Large-N_C inspired approach to hadronic light-by-light scattering in the muon $g - 2$

Andreas Nyffeler*

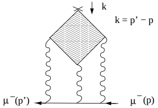
Mainz Institute for Theoretical Physics (MITP) Topical Workshop
Hadronic contributions to the muon anomalous magnetic moment: strategies for improvements of the accuracy of the theoretical prediction
Schloss Waldthausen near Mainz, Germany
April 1-5, 2014

* Hospitality by the Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Switzerland, is gratefully acknowledged.

Outline

- Hadronic light-by-light (HLbL) scattering in the muon $g-2$:
  - Classification of contributions: large-N_C and chiral counting p^2
  - Summary of selected results
- Large-N_C QCD approach: Minimal Hadronic Ansatz (MHA)
- MHA for pion-exchange contribution in HLbL
  - Experimental and theoretical constraints on form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$
  - Lowest meson dominance (LMD), LMD+V
- Limitations and Generalization
- Conclusions

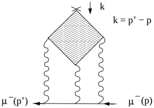
Hadronic light-by-light scattering in the muon $g-2$



$\mathcal{O}(\alpha^3)$ hadronic contribution to muon $g-2$: four-point function $\langle VVVV \rangle$ projected onto a_μ (external soft photon $k \to 0$).

Current approach: use some hadronic model at low energies with exchanges and loops of resonances and some form of (dressed) "quark-loop" at high energies.
Problem: $\langle VVVV \rangle$ depends on several invariant momenta $\Rightarrow$ distinction between low and high energies is not as easy as for two-point function $\langle VV \rangle$ (had. VP).

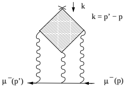
Hadronic light-by-light scattering in the muon $g-2$

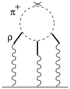


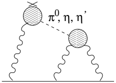
$\mathcal{O}(\alpha^3)$ hadronic contribution to muon $g-2$: four-point function $\langle VVVV \rangle$ projected onto a_μ (external soft photon $k \to 0$).

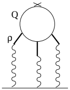
Current approach: use some hadronic model at low energies with exchanges and loops of resonances and some form of (dressed) "quark-loop" at high energies.
Problem: $\langle VVVV \rangle$ depends on several invariant momenta $\Rightarrow$ distinction between low and high energies is not as easy as for two-point function $\langle VV \rangle$ (had. VP).

Classification of de Rafael '94: Chiral counting p^2 (ChPT) and large-N_C counting as guideline (all higher orders in p^2 and N_C contribute):



$=$


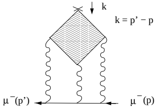
$+ \cdots +$


$+ \cdots +$ Exchanges of other resonances ($f_0, a_1, f_2 \ldots$) $+$


$+ \cdots$

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C
	pion-loop (dressed)	pseudoscalar exchanges		quark-loop (dressed)

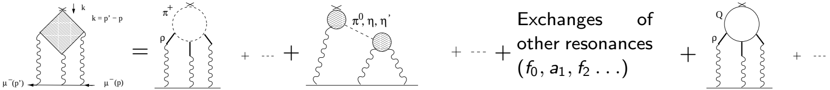
Hadronic light-by-light scattering in the muon $g-2$



$\mathcal{O}(\alpha^3)$ hadronic contribution to muon $g-2$: four-point function $\langle VVVV \rangle$ projected onto a_μ (external soft photon $k \to 0$).

Current approach: use some hadronic model at low energies with exchanges and loops of resonances and some form of (dressed) "quark-loop" at high energies.
Problem: $\langle VVVV \rangle$ depends on several invariant momenta $\Rightarrow$ distinction between low and high energies is not as easy as for two-point function $\langle VV \rangle$ (had. VP).

Classification of de Rafael '94: Chiral counting p^2 (ChPT) and large-N_C counting as guideline (all higher orders in p^2 and N_C contribute):



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C
	pion-loop (dressed)	pseudoscalar exchanges		quark-loop (dressed)

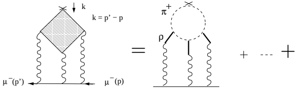
Relevant scales in HLbL ($\langle VVVV \rangle$ with off-shell photons): $0-2$ GeV, i.e. larger than m_μ !

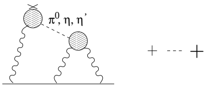
Constrain models using experimental data (form factors of hadrons with photons) and theory (ChPT at low energies; short-distance constraints from pQCD / OPE at high momenta).

Going beyond models: Dispersion relations or Lattice QCD.

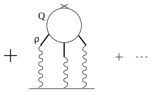
Issue: on-shell versus off-shell form factors. For instance pion-pole versus pion-exchange: How do we define the pion-pole contribution ? Is there a form factor at external vertex ?

HLbL in the muon $g-2$: Summary of selected results





Exchanges of other resonances ($f_0, a_1, f_2 \ldots$)

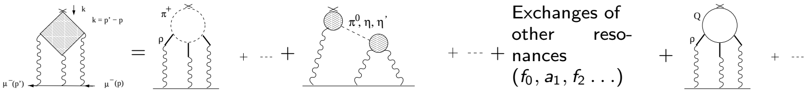


Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN:	+116 (40)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

HLbL in the muon $g-2$: Summary of selected results



Contribution to $a_\mu \times 10^{11}$:

BPP: +83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS: +90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN: +80 (40)		+83 (12)		
MV: +136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007: +110 (40)				
PdRV:+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN: +116 (40)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
	ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02: Extended Nambu-Jona-Lasinio (ENJL) model; but for some contributions also other models used (in particular for pseudoscalars, pion-loop)

HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02: Hidden Local Symmetry (HLS) model (often = VMD)

KN = Knecht, Nyffeler '02: large-N_C QCD for pion-pole (lowest meson dominance LMD, LMD+V)

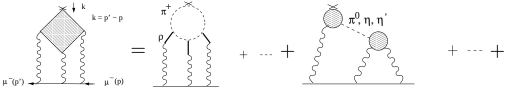
MV = Melnikov, Vainshtein '04: large-N_C QCD, short-distance constraint from $\langle VVVV \rangle$ on pion-pole and axial-vector contribution, mixing of two axial-vector nonets

2007 = Bijnens, Prades; Miller, de Rafael, Roberts (compilation)

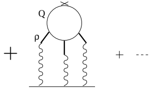
PdRV = Prades, de Rafael, Vainshtein '09 (compilation; adding errors quadratically)

