$\pi \to e^+e^-$ decay

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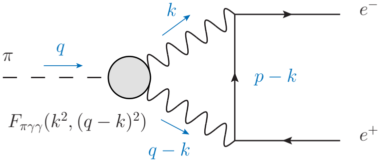
Arbeitsgruppenseminar
27 March 2014

SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

At LO in α_{EM}, this process occurs via 2γ intermediate state.



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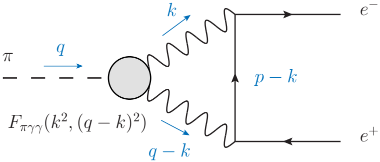

$$\frac{BR(\pi \to e^+e^-)}{BR(\pi \to \gamma\gamma)} = 2\left(\frac{\alpha m_e}{\pi m_\pi}\right)^2 \beta_e(m_\pi^2)|\mathcal{A}(m_\pi^2)|^2,$$

$$\mathcal{A}(q^2) = \frac{2i}{\pi^2 q^2}\int d^4k \, \frac{q^2k^2-(k\cdot q)^2}{k^2(k-q)^2((p-k)^2-m_e^2)} F_{\pi\gamma\gamma}(k^2,(q-k)^2).$$

With normalized $F_{\pi\gamma\gamma}(0,0)=1$. It diverges if $F_{\pi\gamma\gamma}(k_1^2,k_2^2)=const.$

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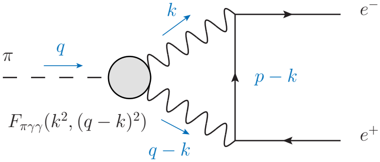

$$\frac{BR(\pi \to e^+e^-)}{BR(\pi \to \gamma\gamma)} = 1.55 \times 10^{-10} |\mathcal{A}(m_\pi^2)|^2,$$

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Drell (1958) obtained the imaginary part based on unitarity ideas:

$$Im(\mathcal{A}) \propto \int d\Pi_{\gamma\gamma} \langle\gamma\gamma|T|\pi\rangle \; \langle\gamma\gamma|T|e^+e^-\rangle^*$$

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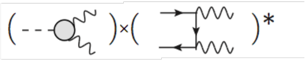

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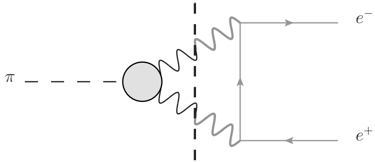
$$Im\mathcal{A}(q^2) = \frac{\pi}{2\beta_e(q^2)} ln\left(\frac{1-\beta_e(q^2)}{1+\beta_e(q^2)}\right); \quad \beta_e(q^2) = \sqrt{1-\frac{4m_e^2}{q^2}}$$

Indeed, this can be deduced also from Cutkosky rules



This result is model-independent, furthermore:

$$|\mathcal{A}|^2 = Re(\mathcal{A})^2 + Im(\mathcal{A})^2 \geq Im(\mathcal{A})^2 = (-17.52)^2; BR(\pi \to e^+e^-) \geq 4.7\times 10^{-8}.$$

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To go further, he postulated an unsubstracted dispersion relation:

$$Re(\mathcal{A}(t)) = \frac{1}{\pi}\int_0^{\Lambda^2} \frac{Im(\mathcal{A}(s))}{s-t} ds,$$

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Leads to $BR = \{6.19, 13.5, 16.7\} \times 10^{-8}$ for $\Lambda = \{0.3, 0.77, 1\}$GeV.

Berman and Geffen (1959) proposed a model for the $F_{\pi\gamma\gamma}(k_1^2, k_2^2)$ based on analytical properties:

$$F_{\pi\gamma\gamma}(k_1^2, k_2^2) = \frac{M^2}{M^2 - k_1^2 - k_2^2}.$$

They calculated the integral to $\mathcal{O}\left(\frac{m_\pi^2}{M^2}, \frac{m_e^2}{M^2}, \frac{m_\pi^2}{M^2} ln\left(\frac{m_e^2}{M^2}\right)\right)$ and obtained very stable results under M^2 variations.

