Using analytical continuation for a_μ^{hvp}

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in collaboration with

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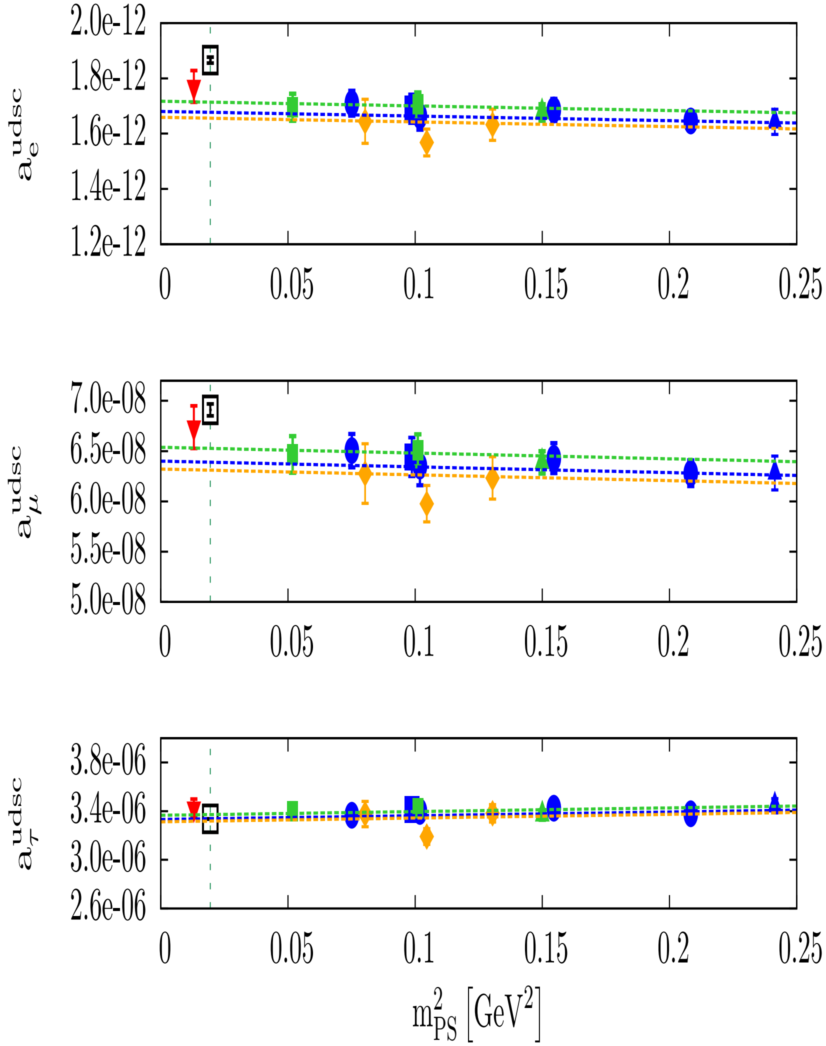
- **Status of standard a_μ^{hvp} calculation**
- **Analytical continuation**
- **Example of a_μ^{hvp}**
- **Conclusion**