N = Nyffeler '09: large-N_C QCD for pion-exchange with off-shell LMD+V form factor, new short-distance constraint at external vertex; JN = Jegerlehner, Nyffeler '09 (compilation; adding errors linearly)

HLbL in the muon $g-2$: Summary of selected results



Exchanges of other resonances ($f_0, a_1, f_2 \ldots$)



Contribution to $a_\mu \times 10^{11}$:

BPP: +83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS: +90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN: +80 (40)		+83 (12)		
MV: +136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007: +110 (40)				
PdRV: +105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN: +116 (40)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
	ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02: Extended Nambu-Jona-Lasinio (ENJL) model; but for some contributions also other models used (in particular for pseudoscalars, pion-loop)

HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02: Hidden Local Symmetry (HLS) model (often = VMD)

KN = Knecht, Nyffeler '02: large-N_C QCD for pion-pole (lowest meson dominance LMD, LMD+V)

MV = Melnikov, Vainshtein '04: large-N_C QCD, short-distance constraint from $\langle VVVV \rangle$ on pion-pole and axial-vector contribution, mixing of two axial-vector nonets

2007 = Bijnens, Prades; Miller, de Rafael, Roberts (compilation)

PdRV = Prades, de Rafael, Vainshtein '09 (compilation; adding errors quadratically)

N = Nyffeler '09: large-N_C QCD for pion-exchange with off-shell LMD+V form factor, new short-distance constraint at external vertex; JN = Jegerlehner, Nyffeler '09 (compilation; adding errors linearly)

Questioned: size of quark-loop ($107(2) \times 10^{-11}$: Goecke et al. '11-'13; Schwinger-Dyson equation) and size of pion-loop ($-(11-71) \times 10^{-11}$: Engel et al. '12, '13; pion-polarizabilities, effect of a_1 meson)

Very recent new estimates: $a_\mu(f_0, f_0', a_0) = [(-0.9 \pm 0.2) \text{ to } (-3.1 \pm 0.8)] \times 10^{-11}$, $a_\mu(f_1, f_1') = (6.4 \pm 2.0) \times 10^{-11}$ and $a_\mu(f_2, f_2', a_2, a_2') = (1.1 \pm 0.1) \times 10^{-11}$ (Pauk, Vanderhaeghen '14).

Large-N_C QCD approach: Minimal Hadronic Ansatz (MHA)

Moussallam, Stern '94; Moussallam '95, '97; Peris et al. '98; Knecht et al. '99; ...

- In QCD, in leading order in N_C, in each channel of a Green's function an infinite tower of narrow resonances contributes $\Rightarrow$ only poles, no cuts (meromorphic functions).
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively. Interpolation works best for order parameters (Green's functions, LEC's) and integrals over Green's functions in Euclidean space. Not suited to describe shape of resonances in physical region.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD).

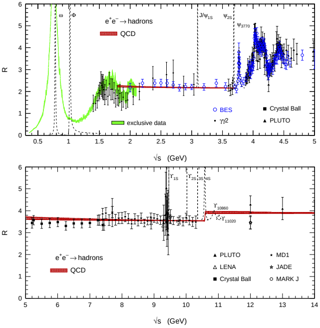
Large-N_C QCD approach: Minimal Hadronic Ansatz (MHA)

Moussallam, Stern '94; Moussallam '95, '97; Peris et al. '98; Knecht et al. '99; ...

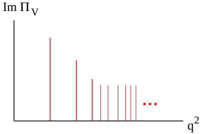
- In QCD, in leading order in N_C, in each channel of a Green's function an infinite tower of narrow resonances contributes $\Rightarrow$ only poles, no cuts (meromorphic functions).
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively. Interpolation works best for order parameters (Green's functions, LEC's) and integrals over Green's functions in Euclidean space. Not suited to describe shape of resonances in physical region.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD).

Example: 2-point function $\langle VV \rangle \to$ spectral function $\mathrm{Im}\Pi_V \sim \sigma(e^+e^- \to \text{hadrons})$

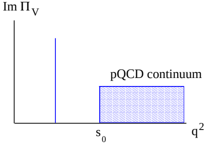
Real world (Davier et al., '03)



Large-N_C QCD ('t Hooft '74)



Minimal Hadronic Ansatz (MHA)



Scale s_0 fixed by the OPE

Minimal Hadronic Ansatz for $\langle VV \rangle$ and HVP contribution to $g-2$

Adapted from de Rafael

Consider Adler function $\mathcal{A}(Q^2) \equiv -Q^2 \frac{\partial \Pi_V(Q^2)}{\partial Q^2}$

$$\mathcal{A}(Q^2)\Big|_{\text{MHA}} = \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) e^2 \left\{ 2 f_V^2 M_V^2 \frac{Q^2}{(Q^2 + M_V^2)^2} + \frac{N_C}{16\pi^2} \frac{4}{3} \frac{Q^2}{Q^2 + s_0} (1 + \ldots) \right\}$$

Chiral loops (two-pion states) subleading in $1/N_C$.

No $1/Q^2$ term in the OPE $\Rightarrow$ fixes s_0: $2 f_V^2 M_V^2 = \frac{N_C}{16\pi^2} \frac{4}{3} s_0 \left(1 + \frac{3}{8} \frac{\alpha_s(s_0)}{\pi} + \ldots\right)$

General relation:

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi} \int_0^1 \frac{dx}{x} (1 - x) \left(1 - \frac{x}{2}\right) \mathcal{A}\left(\frac{x^2}{1 - x} m_\mu^2\right)$$

$$a_\mu^{\text{HVP}}\Big|_{\text{MHA}} = (5700 \pm 1900) \times 10^{-11} \qquad \text{(33\% systematic error from } 1/N_C\text{)}$$

Of course, this error for HVP cannot compete with evaluations based on data on $\sigma(e^+e^- \to \text{hadrons})$ with $\pm 45 \times 10^{-11}$. Imposing further theoretical and experiment constraints on the MHA for the relevant Green's functions can maybe bring down the error to 10-15%. Would be almost enough for HLbL.