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Similar results found later in the $60's$ with the success of VMD models:

$$F_{\pi\gamma\gamma}(k_1^2, k_2^2) = \frac{M_V^4}{(M_V^2 - k_1^2)(M_V^2 - k_2^2)}.$$

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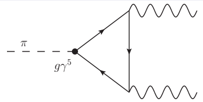

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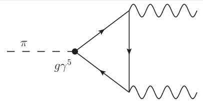

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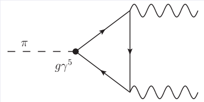

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Large $N_c + \chi PT$:
$Re\mathcal{A} \sim 10, BR \sim 6.1 \times 10^{-8}$.
Knecht et al ('99) PRL83

Current Situation

Most recent experimental result $BR(\pi \rightarrow e^+e^-) = 7.5(4) \times 10^{-8}$ from KTeV collaboration (2007) is still different from theory ($\sim 3\sigma$). The difference is $\sim$ 20% $BR(\pi \rightarrow e^+e^-)^{th.}$

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We will understand this insensitivity to $F_{\pi\gamma\gamma}$ in next section.

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To show this, we obtain $Re(\mathcal{A})$ to $\mathcal{O}\left(\frac{m_\pi^2}{\Lambda^2}, \frac{m_e^2}{\Lambda^2}, \frac{m_\pi^2}{\Lambda^2} ln\left(\frac{m_e^2}{\Lambda^2}\right)\right)$ terms)

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$$Re(\mathcal{A}) \approx \underbrace{\left(\int ...\right)}_{F_{\pi\gamma\gamma}\ \text{dep.}} \underbrace{-\frac{5}{4} + \frac{1}{\beta_e}\left(\frac{\pi^2}{12} + \frac{1}{4} ln^2\left(\frac{1-\beta_e}{1+\beta_e}\right) + Li_2\left(\frac{1-\beta_e}{1+\beta_e}\right)\right)}_{F_{\pi\gamma\gamma}\ \text{indep.}}.$$

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If we want to reproduce the experiment, we need $Re(\mathcal{A}) = 13.5$.

$$Re(\mathcal{A}) \approx 30.7 + \int_0^\infty dQ \frac{3}{Q} \left(\frac{m_e^2}{m_e^2 + Q^2} - F_{\pi\gamma\gamma}(Q^2, Q^2) \right).$$

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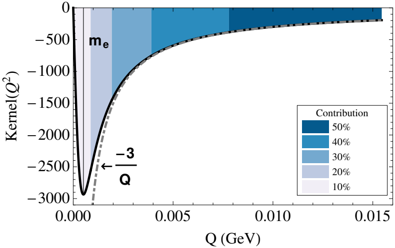

- Negative defined
  - $\to$ Partial cancellation.
- Singularity from $\gamma\gamma$ $\frac{1}{Q}$ suppression
  - $\to$ Low energies relevant.
- Peak at lepton mass
  - $\to$ IR regulator $\sim ln(m_e^2)$.
- High energies, $F_{\pi\gamma\gamma}$ dominates
  - $\to$ UV regulator $\sim -ln(\Lambda^2)$.

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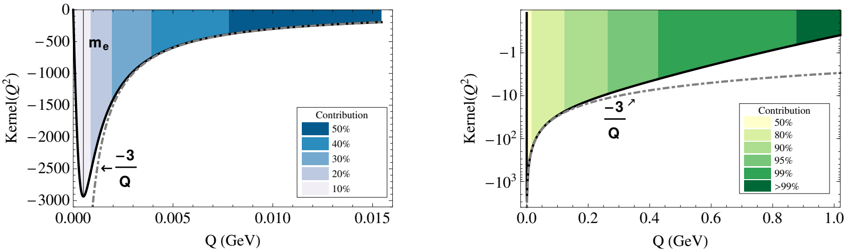

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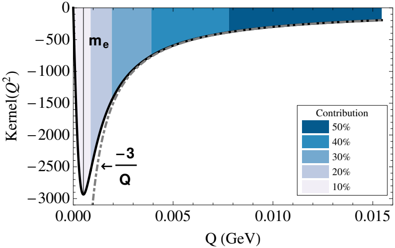

