\gamma^*\gamma^* \to \pi\pi$ system: Watson's theorem

$$\text{disc} f_0(s;q_1^2,q_2^2) = 2i\sigma_s f_0(s;q_1^2,q_2^2) t_0^*(s)$$

$$t_0(s) = \frac{1}{\sigma_s} e^{i\delta_0(s)} \sin\delta_0(s) \qquad \sigma_s = \sqrt{1-\frac{4M_\pi^2}{s}}$$



$\hookrightarrow$ solution in terms of **Omnès function**, e.g. for pion pole only

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|} \qquad N_0(s;q_1^2,q_2^2) = \frac{2L}{\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}$$

$$L = \log\frac{s-q_1^2-q_2^2+\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s-q_1^2-q_2^2-\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \qquad \lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$$

- **Analytic continuation** in q_i^2?

$\gamma^*\gamma^* \to \pi\pi$: analytic continuation

$$L = \log \frac{s-q_1^2-q_2^2+\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s-q_1^2-q_2^2-\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \overset{q_2^2\to 0}{\longrightarrow} \pm\log\frac{1+\sigma_s}{1-\sigma_s}$$

- Singularities of the log: **anomalous thresholds**

$$s_\pm = q_1^2+q_2^2-\frac{q_1^2q_2^2}{2M_\pi^2}\pm\frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2-4M_\pi^2)q_2^2(q_2^2-4M_\pi^2)}$$

$\hookrightarrow$ usual Omnès derivation breaks down

- Idea: consider first the **scalar loop function**

$$C_0(s) \equiv C_0\big((q_1+q_2)^2;q_1^2,q_2^2\big) = \frac{1}{i\pi^2}\int\frac{d^4k}{(k^2-M_\pi^2)\big((k+q_1)^2-M_\pi^2\big)\big((k-q_2)^2-M_\pi^2\big)}$$

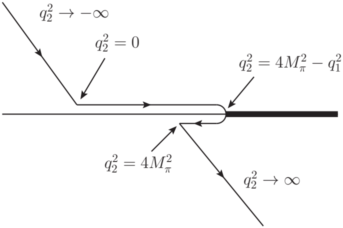
$$\text{disc}\, C_0(s) = -\frac{2\pi i}{\sqrt{\lambda\big(s,q_1^2,q_2^2\big)}}L = -\pi i\,\sigma_s N_0\big(s;q_1^2,q_2^2\big)$$

$\gamma^*\gamma^* \to \pi\pi$: anomalous thresholds

$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2 - 4M_\pi^2)q_2^2(q_2^2 - 4M_\pi^2)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$

  $\hookrightarrow$ moves through unitarity cut onto first sheet



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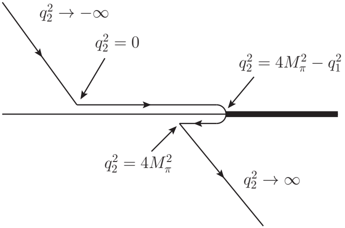
- Need to deform the contour



$$C_0(s) = \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} ds' \frac{\operatorname{disc} C_0(s')}{s'-s}$$

$$+\,\theta\left(q_1^2+q_2^2-4M_\pi^2\right)\frac{1}{2\pi i}\int_0^1 dx \frac{\partial s_x}{\partial x}\frac{\operatorname{disc}_{\rm an} C_0(s_x)}{s_x - s}$$

$$s_x = x4M_\pi^2 + (1-x)s_+ \qquad \operatorname{disc}_{\rm an} C_0(s) = \frac{4\pi^2}{\sqrt{\lambda(s,q_1^2,q_2^2)}}$$

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

- Integrand similar to the scalar-loop example

$$\frac{N_0(s;q_1^2,q_2^2)\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\,C_0(s)}{\pi i\,\sigma_s}\frac{\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\,C_0(s)}{\pi i}\frac{t_0(s)}{\Omega_0(s)}$$

$\hookrightarrow$ Additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

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$\hookrightarrow$ Additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

Omnès representation for $\gamma^*\gamma^* \to \pi\pi$

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

$$+\theta\left(q_1^2+q_2^2-4M_\pi^2\right)\frac{\Omega_0(s)}{2\pi i}\int_0^1 dx \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}} f_0(s_x;q_1^2,q_2^2)}{s_x-s}$$

$$s_x = x4M_\pi^2+(1-x)s_+ \qquad \text{disc}_{\text{an}} f_0(s;q_1^2,q_2^2) = -\frac{8\pi}{\sqrt{\lambda(s,q_1^2,q_2^2)}}\frac{t_0(s)}{\Omega_0(s)}$$