The full four-flavour contribution for leptons

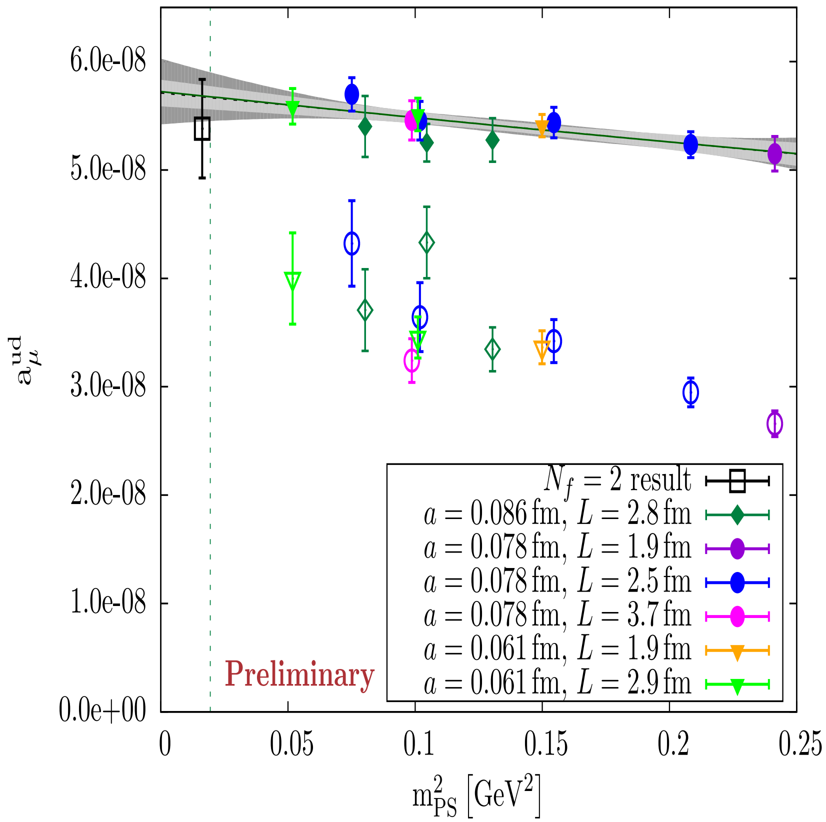


- fit function:

$$a_\mu(m_{\mathrm{PS}}, a) = A + B\ m_{PS}^2 + C\ a^2$$

- maximal twist: only $O(a^2)$ effects
- full analysis of short distance singularities $\rightarrow$ $O(a)$-improvement not spoiled

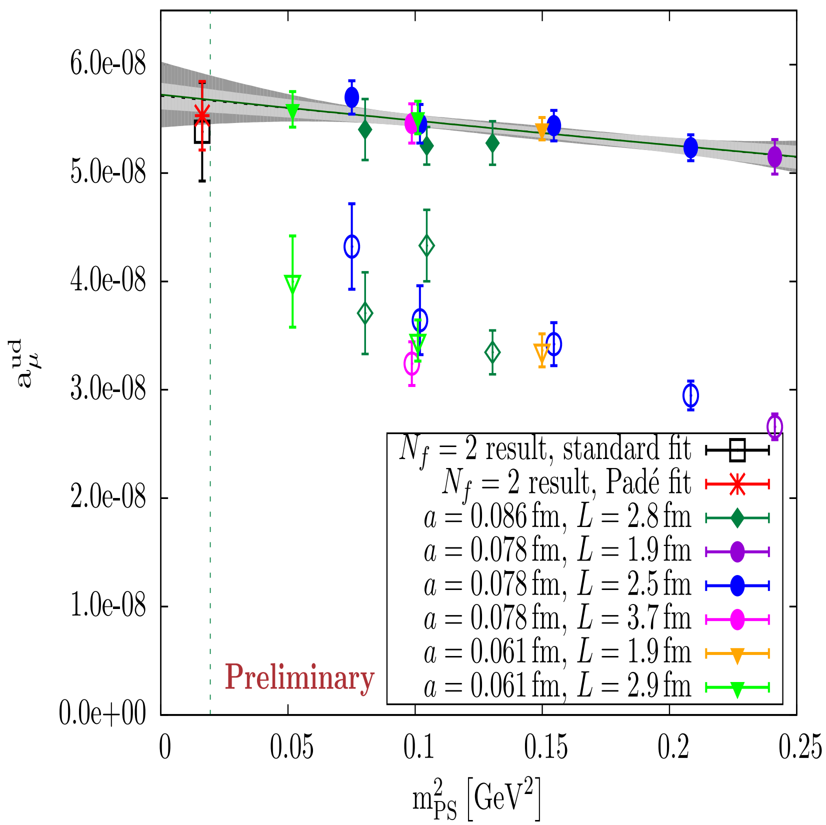
Light contribution at the physical point