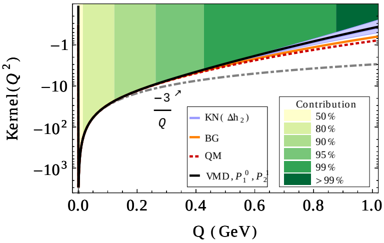

$\mathbf{BG} \sim \frac{M^2}{M^2+2Q^2}$; $\quad \mathbf{P_2^1} \sim \frac{(M_1^2M_2^2+\lambda Q^2)^2}{(M_1^2+Q^2)^2(M_2^2+Q^2)^2}$; $\quad \mathbf{QM} \sim \left(\frac{M_Q}{Q}\right)^2 ln\left(1+\left(\frac{Q}{M_Q}\right)^2\right)$

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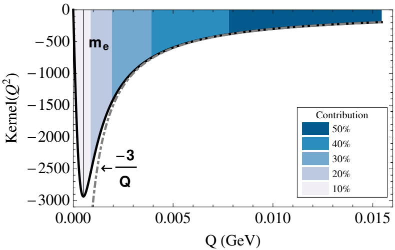

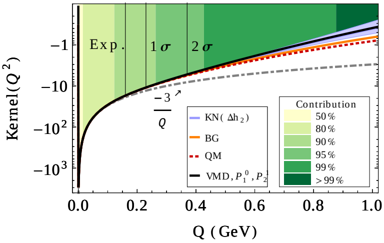

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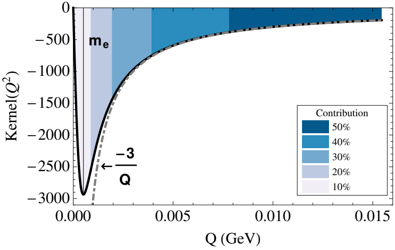

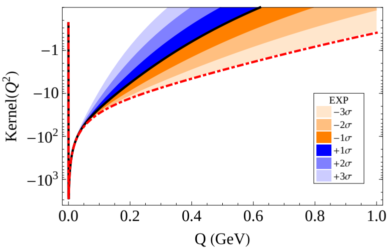

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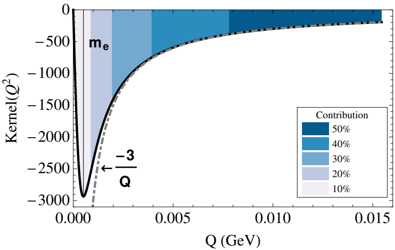

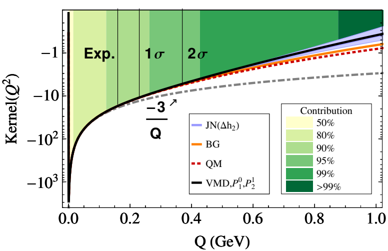

To reproduce the experimental value we need

- Different behavior at low energies (@∼200MeV).
- Rapidly falling $F_{\pi\gamma\gamma}$.

To get an accurate result we need

- Good description of low energy pars. (i.e.: taylor exp. for $F_{P\gamma\gamma}(Q^2,Q^2)$).

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$$P_1^0 : Re(\mathcal{A}) = 10.46(12); \quad P_2^1 : Re(\mathcal{A}) = 10.46(16)$$

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$$P_1^0 : BR = 6.36(4) \times 10^{-8}; \quad P_2^1 : BR = 6.36(5) \times 10^{-8}$$

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  $\rightarrow$ Not available.
- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\rightarrow F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).
- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\to$ Not available.
- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).
- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.
  $$BR = 6.36(5) \times 10^{-8}$$
- To compare other models, use the same slope for all of them:
  $$\textbf{BG}: \; BR = 6.22 \times 10^{-8}, \quad \textbf{QM}: \sim 6.08 \times 10^{-8}.$$

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\to$ Not available.
- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).
- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- To compare other models, use the same slope for all of them:

$$\textbf{BG}: \; BR = 6.22 \times 10^{-8}, \quad \textbf{QM}: \sim 6.08 \times 10^{-8}.$$

- Differences because of different (non-factorizable) Q^2 dependence at low energies.

