Fits and Related Systematics for the Hadronic Vacuum Polarization on the Lattice

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Based on coll. with Christopher Aubin (Fordham U.), Tom Blum (Connecticut U.),

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Lots of activity !

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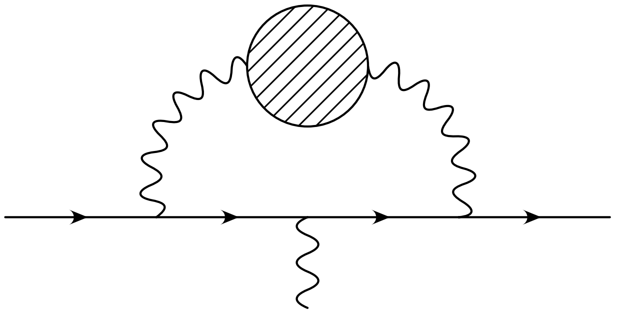
- ...

Introduction

Lautrup-de Rafael '69
Blum '02

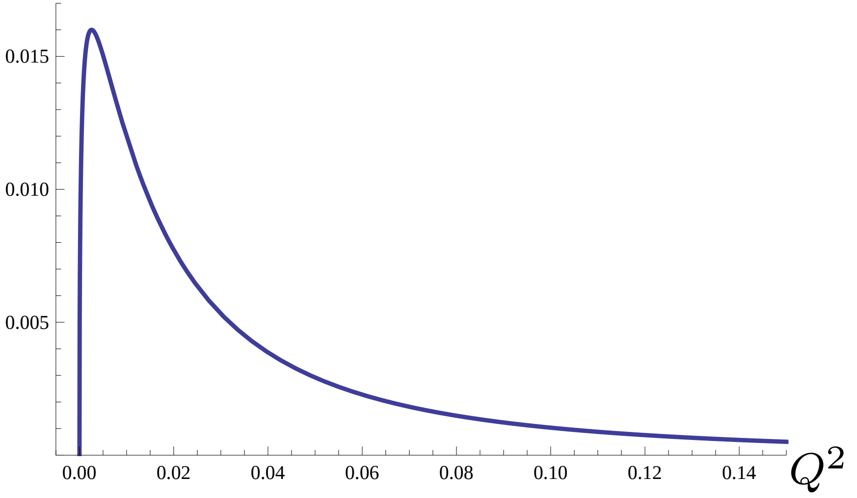


$$(g-2)_\mu^{HVP} \sim \int_0^\infty dQ^2 \underbrace{f(Q^2)}_{\text{known}} \left[\Pi(Q^2) - \Pi(0)\right]$$

$$\Pi(Q^2) - \Pi(0) \Longrightarrow (g-2)_\mu^{HVP}$$

★ integrand strongly peaked at $Q^2 \sim m_\mu^2/4 \sim 0.003\ \mathrm{GeV}^2$

(g-2) integrand



Introduction

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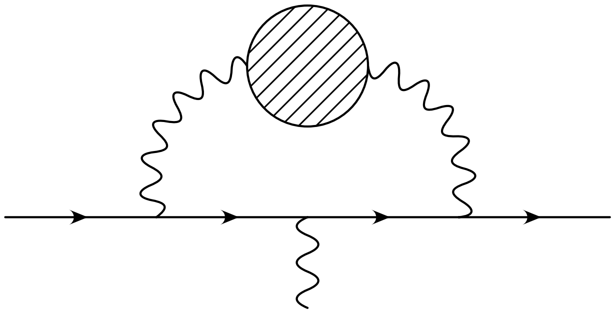