- modified method
- standard method

- VMD and polynomial fit

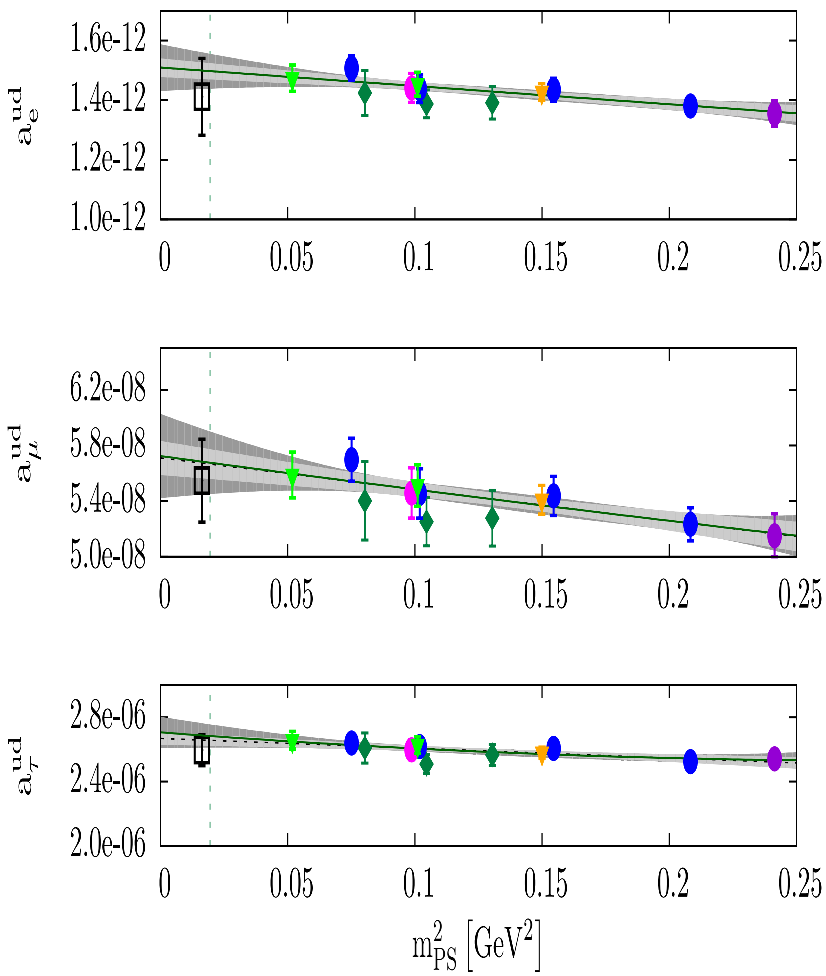
Light contribution at the physical point



- modified method
- standard method

- VMD and polynomial fit
- compare to Padé fit

Light contribution all leptons



$$a_{\rm e}^{\rm hvp} = 1.50(03)10^{-12} \quad (N_f = 2+1+1)$$

$$a_\mu^{\rm hvp} = 5.67(11)10^{-8} \quad (N_f = 2+1+1)$$

$$a_\tau^{\rm hvp} = 2.66(02)10^{-6} \quad (N_f = 2+1+1)$$

- fit function:

$$a_\mu(m_{\rm PS}, a) = A + B\ m_{PS}^2 + C\ a^2$$

Alternative method: analytic continuation

Compute HVP function via analytic continuation

$$\bar{\Pi}(K^2)(K_\mu K_\nu - \delta_{\mu\nu}K^2) = \int dt\, e^{\omega t} \int d^3\vec{x}\, e^{i\vec{k}\vec{x}}\, \langle\Omega|T\{J_\mu^E(\vec{x},t)J_\nu^E(\vec{0},0)\}|\Omega\rangle$$

- $J_\mu^E(X)$ electromagentic current
- $K = (\vec{k}, -i\omega)$, $\vec{k}$ spatial momentum, ω the photon energy (input)

Advantage

- vary $\omega \rightarrow$ smooth values for $K^2 = -\omega^2 + \vec{k}^2$
- can cover space-like and time-like momentum regions
- can reach small momenta and even zero momentum
- important condition:

$$-K^2 = \omega^2 - \vec{k}^2 < M_V^2\ , \quad \text{or} \quad \omega < E_{\mathrm{vector}}$$

- make use of ideas: (Ji; Meyer; X. Feng, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, J. Noaki, E. Shintani; G. de Divitiis, R. Petronzio, N. Tantalo)

Fourier Transformation

- spatial transformation

$$C_{\mu\nu}(\vec{k},t) = \sum_{\vec{x}} e^{-i\vec{k}(\vec{x}+a\hat{\mu}/2-a\hat{\nu}/2)} \, \langle J_\mu^E(\vec{x},t) J_\nu^E(\vec{0},0)\rangle \ ,$$