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\to$ Not available.

- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).

- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- To compare other models, use the same slope for all of them:

$$\mathbf{BG}: \; BR = 6.22 \times 10^{-8}, \quad \mathbf{QM}: \sim 6.08 \times 10^{-8}.$$

- Differences because of different (non-factorizable) Q^2 dependence at low energies.
- Assume behavior may be parametrized by C_1^0 Chisholm approximants

$$C_1^0(q_1^2, q_2^2) = P_1^0(q_1^2 + q_2^2).$$

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\to$ Not available.
- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).

- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- Compute result for Chisholm approximants.

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\to$ Not available.

- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).

- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- Compute result for Chisholm approximants.

$$P_1^0 : Re(\mathcal{A}) = 10.00(12);$$

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\to$ Not available.

- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).

- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- Compute result for Chisholm approximants.

$$P_1^0 : Re(\mathcal{A}) = 10.00(12); \quad P_2^1 : Re(\mathcal{A}) = 9.98(19)$$

PREDICTIONS FROM A LOW ENERGY APPROACH

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\to$ Not available.

- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).

- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- Compute result for Chisholm approximants.

$$P_1^0 : BR = 6.22(4)10 \times 10^{-8}; \quad P_2^1 : BR = 6.21(6) \times 10^{-8}$$

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data. $\to$ Not available.
- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86) $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).
- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- Compute result for Chisholm approximants.

$$BR = 6.22(7)10 \times 10^{-8}$$

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data. $\to$ Not available.
- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86) $\to F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).
- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

- Compute result for Chisholm approximants.

$$BR = 6.22(7)10 \times 10^{-8}$$

We get then $BR = (6.36(5) \div 6.22(7)) \times 10^{-8}$
Previously, most precise value $BR = 6.23(12) \times 10^{-8}$ (Dorokhov, PLB677).

Predictions from a low energy approach

- Reconstruct $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ from low energy pars. from exp. data.
  $\rightarrow$ Not available.

- Reconstruct $F_{\pi\gamma\gamma}(Q^2, 0) \equiv F_{\pi\gamma}(Q^2)$ instead (P.Masjuan, Phys.Rev.D86)
  $\rightarrow F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = F_{\pi\gamma}(Q_1^2) \times F_{\pi\gamma}(Q_2^2)$ (factorization + Padé App.).

- Compute exact numerical result: $\mathcal{O}(2,1)\%$, corrections to $(Re(\mathcal{A}), BR)$.

$$BR = 6.36(5) \times 10^{-8}$$

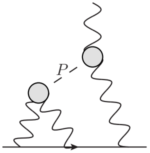
- Compute result for Chisholm approximants.

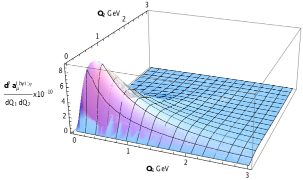
$$BR = 6.22(7)10 \times 10^{-8}$$

We get then $BR = (6.36(5) \div 6.22(7)) \times 10^{-8}$
Previously, most precise value $BR = 6.23(12) \times 10^{-8}$ (Dorokhov, PLB677).

- Future measurements on $F_{\pi\gamma\gamma}(Q_1^2, Q_2^2)$ could test these assumptions.
- Straightforward extension to the η.

DETAILS ON $HLbyL^{\pi^0-exchange}$





- $F_{\pi\gamma\gamma}(m^2; Q_1^2, Q_2^2) \times F_{\pi\gamma\gamma}(m^2; Q_3^2, 0)$ combination appears in loops.
- At least we can test $F_{\pi\gamma\gamma}(m_\pi^2; Q_1^2, Q_2^2)$ in a loop (but no π-off-shellness).

- Dominant contribution with positive defined kernel.
- Peaked at low energies too.
- Falls rapidly with $F_{\pi\gamma\gamma}$.
- Fast(slow) decreasing $\to$ lower(higher) value $\to$ opposite to $\pi \to e^+e^-$.