MHA for pion-exchange contribution in HLbL

1. Experimental and theoretical constraints on form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$
2. Lowest meson dominance (LMD), LMD+V

Off-shell pion form factor from $\langle VVP \rangle$

- Following Bijnens, Pallante, Prades '96; Hayakawa, Kinoshita, Sanda '96; Hayakawa, Kinoshita '98, we can define off-shell form factor for π^0:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2 \cdot y)} \langle 0 | T\{ j_\mu(x) j_\nu(y) P^3(0)\} | 0 \rangle$$
$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta \, \frac{i\langle \overline{\psi}\psi\rangle}{F_\pi} \, \frac{i}{(q_1+q_2)^2 - m_\pi^2} \, \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) + \ldots$$

Up to small mixing effects of P^3 with η and η' and neglecting exchanges of heavier states like $\pi^{0\prime}, \pi^{0\prime\prime}, \ldots$

$$j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x), \quad \psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \hat{Q} = \mathrm{diag}(2,-1,-1)/3$$

(light quark part of electromagnetic current)

$$P^3 = \overline{\psi} i \gamma_5 \frac{\lambda^3}{2}\psi = \left(\overline{u} i\gamma_5 u - \overline{d} i \gamma_5 d\right)/2, \quad \langle \overline{\psi}\psi\rangle = \text{single flavor quark condensate}$$

Bose symmetry: $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_2^2, q_1^2)$

- Note: for off-shell pions, instead of $P^3(x)$, we could use any other suitable interpolating field, like $(\partial^\mu A_\mu^3)(x)$ or even an elementary pion field $\pi^3(x)$! Off-shell form factor is therefore model dependent and not a physical quantity !

On-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ and transition form factor $F(Q^2)$

- On-shell $\pi^0\gamma^*\gamma^*$ form factor between an on-shell pion and two off-shell photons:

$$i \int d^4x \, e^{iq_1 \cdot x} \langle 0 | T\{j_\mu(x) j_\nu(0)\} | \pi^0(q_1 + q_2) \rangle = \varepsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$

  Relation to off-shell form factor:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$$

  Form factor for real photons is related to $\pi^0 \to \gamma\gamma$ decay width:

$$\mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(q_1^2 = 0, q_2^2 = 0) = \frac{4}{\pi\alpha^2 m_\pi^3} \Gamma_{\pi^0 \to \gamma\gamma}$$

  Often normalization with chiral anomaly is used:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, 0) = -\frac{1}{4\pi^2 F_\pi}$$

- Pion-photon transition form factor:

$$F(Q^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, q_2^2 = 0), \qquad Q^2 \equiv -q_1^2$$

  Note that $q_2^2 = 0$, but $\vec{q}_2 \neq \vec{0}$ for on-shell photon !

Experimental constraints on (on-shell) $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

- Any hadronic model of the form factor has to reproduce the $\pi^0 \to \gamma\gamma$ decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}[1 + \mathcal{O}(m_q)]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small), see also Kampf, Moussallam '09. Leads to normalization:

$$\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = -\frac{N_C}{12\pi^2 F_\pi}$$

For $F_\pi = 92.4$ MeV, this reproduces very well the decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.49)$ eV (PDG 2010, 6.3% precision). More recently the PrimEx Collaboration (Larin et al. '11) presented the measurement $\Gamma(\pi^0 \to \gamma\gamma) = (7.82 \pm 0.23)$ eV (2.8% precision).

Note: Uncertainty in neutral pion contribution to HLbL originating from $\Gamma(\pi^0 \to \gamma\gamma)$ has not been taken into account in most evaluations ! F_π is used without any error attached to it.

Experimental constraints on (on-shell) $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

• Any hadronic model of the form factor has to reproduce the $\pi^0 \to \gamma\gamma$ decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}[1 + \mathcal{O}(m_q)]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small), see also Kampf, Moussallam '09. Leads to normalization:

$$\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = -\frac{N_C}{12\pi^2 F_\pi}$$

For $F_\pi = 92.4$ MeV, this reproduces very well the decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.49)$ eV (PDG 2010, 6.3% precision). More recently the PrimEx Collaboration (Larin et al. '11) presented the measurement $\Gamma(\pi^0 \to \gamma\gamma) = (7.82 \pm 0.23)$ eV (2.8% precision).

Note: Uncertainty in neutral pion contribution to HLbL originating from $\Gamma(\pi^0 \to \gamma\gamma)$ has not been taken into account in most evaluations ! F_π is used without any error attached to it.

• Information on the $\pi^0 - \gamma$ transition form factor with one on-shell and one off-shell photon from the process $e^+e^- \to e^+e^-\pi^0$

Brodsky-Lepage '79-'81 predict the following behavior:

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$$

Maybe with slightly different prefactor !

Data from CELLO '90 and CLEO '08 fairly well confirm this behavior, although $Q^2 \leq 9$ GeV^2 maybe not yet large enough. Data from BABAR '09 in range 4 GeV$^2 \leq Q^2 \leq 40$ GeV^2 do not show this fall-off. But data from BELLE '12 in same range seem to fall off.

Theory: QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

Knecht, Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

• When the space-time arguments of all three currents approach each other one obtains with the Operator Product Expansion (OPE), up to corrections $\mathcal{O}(\alpha_s)$:

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3} \frac{1}{\lambda^2} \frac{q_1^2 + q_2^2 + (q_1 + q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

Theory: QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

Knecht, Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

• When the space-time arguments of all three currents approach each other one obtains with the Operator Product Expansion (OPE), up to corrections $\mathcal{O}(\alpha_s)$:

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3}\frac{1}{\lambda^2}\frac{q_1^2 + q_2^2 + (q_1+q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

• When the space-time arguments of the two vector currents in $\langle VVP \rangle$ approach each other the OPE leads to Green's function $\langle AP \rangle$ and one obtains:

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, (\lambda q_1)^2, (q_2 - \lambda q_1)^2) = \frac{2F_0}{3}\frac{1}{\lambda^2}\frac{1}{q_1^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

As pointed out in Melnikov, Vainshtein '04, higher twist corrections have been worked out in Shuryak, Vainshtein '82, Novikov et al. '84 (in chiral limit):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0}\gamma^*\gamma^*}(0, (\lambda q_1)^2, (\lambda q_1)^2) = \frac{2F_0}{3}\left\{\frac{1}{\lambda^2 q_1^2} + \frac{8}{9}\frac{\delta^2}{\lambda^4 q_1^4} + \mathcal{O}\left(\frac{1}{\lambda^6}\right)\right\}$$

δ^2 parametrizes the relevant higher-twist matrix element.