- discrete momenta

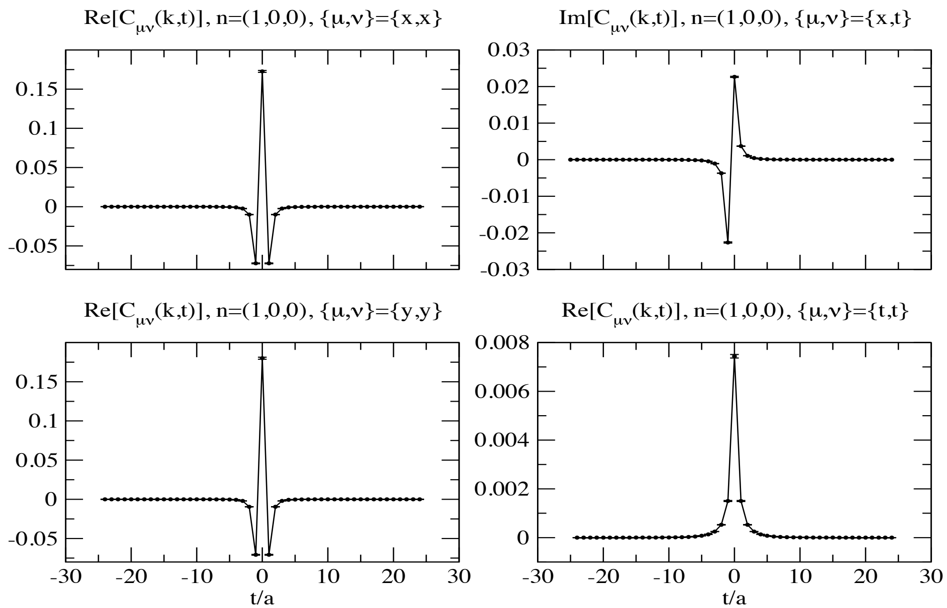
$$\vec{k} = (2\pi/L)\vec{n}$$

- transformation in time

$$\bar{\Pi}_{\mu\nu}(\vec{k},\omega;T) = \sum_{t=-T/2}^{T/2} e^{\omega(t+a(\delta_{\mu,t}-\delta_{\nu,t})/2)} C_{\mu\nu}(\vec{k},t)$$

$$\bar{\Pi}(K^2;T)\left(K_\mu K_\nu - \delta_{\mu\nu}K^2\right) = \bar{\Pi}_{\mu\nu}(\vec{k},\omega;T)$$

Correlators for different polarization



- very different behaviour for different μ, ν
- all lead to the same result eventually

Truncating of timeline transformation: introducing a finite size effect

- problem for large t: correlator very noisy

- truncate time summation: $t_{\max} = \eta T/2$

$$\bar{\Pi}(K^2; t_{\max}) \left(K_\mu K_\nu - \delta_{\mu\nu} K^2\right) = \bar{\Pi}_{\mu\nu}(\vec{k}, \omega; t_{\max})$$

$$\bar{\Pi}_{\mu\nu}(\vec{k}, \omega; t_{\max}) = \sum_{t=-t_{\max}}^{t_{\max}-a(\delta_{\mu,t}-\delta_{\nu,t})} e^{\omega(t+a(\delta_{\mu,t}-\delta_{\nu,t})/2)} C_{\mu\nu}(\vec{k}, t)$$