DETAILS ON $HLbyL^{\pi^0-exchange}$

We show the implication on different models

	Model	Published model		Modified model	
		$\pi^0 \to e^+e^-$ ($\times 10^8$)	*HLBL* ($\times 10^{10}$)	$\pi^0 \to e^+e^-$ ($\times 10^8$)	*HLBL* ($\times 10^{10}$)
Knecht and Nyffeler	LMD+V	6.32	6.29	6.40	5.22
Dorokhov et al	VMD	6.34	5.64	7.50	2.44
Our proposal	PA + non-fact.	6.36	5.55	6.41	5.54

In the first model we modify h_2 term (originally unconstrained) to 12.16.

In the second we take the only parameter $M_{VMD} = 330$ MeV.

Last, we add to our factorized expr. an additional non-fact. term.

Conclusions & Outlook

- $\pi \to e^+e^-$ and $a_\mu^{HLbL\pi^0}$ complementary and peaked at low energies.
- Syst. approach based on low energy pars. is an ideal tool.
- Factorization or not? (OPE and Sum Rule constrains).
- To be probed by experiments: BESIII, KLOE-2?
- Latest Exp. result (KTeV) shows $\sim 3\sigma$ discrepancy.
- Radiative corrections are important too.
- Precision a first step towards constraining new physics.
- Extension to the η shows slight disagreement too.
- Extension to the η'.
- Has the lattice a word on it?

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 2\left(\frac{\alpha m_\mu}{\pi m_\eta}\right)^2 \beta_\mu(m_\eta^2)|\mathcal{A}(m_\eta^2)|^2,$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + Im(\mathcal{A})^2$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result:$\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

  $P_1^0 : Re(\mathcal{A}) = -0.99(5);$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7}|\mathcal{A}(m_\eta^2)|^2,$$
$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$P_1^0 : Re(\mathcal{A}) = -0.99(5); \quad P_2^1 : Re(\mathcal{A}) = -1.01(6)$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$P_1^0 : BR = 4.51(1) \times 10^{-6}; \quad P_2^1 : BR = 4.51(2) \times 10^{-6}$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 4.51(2) \times 10^{-6}$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 4.51(2) \times 10^{-6}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$
$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 4.51(2) \times 10^{-6}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7}|\mathcal{A}(m_\eta^2)|^2,$$
$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$
$$BR = 4.51(2) \times 10^{-6}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.
$$P_1^0 : Re(\mathcal{A}) = -1.52(5);$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 4.51(2) \times 10^{-6}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.

$$P_1^0 : Re(\mathcal{A}) = -1.52(5); \quad P_2^1 : Re(\mathcal{A}) = -1.57(7)$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 4.51(2) \times 10^{-6}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.

$$P_1^0 : BR = 4.70(2)10 \times 10^{-6}; \quad P_2^1 : BR = 4.73(3) \times 10^{-6}$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$
$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$
$$BR = 4.51(2) \times 10^{-6}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.
$$BR = 4.71(3)10 \times 10^{-6}$$

$\eta \to \mu^+\mu^-$

$$\frac{BR(\eta \to \mu^+\mu^-)}{BR(\eta \to \gamma\gamma)} = 3.7 \times 10^{-7} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 5.47^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 4.51(2) \times 10^{-6}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.

$$BR = 4.71(3)10 \times 10^{-6}$$

We get then $BR = 4.51(2) \div 4.71(3) \times 10^{-6}$.

$BR^{Exp} = 5.8(8) \times 10^{-6}$.

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 2\left(\frac{\alpha m_e}{\pi m_\eta}\right)^2 \beta_e(m_\eta^2)|\mathcal{A}(m_\eta^2)|^2,$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12} |\mathcal{A}(m_\eta^2)|^2,$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + Im(\mathcal{A})^2$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result:$\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

  $P_1^0 : Re(\mathcal{A}) = 31.51(11);$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$P_1^0 : Re(\mathcal{A}) = 31.51(11); \quad P_2^1 : Re(\mathcal{A}) = 31.50(11)$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$P_1^0 : BR = 5.45(2) \times 10^{-9}; \quad P_2^1 : BR = 5.45(3) \times 10^{-9}$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 5.45(3) \times 10^{-9}$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 5.45(3) \times 10^{-9}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 5.45(3) \times 10^{-9}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12} |\mathcal{A}(m_\eta^2)|^2,$$

$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$

$$BR = 5.45(3) \times 10^{-9}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.