The sum-rule estimate in Novikov et al. '84 yielded $\delta^2 = (0.2 \pm 0.02)$ GeV^2

Short-distance constraint on form factor at external vertex

- When the space-time argument of one of the vector currents approaches the argument of the pseudoscalar density in $\langle VVP \rangle$ one obtains (Knecht, Nyffeler, EPJC '01):

$$\langle V \underbrace{VP}_{\text{OPE}} \rangle \to \langle VT \rangle \qquad \text{Vector-Tensor two-point function}$$

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + q_2)^2, (\lambda q_1)^2, q_2^2) = -\frac{2}{3} \frac{F_0}{\langle \overline{\psi}\psi \rangle_0} \Pi_{\mathrm{VT}}(q_2^2) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

The vector-tensor two-point function Π_{VT} is defined by:

$$\delta^{ab}(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = \int d^4x e^{ip\cdot x} \langle 0 | T\{V_\mu^a(x)(\overline{\psi}\,\sigma_{\rho\sigma}\frac{\lambda^b}{2}\,\psi)(0)\}|0\rangle, \quad \sigma_{\rho\sigma} = \frac{i}{2}[\gamma_\rho, \gamma_\sigma]$$

$$(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = (p_\rho \eta_{\mu\sigma} - p_\sigma \eta_{\mu\rho})\,\Pi_{\mathrm{VT}}(p^2), \quad \text{CVC, parity invariance}$$

At the external vertex in light-by-light scattering the following limit is relevant (soft photon $q_2 \to 0$)

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = -\frac{2}{3} \frac{F_0}{\langle \overline{\psi}\psi \rangle_0} \Pi_{\mathrm{VT}}(0) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Short-distance constraint at the external vertex (cont.)

Ioffe, Smilga '84 defined the quark condensate magnetic susceptibility χ of QCD in the presence of a constant external electromagnetic field

$$\langle 0 | \bar{q} \sigma_{\mu\nu} q | 0 \rangle_F = e\, e_q\, \chi\, \langle \overline{\psi}\psi \rangle_0\, F_{\mu\nu}, \qquad e_u = 2/3, \;\; e_d = -1/3$$

Belyaev, Kogan '84 then showed that

$$\Pi_{\mathrm{VT}}(0) = -\frac{\langle \overline{\psi}\psi \rangle_0}{2}\, \chi$$

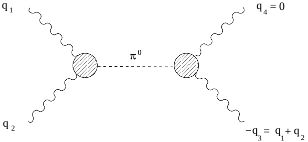
⇒ Short-distance constraint on off-shell form factor at external vertex (Nyffeler '09):

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3}\, \chi \; + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad (*)$$

- Note that there is no falloff in OPE in $(*)$, unless χ vanishes !
  A constituent quark model for the form factor would lead to a $1/q_1^2$ fall-off.
- Corrections of $\mathcal{O}(\alpha_s)$ in OPE ⇒ χ depends on renormalization scale μ. Not obvious, what is the "correct" scale μ in HLbL.
- Unfortunately there is no agreement in the literature what the value of $\chi(\mu)$ should be !
  Range of values from $\chi(\mu \sim 0.5 \text{ GeV}) \approx -9 \text{ GeV}^{-2}$ (Ioffe, Smilga '84; Vainshtein '03, ..., Narison '08) to $\chi(\mu \sim 1 \text{ GeV}) \approx -3 \text{ GeV}^{-2}$ (Balitsky, Yung '83; Ball et al. '03; ...; Ioffe '09). Running with μ cannot explain such a difference. Recent result from Lattice QCD: $\chi_u^{\overline{\mathrm{MS}}}(\mu = 2 \text{ GeV}) = -(2.08 \pm 0.08) \text{ GeV}^{-2}$ (Bali et al. '12).

QCD short-distance constraint on $\langle VVVV \rangle$ in $g-2$

- Melnikov, Vainshtein '04 found QCD short-distance constraint on whole 4-point function:



$$\langle \underbrace{VV}_{\text{OPE}} V|\gamma\rangle \overset{q_1^2\sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

- From this they deduced for the LbyL scattering amplitude for finite $q_1^2, q_2^2, -q_3 = q_1 + q_2$ (Eq. (18) in MV '04, using our normalization for form factor; Minkowski space notation):

$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu}\tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1,2,3$. For external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$. Except in $\tilde{f}_{\rho\sigma}$, $q_4 \to 0$ is understood.

- Expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$. No form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by constant WZW form factor $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0) \approx \mathcal{F}_{\pi^0\gamma\gamma}(0,0)$! $\Rightarrow$ MV '04 consider the pion-pole contribution !
- If one then studies the behavior for large q_3^2, one obtains from the pion propagator an overall $1/q_3^2$ behavior (apart from $f_3^{\sigma\rho}$). According to MV '04 this agrees exactly with behavior of quark-loop in perturbative QCD for large momenta.
- From quark-hadron duality in large-N_C QCD it follows that the sum of all resonance exchanges has to match with the quark-loop ! But why should already the pion-pole contribution alone match with the quark-loop ?

The LMD and LMD+V form factors

Knecht, Nyffeler, EPJC '01; Nyffeler '09

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in large-N_C QCD in chiral limit with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all leading (and some subleading) QCD short-distance constraint from OPE
- Reproduces Brodsky-Lepage (BL): $\lim_{Q^2 \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$

  (OPE and BL cannot be fulfilled simultaneously with only one vector resonance, LMD)
- Normalized to decay width $\Gamma_{\pi^0 \to \gamma\gamma}$

The LMD and LMD+V form factors

Knecht, Nyffeler, EPJC '01; Nyffeler '09

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in large-N_C QCD in chiral limit with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all leading (and some subleading) QCD short-distance constraint from OPE
- Reproduces Brodsky-Lepage (BL): $\lim_{Q^2\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$

  (OPE and BL cannot be fulfilled simultaneously with only one vector resonance, LMD)
- Normalized to decay width $\Gamma_{\pi^0\to\gamma\gamma}$

LMD form factors (off-shell, on-shell, transition form factor):

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^{0*}\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2 + q_2^2 + q_3^2 - c_V}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}, \qquad q_3^2 = (q_1 + q_2)^2$$

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2 + q_2^2 - \bar{c}_V}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}, \qquad \bar{c}_V = c_V - m_\pi^2$$

$$F^{\mathrm{LMD}}(Q^2) = -\frac{F_\pi}{3M_V^2} \frac{Q^2 + \bar{c}_V}{Q^2 + M_V^2}$$

Note that the LMD transition form factor does not fall off like $1/Q^2$ for large Q^2.