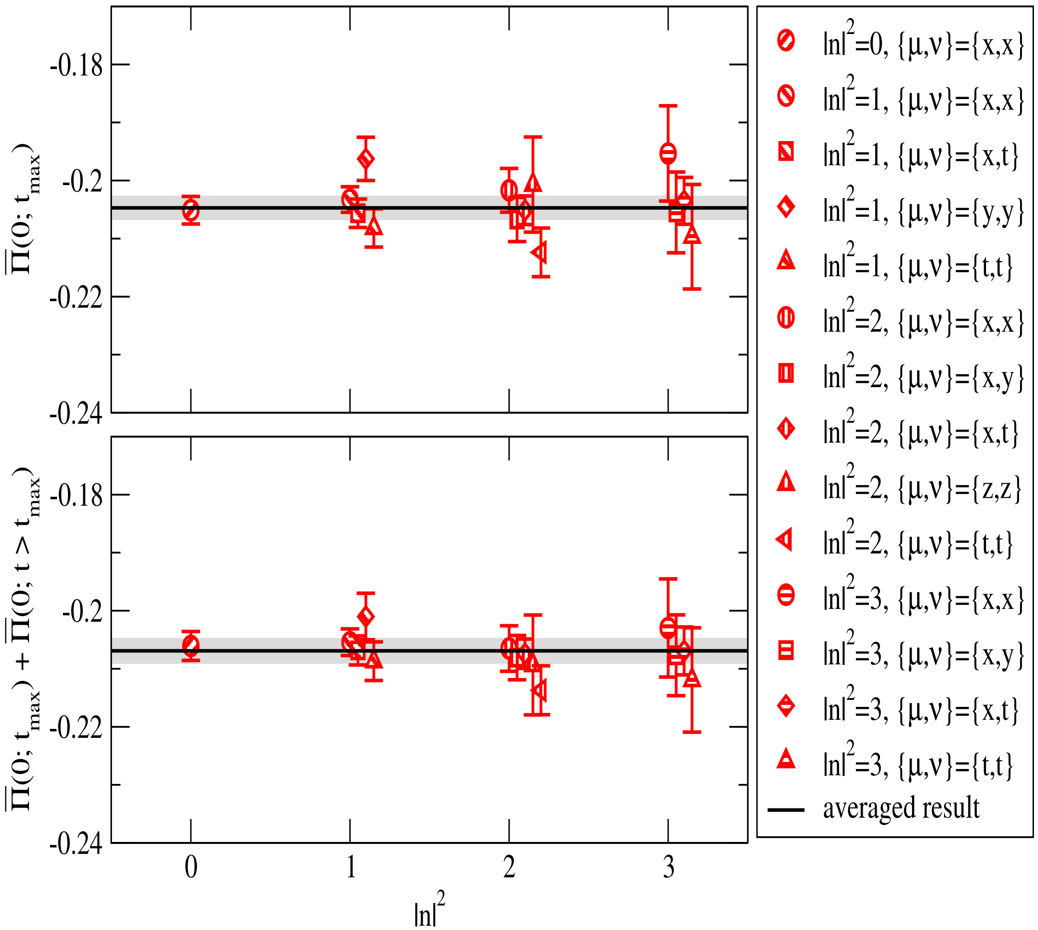
  - for each fixed η method correct for $T \to \infty$
  - for $\eta \neq 1$ introduce a *finite size effect*

- for $t > t_{\max}$: describe data by model

- Here:

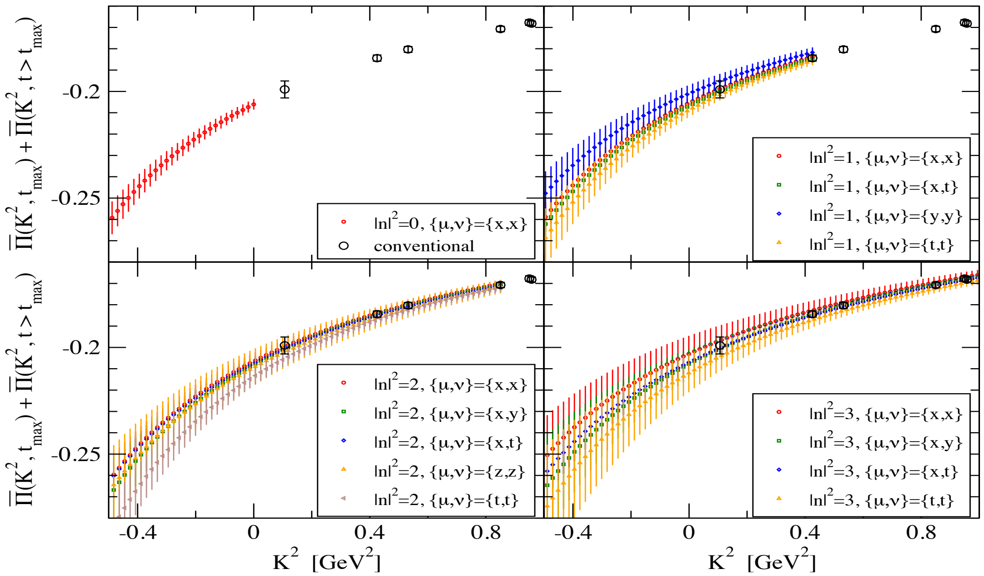
  - choice of $\eta = 3/4$
  - assume ground state dominance for large t (ρ-mass)