$P_1^0 : Re(\mathcal{A}) = 30.95(11);$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12}|\mathcal{A}(m_\eta^2)|^2,$$
$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$
$$BR = 5.45(3) \times 10^{-9}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

- Compute result for Chisholm approximants.
$$P_1^0 : Re(\mathcal{A}) = 30.95(11); \quad P_2^1 : Re(\mathcal{A}) = 30.34(12)$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12} |\mathcal{A}(m_\eta^2)|^2,$$
$$|\mathcal{A}(m_\eta^2)| = Re(\mathcal{A})^2 + 21.92^2$$

- Compute exact numerical result: $\mathcal{O}(90, 10)\%$ corrections to $(Re(\mathcal{A}), BR)$
$$BR = 5.45(3) \times 10^{-9}$$

Previously, most precise value $\{5.27(12), 4.64\} \times 10^{-6}$ (Dorokhov, PLB677).

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$$P_1^0 : BR = 5.32(2)10 \times 10^{-9}; \quad P_2^1 : BR = 5.18(3) \times 10^{-9}$$

$\eta \to e^+e^-$

$$\frac{BR(\eta \to e^+e^-)}{BR(\eta \to \gamma\gamma)} = 9.4 \times 10^{-12} |\mathcal{A}(m_\eta^2)|^2,$$

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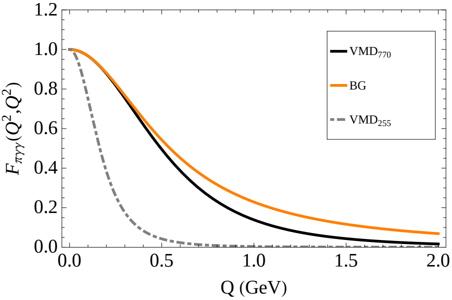
We get then $BR = 5.45(3) \div 5.25(9) \times 10^{-9}$.
$BR^{Exp} \leq 5.6 \times 10^{-6}$.

$F_{\pi\gamma\gamma}(Q^2, Q^2)$ Plots



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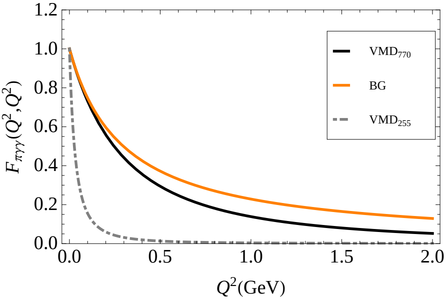

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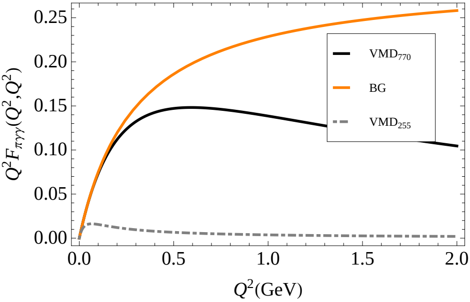

Details on Knecht and Nyffeler model

$$\mathbf{KN} = \frac{4\pi^2 F_\pi^2}{3} \frac{\frac{3M_{V1}^4 M_{V2}^4}{4\pi^2 F_\pi^2} + h_5(Q_1^2 + Q_2^2) - h_2 Q_1^2 Q_2^2 + Q_1^2 Q_2^2 (Q_1^2 + Q_2^2)}{(M_{V1}^2 + Q_1^2)(M_{V2}^2 + Q_1^2)(M_{V1}^2 + Q_2^2)(M_{V2}^2 + Q_2^2)}$$

Their parameter h_2 is, in general, non-factorizable
Knecht Nyffeler used $h_2 \in \{-10, 0, 10\} \to a_\mu^{LbyL;\pi^0} \in \{6.3, 5.5, 5.3\}$
Favoring $\pi \to e^+e^-$ implies $h_2 > 0$, its max. $12.2 \to a_\mu^{LbyL;\pi^0} = 5.22$

Padé Extension

$$\frac{(M_1^2 M_2^2 + \lambda Q_1^2)(M_1^2 M_2^2 + \lambda Q_2^2) + \gamma Q_1^2 Q_2^2}{(M_1^2 + Q_1^2)(M_2^2 + Q_1^2)(M_1^2 + Q_2^2)(M_2^2 + Q_2^2)}$$

The lowest value for γ is $-\lambda^2$, which gives the closest value to the Exp. BR.