The LMD+V form factor

Off-shell LMD+V form factor:

$$\begin{aligned}
\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_3^2, q_1^2, q_2^2) &= \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)} \\
P_H^V(q_1^2, q_2^2, q_3^2) &= h_1\,(q_1^2 + q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2 + q_2^2)\, q_3^2 + h_4\, q_3^4 \\
&\quad + h_5\,(q_1^2 + q_2^2) + h_6\, q_3^2 + h_7 \\
q_3^2 &= (q_1 + q_2)^2
\end{aligned}$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV, Free parameters: h_i

On-shell LMD+V form factor:

$$\begin{aligned}
\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_1^2, q_2^2) &= \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2) + h_1\,(q_1^2 + q_2^2)^2 + \bar{h}_2\, q_1^2\, q_2^2 + \bar{h}_5\,(q_1^2 + q_2^2) + \bar{h}_7}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)} \\
\bar{h}_2 &= h_2 + m_\pi^2, \qquad \bar{h}_5 = h_5 + h_3 m_\pi^2, \qquad \bar{h}_7 = h_7 + h_6 m_\pi^2 + h_4 m_\pi^4
\end{aligned}$$

Transition form factor:

$$F^{\mathrm{LMD+V}}(Q^2) = \frac{F_\pi}{3} \frac{1}{M_{V_1}^2 M_{V_2}^2} \frac{h_1 Q^4 - \bar{h}_5 Q^2 + \bar{h}_7}{(Q^2 + M_{V_1}^2)(Q^2 + M_{V_2}^2)}$$

- $h_1 = 0$ in order to reproduce Brodsky-Lepage behavior.
- Can treat h_1 as free parameter to fit the BABAR data, but the form factor does then not vanish for $Q^2 \to \infty$, if $h_1 \neq 0$.

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (or normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to BABAR data: $h_1 = (-0.17 \pm 0.02)$ GeV^2, $h_5 = (6.51 \pm 0.20)$ GeV$^4 - h_3 m_\pi^2$ with $\chi^2/\mathrm{dof} = 15.0/15 = 1.0$

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4 / (4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (or normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to BABAR data: $h_1 = (-0.17 \pm 0.02)$ GeV^2, $h_5 = (6.51 \pm 0.20)$ GeV$^4 - h_3 m_\pi^2$ with $\chi^2/\mathrm{dof} = 15.0/15 = 1.0$

h_3, h_4, h_6 are unknown / less constrained:

- New short-distance constraint $\Rightarrow h_1 + h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \, \chi$ $(*)$
  LMD ansatz for $\langle VT \rangle \Rightarrow \chi^{\mathrm{LMD}} = -2/M_V^2 = -3.3$ GeV^{-2} (Balitsky, Yung '83)
  Close to $\chi(\mu = 1$ GeV$) = -(3.15 \pm 0.30)$ GeV^{-2} (Ball et al. '03)
  Assume large-N_C (LMD/LMD+V) framework is self-consistent
  $\Rightarrow \chi = -(3.3 \pm 1.1)$ GeV^{-2}
  $\Rightarrow$ vary $h_3 = (0 \pm 10)$ GeV^2 and determine h_4 from relation $(*)$ and vice versa
- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to h_6
  Assume that LMD/LMD+V estimates of low-energy constants from chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$ are self-consistent. Assume 100% error on estimate for the relevant, presumably small low-energy constant $\Rightarrow h_6 = (5 \pm 5)$ GeV^4

Limitations of MHA approach

- Cannot fulfill all short-distance constraints on Green's functions (or form factors) with finite number of resonances (Bijnens et al. '03) $\Rightarrow$ needs to choose among constraints and make compromises.
- Beyond LMD: many parameters in ansatz, not all of them can be fixed from experimental or short-distance constraints.
- Most of the work done in chiral limit, assuming octet symmetry.
- Difficult to treat terms subleading in N_C: width of resonances, loops with resonances (pion loop in HLbL).

Generalization: Resonance Chiral Theory (RχT)

- Gasser, Leutwyler, '84; Donoghue et al. '89; Ecker et al. NPB '89: use of resonance Lagrangian with vector mesons, axial vector mesons, heavy scalars ($f_0(980)$) and heavy pseudoscalars to estimate low-energy constants (LEC) in ChPT at order p^4 when integrating out the resonances at tree level (explains success of vector meson dominance).
- Ecker et al. PLB '89: Imposing short-distance constraints on resonance Lagrangian leads to unique estimates for LEC's at order p^4, at least for vector mesons in different representations (vector field, tensor field, gauged vector fields, massive vector fields).
  Note: Imposing short-distance constraints on resonance Lagrangian does not uniquely determine LEC's at order p^6 (Moussallam, Stern '94; Knecht, Nyffeler '01).
- Advantage of resonance Lagrangian: can easily identify in which Green's functions (processes) the parameters in Lagrangian enter.
  Problem: in general many terms in Lagrangian allowed by chiral symmetry, not all can be determined from short-distance constraints.
- RχT Lagrangian for odd intrinsic parity sector which fulfills various QCD short-distance constraints to fix parameters in the Lagrangian: Pallante, Petronzio '93; Prades '94; Ananthanarayan, Moussallam '02; Ruiz-Femenia, Pich, Portoles '03; Kampf, Moussallam '09; Kampf, Novotny '11; Roig, Sanz Cillero '13.
  RχT with two vector resonances: Mateu, Portoles '07; Czyz et al. '12.
- Attempts to go beyond leading order in N_C: Loops with resonances, including renormalization (Pich et al.).

Pseudoscalar pole / exchange in large-N_C QCD

Model for $\mathcal{F}_{P^{(*)}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
VMD	57	83
LMD (on-shell) [KN]	73	–
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant 2nd FF) [MV]	77(7)	114(10)
LMD+V (off-shell) [N]	72(12)	99(16)
LMD+P (off-shell) [KaNo]	65.8(1.2)	–
LMD+P (off-shell) [RGL]	66.5(1.9)	104.3(5.2)
LMD+P (on-shell) [RGL]	57.5(0.5)	82.7(2.8)

KN = Knecht, Nyffeler '02
MV = Melnikov, Vainshtein '04
N = Nyffeler '09
KaNo = Kampf, Novotny '11 (RχT)
RGL = Roig, Guevara, Lopez Castro '14 (RχT)

Note: KN, MV, N use VMD FF for η, η'