Demonstration of $\vec{n}$ indpendence



- increasing error for larger $\vec{n}^2$

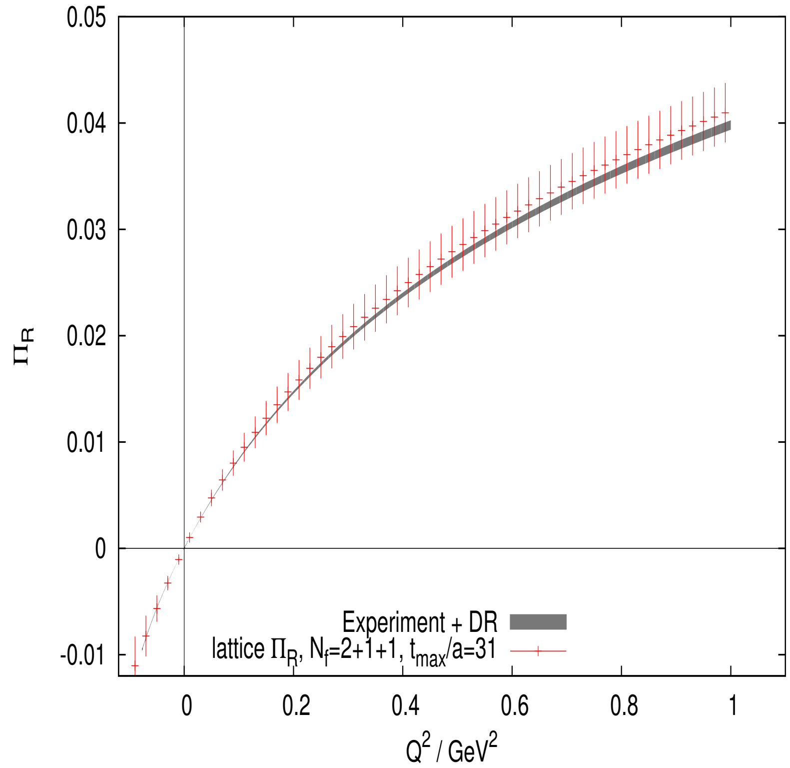
HVP from analytical continuation

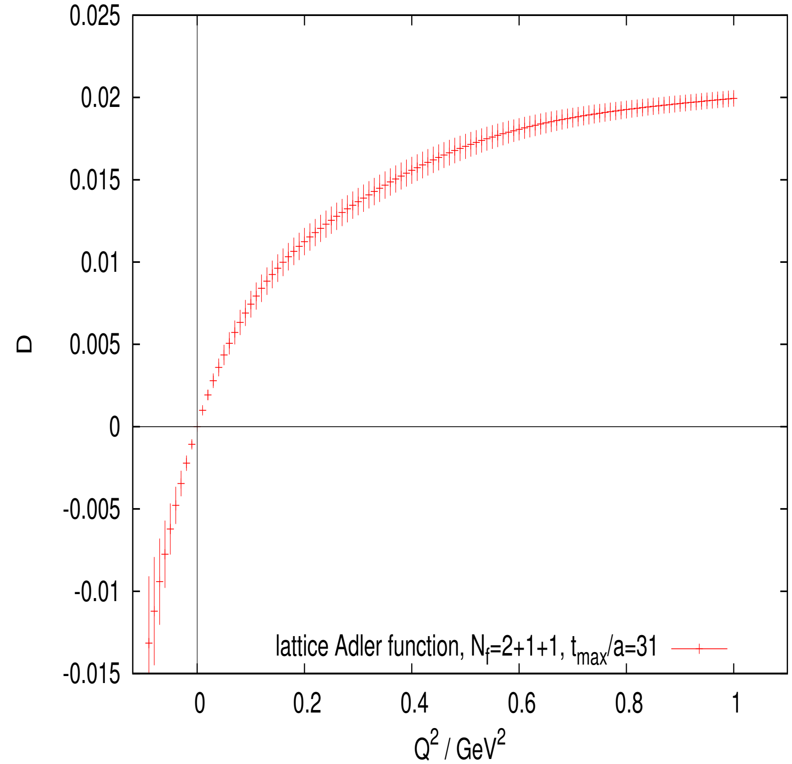


- different $\vec{n}$ lead to consistent results
- agreement with standard calculation
- however, larger errors for $|\vec{n}| > 0$