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The parameter Λ gives the scale of the OPE behavior.
For $\Lambda = 0 \approx$ BG, for $\Lambda = \infty \approx$ VMD.

Remarks on OPE Behavior

$$F_{\pi\gamma\gamma}(Q^2, Q^2) = \frac{M^4 + \frac{M^2 Q^4}{3(Q^2+\Lambda^2/2)}}{(M^2+Q^2)^2}$$

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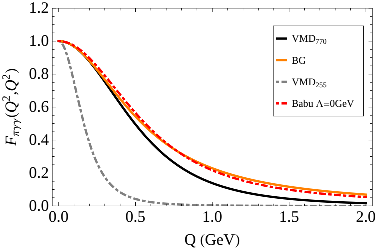
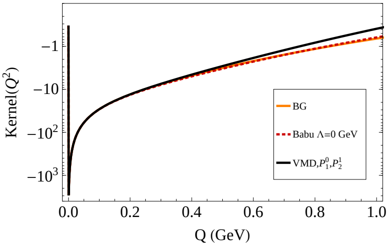


Figure: $\Lambda = 0\ GeV$.



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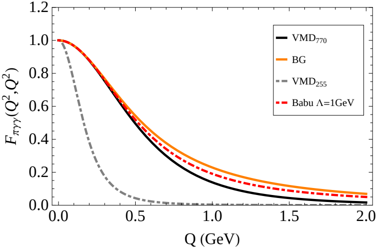
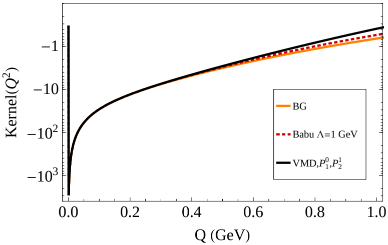


Figure: $\Lambda = 1$ GeV.



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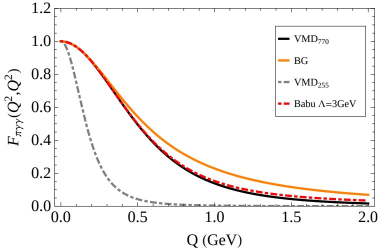
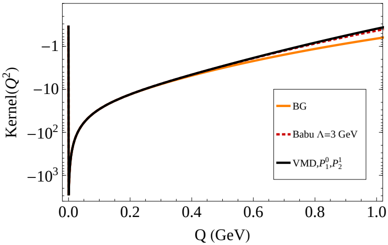


Figure: $\Lambda = 3$ GeV.



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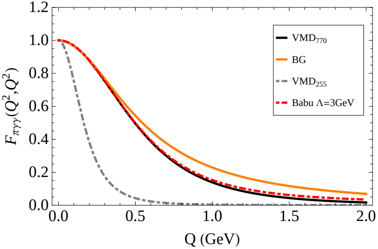

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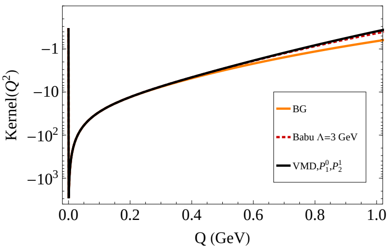

Figure: $\Lambda = 3$ GeV.

For $\Lambda \ll M$ correction to VMD $\approx -\frac{1}{2} + \frac{\Lambda^2}{4M^2}(ln\frac{2M^2}{\Lambda^2} - 1)$.
For $\Lambda \gg M$ correction to VMD $\approx 0 - \frac{M^2}{\Lambda^2}(ln(\frac{\Lambda^2}{2M^2}) - 1)$.