Conclusions

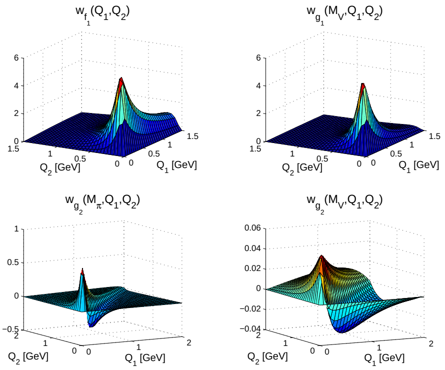
- HLbL in muon $g-2$: can always perform Wick rotation to Euclidean momenta $\Rightarrow$ MHA in large N_C QCD should be a good approximation for the exchange of narrow resonances, if we have good matching with pQCD.
- Problem: terms subleading in N_C, like pion-loop, or broad resonances like σ-meson (double counting with pion-loop ?).
- Needed: more short-distance constraints ! Will also be useful for other approaches using resonances Lagrangians or dispersion relations.
- Short-distance analysis of full 4-point $\langle VVVV \rangle$ (Knecht, Nyffeler, work in progress).
- Needed: more experimental constraints from resonance decays, form factors, cross-sections, e.g. $\pi^0 \to e^+e^-\gamma, \pi^0 \to e^+e^-e^+e^-$ to get doubly off-shell transition form factor, $a_1 \to \pi\gamma, \rho\pi$ to construct Lagrangian with axial vectors.
- Resonance Chiral Theory framework: relations betweens various processes with resonances to fix (some of) the parameters.
- Study relevant momentum regions in HLbL (as model independent as possible) (Knecht, Nyffeler, work in progress).

Relevant momentum regions in $a_\mu^{\mathrm{LbyL};\pi^0}$

- In Knecht, Nyffeler '02, a 2-dimensional integral representation was derived for a certain class (VMD-like) of form factors (schematically):

$$a_\mu^{\mathrm{LbyL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\, f_i(Q_1, Q_2)$$

  with universal weight functions w_i. Dependence on form factors resides in the f_i.
- Expressions with on-shell form factors are maybe not valid as they stand. Maybe one needs to set form factor at external vertex to a constant to obtain pion-pole contribution (Melnikov, Vainshtein '04). Expressions valid for WZW and off-shell VMD form factors.
- Plot of weight functions w_i from Knecht, Nyffeler '02:



- Relevant momentum regions around 0.25 – 1.25 GeV. As long as form factors in different models lead to damping, expect comparable results for $a_\mu^{\mathrm{LbyL};\pi^0}$, at level of 20%.
- Jegerlehner, Nyffeler '09 derived 3-dimensional integral representation for general (off-shell) form factors (hyperspherical approach). Integration over $Q_1^2, Q_2^2, \cos\theta$, where $Q_1 \cdot Q_2 = |Q_1||Q_2| \cos\theta$.
- Idea taken up by Dorokhov et al. '12 (for scalars) and Bijnens, Zahiri Abyaneh '12, '13 (for all contributions, work in progress). See also: Pauk, Vanderhaeghen '14 (for axial vectors).

Backup

Our estimate for pseudoscalar-exchange contribution

Nyffeler '09; Jegerlehner, Nyffeler '09

- π^0
  - Estimate with off-shell form factor $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma^*}$ which obeys new short-distance constraint at external vertex:

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

  - Largest uncertainty from $h_6 = (5 \pm 5)\ \mathrm{GeV}^4 \Rightarrow \pm 6.4 \times 10^{-11}$ in $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$

    If we would vary $h_6 = (0 \pm 10)\ \mathrm{GeV}^4 \Rightarrow \pm 12 \times 10^{-11}$!
  - Varying $\chi = -(3.3 \pm 1.1)\ \mathrm{GeV}^{-2} \Rightarrow \pm 2.1 \times 10^{-11}$

    Value of χ not so important, but range does not include Vainshtein's estimate $\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9\ \mathrm{GeV}^{-2}$
  - Varying $h_3 = (0 \pm 10)\ \mathrm{GeV}^2 \Rightarrow \pm 2.5 \times 10^{-11}$ (h_4 via $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$)
  - With h_1, h_5 from fit to BABAR data: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = 71.8 \times 10^{-11} \to$ result unchanged !

- η, η'
  - Short-distance analysis of LMD+V form factor performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
  - Simplified approach: VMD normalized to decay width $\Gamma(\mathrm{PS} \to \gamma\gamma)$.

$$\mathcal{F}^{\mathrm{VMD}}_{\mathrm{PS}^*\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \mathrm{PS} = \eta, \eta'$$

  - $\Rightarrow a_\mu^{\mathrm{LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5 \times 10^{-11}$

    Not taking pole-approximation as done in Melnikov, Vainshtein '04 !

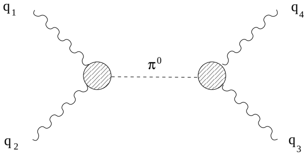
    Note: VMD form factor has too strong damping at large momenta $\to$ values might be a bit too small !

- Estimate for sum of all light pseudoscalars:

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

Pion-pole in $\langle VVVV \rangle$ versus pion-exchange in $a_\mu^{\mathrm{LbyL};\pi^0}$

- To uniquely identify contribution of exchanged neutral pion π^0 in Green's function $\langle VVVV \rangle$, we need to pick out pion-pole:

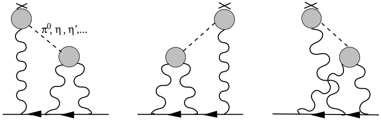


+ crossed diagrams

$$\lim_{(q_1+q_2)^2 \to m_\pi^2} ((q_1+q_2)^2 - m_\pi^2)\langle VVVV \rangle$$

Residue of pole: on-shell vertex function $\langle 0|VV|\pi \rangle \to$ on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

- But in contribution to muon $g-2$, we evaluate Feynman diagrams, integrating over photon momenta with exchanged off-shell pions.
For all the pseudoscalars:



Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ where $\mathrm{PS} = \pi^0, \eta, \eta', \pi^{0\prime}, \ldots$

Off-shell form factors are either inserted "by hand" starting from constant, pointlike Wess-Zumino-Witten (WZW) form factor or using e.g. some resonance Lagrangian.

- Similar statements apply for exchanges (or loops) of other resonances.