Direct application to vacuum polarization function

parameters: $(a \approx 0.078\,\mathrm{fm},\, V = (2.5\mathrm{fm})^3)$



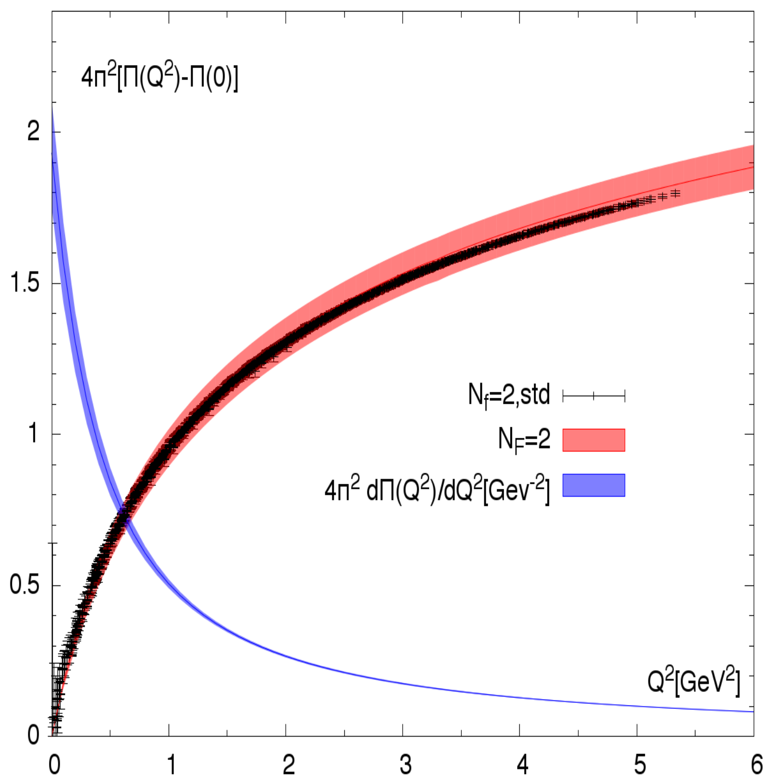


renormalized HVP
dispersion relation (Jegerlehner, 2011)

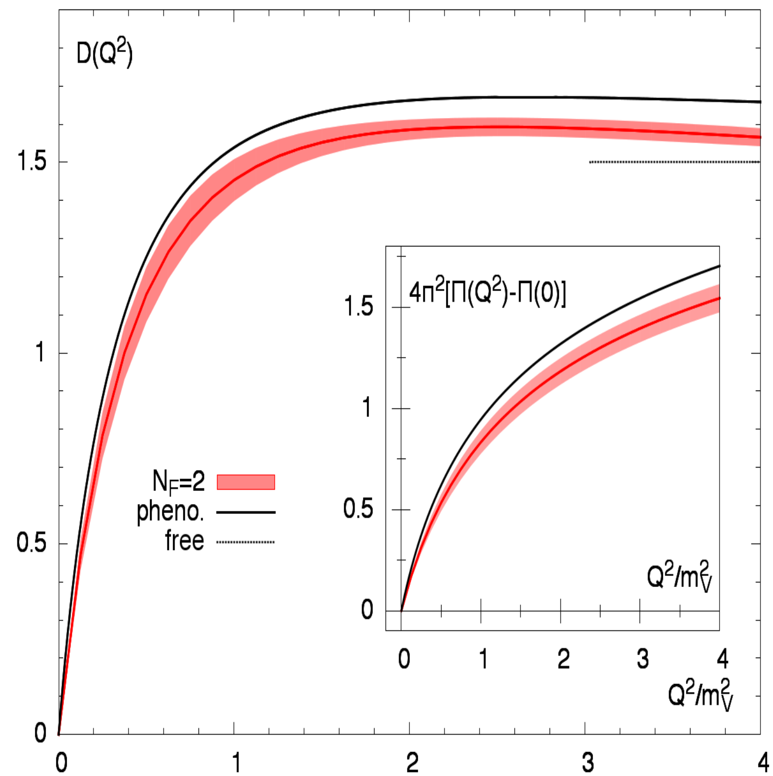
Adlerfunction

Mixed time-momentum representation

(A. Francis, B. Jäger, H. Meyer, H. Wittig)