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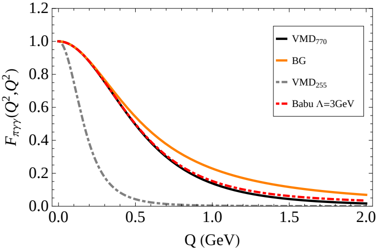

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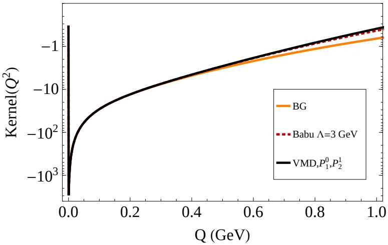

Figure: $\Lambda = 3$ GeV.

The relevant point is to identify in which region the $\frac{1}{Q^2}$ behavior is relevant.
Even if is not exactly the OPE (i.e. BG).

Radiative Corrections

Experimentally, there is huge background from Dalitz decay.

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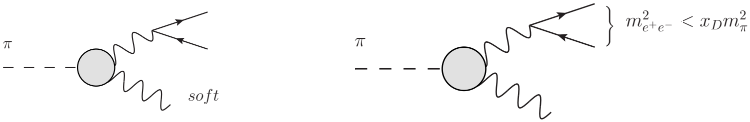

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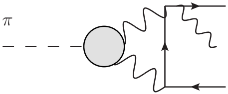

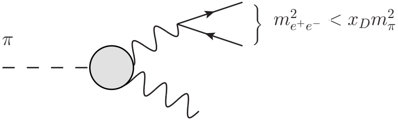

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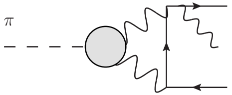
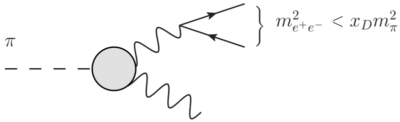




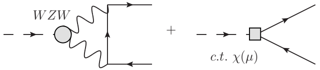
But also contribution from our process.

In exp., $BR^{x_D=0.95} = 6.44(25)(22)10^{-8} \xrightarrow{RC} BR^{KTeV} 7.48(29)(25) \times 10^{-8}$

RC from Bergström ZPC20 (1983); Dorokhov et al, EPJC55 (2008); agree.
Result from Vasko and Novotny, JHEP10 (2011): smaller correction.

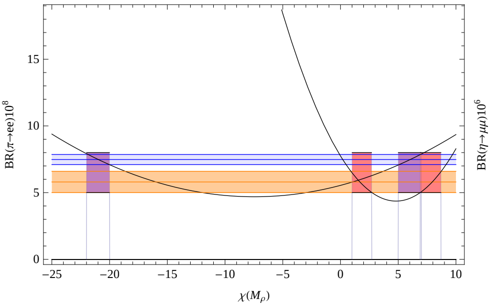
χPT Approach

In χPT we have the WZW constant $F_{\pi\gamma\gamma}$ and a counterterm from higher order



$$Re(\mathcal{A}) \approx -\frac{5}{2} + \frac{3}{2} ln\left(\frac{m_e^2}{\mu^2}\right) + \frac{\pi^2}{12} + \left(\frac{m_e^2}{m_\pi^2}\right) + \chi(\mu)$$

Obtain from one pseudoscalar decay assuming $SU(3)$ symmetry.



Dispersion Formalism

The imaginary part from the photon channel only is

$$Im\mathcal{A}(q^2) = \frac{\pi}{2\beta_e(q^2)} ln\left(\frac{1-\beta_e(q^2)}{1+\beta_e(q^2)}\right); \quad \beta_e(q^2) = \sqrt{1-\frac{4m_e^2}{q^2}}$$

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→ Use of substractions.

$$Re(\mathcal{A}(q^2)) = \mathcal{A}(0) + \frac{q^2}{\pi}\int_0^\infty \frac{Im(\mathcal{A}(s))}{s(s-q^2)} ds.$$

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Substraction constant must be theoretically calculated.
Some corrections from non-included hadronic channels.