Integral representation for pion-exchange contribution

Projection onto the muon $g-2$

(Knecht, Nyffeler '02 (pion-pole with two on-shell form factors); adapted in Jegerlehner '07, '08; Nyffeler '09; Jegerlehner, Nyffeler '09)

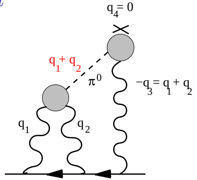
$$
\begin{aligned}
a_\mu^{\mathrm{LbyL};\pi^0} = & -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\
& \times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, q_1^2, (q_1+q_2)^2) \;\; \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_2^2, q_2^2, 0)}{q_2^2 - m_\pi^2} \; T_1(q_1, q_2; p) \right. \\
& \left. + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \;\; \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2} \; T_2(q_1, q_2; p) \right]
\end{aligned}
$$

$$
\begin{aligned}
T_1(q_1, q_2; p) = & \frac{16}{3} (p \cdot q_1)(p \cdot q_2)(q_1 \cdot q_2) - \frac{16}{3} (p \cdot q_2)^2 q_1^2 - \frac{8}{3} (p \cdot q_1)(q_1 \cdot q_2) q_2^2 \\
& + 8 (p \cdot q_2) q_1^2 q_2^2 - \frac{16}{3} (p \cdot q_2)(q_1 \cdot q_2)^2 + \frac{16}{3} m_\mu^2 q_1^2 q_2^2 - \frac{16}{3} m_\mu^2 (q_1 \cdot q_2)^2 \\
T_2(q_1, q_2; p) = & \frac{16}{3} (p \cdot q_1)(p \cdot q_2)(q_1 \cdot q_2) - \frac{16}{3} (p \cdot q_1)^2 q_2^2 + \frac{8}{3} (p \cdot q_1)(q_1 \cdot q_2) q_2^2 \\
& + \frac{8}{3} (p \cdot q_1) q_1^2 q_2^2 + \frac{8}{3} m_\mu^2 q_1^2 q_2^2 - \frac{8}{3} m_\mu^2 (q_1 \cdot q_2)^2
\end{aligned}
$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon).

Pion-exchange versus pion-pole contribution to $a_\mu^{\mathrm{LbyL};\pi^0}$

- Off-shell form factors have been used to evaluate the pion-exchange contribution in Bijnens, Pallante, Prades '96 and Hayakawa, Kinoshita, Sanda '96, '98. "Rediscovered" by Jegerlehner in '07, '08. Consider diagram:



$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \ \times\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

- On the other hand, Knecht, Nyffeler '02 used on-shell form factors:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \ \times\ \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following misleading notation was used:
  $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$
  At external vertex identification with transition form factor was made (wrongly !).
- Melnikov, Vainshtein '04 had observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \ \times\ \mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the WZW term.
- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution !
- The pion-exchange contribution with off-shell pions is model dependent. Only the sum of all contributions in a given model is relevant.

Relevant momentum regions in $a_\mu^{\mathrm{LbyL;PS}}$

Result for pseudoscalar exchange contribution $a_\mu^{\mathrm{LbyL;PS}} \times 10^{11}$ for off-shell LMD+V and VMD form factors obtained with momentum cutoff Λ in 3-dimensional integral representation of Jegerlehner, Nyffeler '09 (integration over square). In brackets, relative contribution of the total obtained with $\Lambda = 20$ GeV.

Λ [GeV]	π^0 LMD+V ($h_3 = 0$)	π^0 LMD+V ($h_4 = 0$)	π^0 VMD	η VMD	η' VMD
0.25	14.8 (20.6%)	14.8 (20.3%)	14.4 (25.2%)	1.76 (12.1%)	0.99 (7.9%)
0.5	38.6 (53.8%)	38.8 (53.2%)	36.6 (64.2%)	6.90 (47.5%)	4.52 (36.1%)
0.75	51.9 (72.2%)	52.2 (71.7%)	47.7 (83.8%)	10.7 (73.4%)	7.83 (62.5%)
1.0	58.7 (81.7%)	59.2 (81.4%)	52.6 (92.3%)	12.6 (86.6%)	9.90 (79.1%)
1.5	64.9 (90.2%)	65.6 (90.1%)	55.8 (97.8%)	14.0 (96.1%)	11.7 (93.2%)
2.0	67.5 (93.9%)	68.3 (93.8%)	56.5 (99.2%)	14.3 (98.6%)	12.2 (97.4%)
5.0	71.0 (98.8%)	71.9 (98.8%)	56.9 (99.9%)	14.5 (99.9%)	12.5 (99.9%)
20.0	71.9 (100%)	72.8 (100%)	57.0 (100%)	14.5 (100%)	12.5 (100%)

π^0:

- Although weight functions plotted earlier are not applicable to off-shell LMD+V form factor, region below $\Lambda = 1$ GeV gives the bulk of the result: 82% for LMD+V, 92% for VMD.
- No damping from off-shell LMD+V form factor at external vertex since $\chi \neq 0$ (new short-distance constraint). Note: VMD falls off too fast, compared to OPE.

η, η':

- Mass of intermediate pseudoscalar is higher than pion mass $\to$ expect a stronger suppression from propagator.
- Peak of relevant weight functions shifted to higher values of Q_i. For η', vector meson mass is also higher $M_V = 859$ MeV. Saturation effect and the suppression from the VMD form factor only fully set in around $\Lambda = 1.5$ GeV: 96% of total for η, 93% for η'.

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$

Integral representation for general off-shell form factors (Jegerlehner, Nyffeler '09):

$$a_\mu^{\mathrm{LbL};\pi^0} = -\frac{2\alpha^3}{3\pi^2}\int_0^\infty \mathrm{d}Q_1\mathrm{d}Q_2\int_{-1}^{+1}\mathrm{d}t\,\sqrt{1-t^2}\,Q_1^3\,Q_2^3 \times\left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_2^2,-Q_1^2,-Q_3^2)\;\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_2^2,-Q_2^2,0)}{(Q_2^2+m_\pi^2)}\,I_1(Q_1,Q_2,t) + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_3^2,-Q_1^2,-Q_2^2)\;\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_3^2,-Q_3^2,0)}{(Q_3^2+m_\pi^2)}\,I_2(Q_1,Q_2,t)\right]$$

where $Q_3^2=(Q_1+Q_2)^2,\quad Q_1\cdot Q_2 = Q_1Q_2\cos\theta,\quad t=\cos\theta$