renormalized HVP



Adlerfunction

Application to $a_\mu^{\rm hvp}$

split in three pieces

- $a_{\bar\mu}^{(1)}$ directly calculable from lattice data

$$a_{\bar\mu}^{(1)} = \alpha^2 \int_0^{K^2_{\rm max}} dK^2 \, \frac{1}{K^2} f\left(\frac{K^2}{m_\mu^2}\frac{m_\rho^2}{m_V^2}\right)\left(\Pi(K^2) - \Pi(0)\right)$$

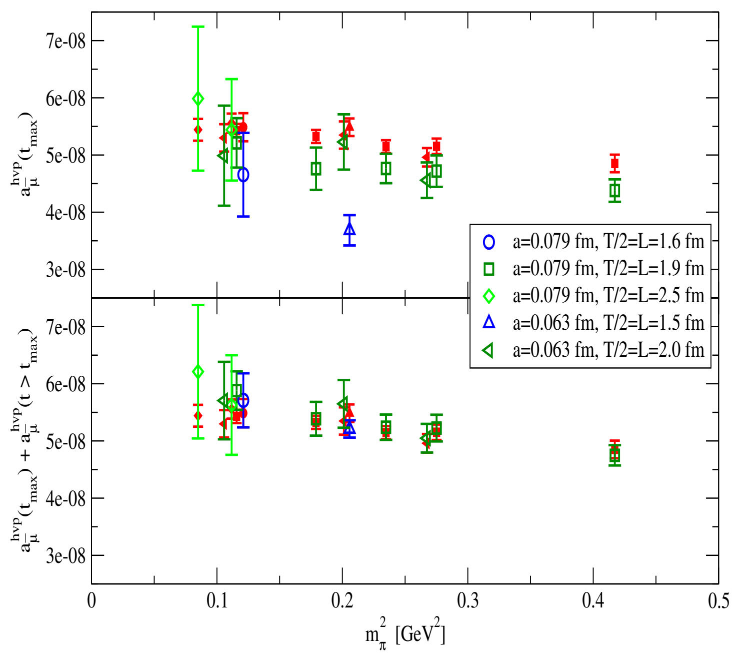
- $a_{\bar\mu}^{(2)}$ only large momentum region: model dependence

$$a_{\bar\mu}^{(2)} = \alpha^2 \int_{K^2_{\rm max}}^{\infty} dK^2 \, \frac{1}{K^2} f\left(\frac{K^2}{m_\mu^2}\frac{m_\rho^2}{m_V^2}\right)\left(\Pi(K^2) - \Pi(K^2_{\rm max})\right)$$

- $a_{\bar\mu}^{(3)}$ correction term

$$a_{\bar\mu}^{(2)} = \alpha^2 \int_{K^2_{\rm max}}^{\infty} dK^2 \, \frac{1}{K^2} f\left(\frac{K^2}{m_\mu^2}\frac{m_\rho^2}{m_V^2}\right)\left(\Pi(K^2_{\rm max}) - \Pi(0)\right)$$

Comparison to standard calculation



without FSE

with FSE

- open symbols: analytic continuation
- filled symbols: standard calculation of $a_{\bar{\mu}}$
- averaged over different polarizations

Summary

- Tested idea of analytical continuation method for computing vacuum polarisation function
  - validity of method demonstrated in 1305.5878
  - method works in practise
- difficulties
  - had to truncate time summation $\rightarrow$ induce finite size effect
  - method only applicable for momenta $K < K_{\rm max}$ with $-K^2 = \omega^2 = k^2 < M_V^2$ (or, $\omega < E_V$)
  - larger errors than standard method for $|\vec{n}| > 0$
- my present view on analytical continuation method:
  - it is clearly an alternative for cross-checking, e.g. $a_\mu^{\rm hvp}$
  - it allows a direct comparison to the hvp function from phenomenological analysis of data
  - maybe method of choice at physical pion mass?
- can it be applied to describe momentum dependence where value at $Q^2 = 0$ is not available?