$$\begin{aligned} I_1(Q_1,Q_2,t) = X(Q_1,Q_2,t)\Big(&8\,P_1\,P_2\,(Q_1\cdot Q_2) - 2\,P_1\,P_3\,(Q_2^4/m_\mu^2 - 2\,Q_2^2) - 2\,P_1\,(2 - Q_2^2/m_\mu^2 + 2\,(Q_1\cdot Q_2)/m_\mu^2) \\ &+4\,P_2\,P_3\,Q_1^2 - 4\,P_2 - 2\,P_3\,(4 + Q_1^2/m_\mu^2 - 2\,Q_2^2/m_\mu^2) + 2/m_\mu^2\Big) \\ &-2\,P_1\,P_2\,(1 + (1-R_{m1})\,(Q_1\cdot Q_2)/m_\mu^2) + P_1\,P_3\,(2 - (1-R_{m1})\,Q_2^2/m_\mu^2) + P_1\,(1-R_{m1})/m_\mu^2 \\ &+P_2\,P_3\,(2 + (1-R_{m1})^2\,(Q_1\cdot Q_2)/m_\mu^2) + 3\,P_3\,(1-R_{m1})/m_\mu^2 \end{aligned}$$

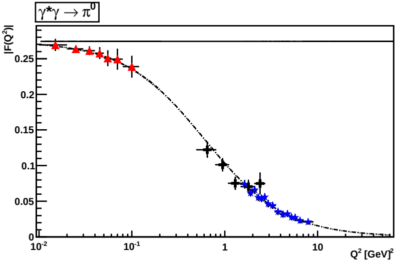
$$\begin{aligned} I_2(Q_1,Q_2,t) = X(Q_1,Q_2,t)\Big(&4\,P_1\,P_2\,(Q_1\cdot Q_2) + 2\,P_1\,P_3\,Q_2^2 - 2\,P_1 + 2\,P_2\,P_3\,Q_1^2 - 2\,P_2 - 4P_3 - 4/m_\mu^2\Big) \\ &-2\,P_1\,P_2 - 3\,P_1\,(1-R_{m2})/(2m_\mu^2) - 3\,P_2\,(1-R_{m1})/(2m_\mu^2) - P_3\,(2 - R_{m1} - R_{m2})/(2m_\mu^2) \\ &+P_1\,P_3\,(2 + 3\,(1-R_{m2})\,Q_2^2/(2m_\mu^2) + (1-R_{m2})^2\,(Q_1\cdot Q_2)/(2m_\mu^2)) \\ &+P_2\,P_3\,(2 + 3\,(1-R_{m1})\,Q_1^2/(2m_\mu^2) + (1-R_{m1})^2\,(Q_1\cdot Q_2)/(2m_\mu^2)) \end{aligned}$$

where $P_1^2 = 1/Q_1^2,\ P_2^2 = 1/Q_2^2,\ P_3^2 = 1/Q_3^2,\ X(Q_1,Q_2,t) = \frac{1}{Q_1Q_2\,x}\arctan\left(\frac{zx}{1-zt}\right),$
$x = \sqrt{1-t^2},\ z = \frac{Q_1Q_2}{4m_\mu^2}\,(1-R_{m1})\,(1-R_{m2}),\ R_{mi} = \sqrt{1+4m_\mu^2/Q_i^2}$

Impact of form factor measurements: example KLOE-2

On the possibility to measure the $\pi^0 \to \gamma\gamma$ decay width and the $\gamma^*\gamma \to \pi^0$ transition form factor with the KLOE-2 experiment

Babusci et al. '12



Simulation of KLOE-2 measurement of $F(Q^2)$ (red triangles). MC program EKHARA 2.0 (Czyż, Ivashyn '11) and detailed detector simulation.
Solid line: $F(0)$ given by chiral anomaly (WZW).
Dashed line: form factor according to on-shell LMD+V model (Knecht, Nyffeler '01).
CELLO (black crosses) and CLEO (blue stars) data at higher Q^2.

Within 1 year of data taking, collecting 5 fb^{-1}, KLOE-2 will be able to measure:

- $\Gamma_{\pi^0 \to \gamma\gamma}$ to 1% statistical precision.
- $\gamma^*\gamma \to \pi^0$ transition form factor $F(Q^2)$ in the region of very low, space-like momenta $0.01~\mathrm{GeV}^2 \leq Q^2 \leq 0.1~\mathrm{GeV}^2$ with a statistical precision of less than 6% in each bin.
  KLOE-2 can (almost) directly measure slope of form factor at origin (note: logarithmic scale in Q^2 in plot !).

Impact of form factor measurements: example KLOE-2 (continued)

- Error in $a_\mu^{\mathrm{LbyL};\pi^0}$ related to the model parameters determined by $\Gamma_{\pi^0\to\gamma\gamma}$ (normalization of form factor; not taken into account in most papers) and $F(Q^2)$ will be reduced as follows:

  - $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx 4 \times 10^{-11}$ (with current data for $F(Q^2) + \Gamma^{\mathrm{PDG}}_{\pi^0\to\gamma\gamma}$)
  - $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx 2 \times 10^{-11}$ ($+\ \Gamma^{\mathrm{PrimEx}}_{\pi^0\to\gamma\gamma}$)
  - $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx (0.7 - 1.1) \times 10^{-11}$ (+ KLOE-2 data)

- Note that this error does not account for other potential uncertainties in $a_\mu^{\mathrm{LbyL};\pi^0}$, e.g. related to the off-shellness of the pion or the choice of model.

- Simple models with few parameters, like VMD (two parameters: F_π, M_V), which are completely determined by the data on $\Gamma_{\pi^0\to\gamma\gamma}$ and $F(Q^2)$, can lead to very small errors in $a_\mu^{\mathrm{LbyL};\pi^0}$. For illustration:

  $a_{\mu;\mathrm{VMD}}^{\mathrm{LbyL};\pi^0} = (57.3 \pm 1.1) \times 10^{-11}$

  $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$ (off-shell LMD+V form factor, including all errors)

  But this might be misleading ! Results differ by about 20% ! VMD form factor has wrong high-energy behavior $\Rightarrow$ too strong damping.

The VMD form factor

Vector Meson Dominance:

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = -\frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{q_1^2 - M_V^2} \frac{M_V^2}{q_2^2 - M_V^2}$$

on-shell = off-shell form factor !

Only two model parameters even for off-shell form factor: F_π and M_V

Note: $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^4$, for large q^2, i.e. falls off too fast compared to OPE prediction $\mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^2$.

Transition form factor:

$$F^{\mathrm{VMD}}(Q^2) = -\frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{Q^2 + M_V^2}$$