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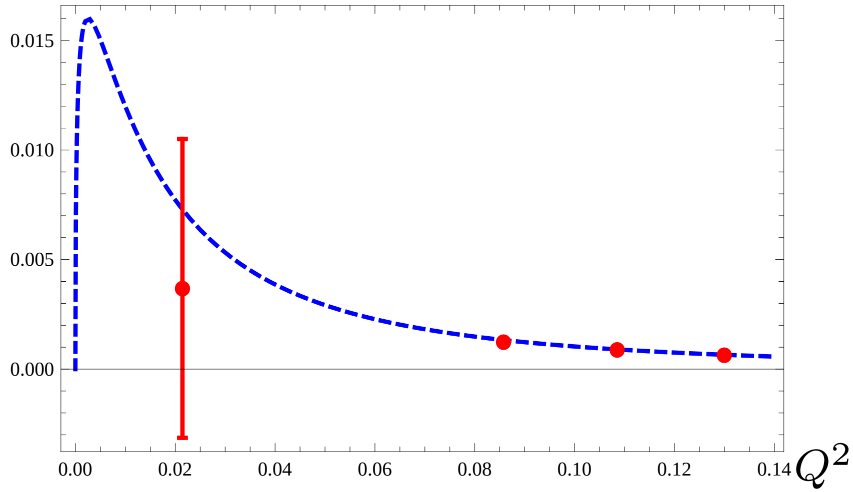
$$\Pi(Q^2) - \Pi(0) \Longrightarrow (g-2)_\mu^{HVP}$$

★ integrand strongly peaked at $Q^2 \sim m_\mu^2/4 \sim 0.003 \text{ GeV}^2$

if no good data in region of curvature $\Longrightarrow$ possibly wrong results !

(g-2) integrand

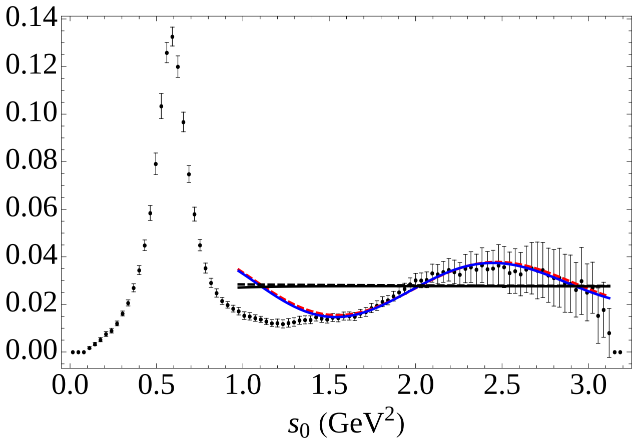


(even with good χ^2)

Need reliable fitting function !

how to test this theoretical error?

A τ-based model for $I = 1$ contributions



Boito, Cata, Golterman, Jamin, Mahdavi, Maltman, Osborne, SP '11 + '12

$t \leq t_{min}$ GeV^2 $\rightarrow$ OPAL data.
$t \geq t_{min}$ GeV^2:

$$\mathrm{Im}\Pi(t) = \rho_{\text{ Pert.Th.}}(t) + e^{-\delta - \gamma\ t} \sin(\alpha + \beta t)$$

$$\Pi(Q^2) = -Q^2 \int_{4m_\pi^2}^{\infty} \frac{dt}{\pi} \frac{\mathrm{Im}\Pi(t)}{t(t+Q^2)}$$

We take $t_{min} = 1.5$ GeV^2.

GOAL: test fitting ansätze accuracy in lattice determinations

- Take typical lattice Q^2 values + lattice covariance matrix

  (e.g., Aubin et al. '12, $64^3 \times 144$ lattice, a = 0.06 fm, $m_\pi = 220\mathrm{MeV}$, periodic BCs)

- Generate fake lattice data for $\Pi(Q^2) - \Pi(0)$ and compare with true answer from model.

- You should try this model to check your systematics: it's very physically motivated !

Fitting functions

Aubin, Blum, Golterman, SP '12

★ Padés, model independent, they enjoy a convergence theorem for $N \to \infty$:

$$\Pi(Q^2) = \Pi(0) + Q^2 \underbrace{\left(a_0 + \sum_{r=1}^{N} \frac{a_r}{Q^2 + b_r} \right)}_{\text{Pade}}$$

$\Pi(0)$, $a's$ and $b's$ are fitting parameters.

★ VMD is not a Pade, since you fix $b_1 = M_\rho^2$. (true $\Pi(Q^2)$ has cut starting at $4m_\pi^2$...)

We have: $a_0 \neq 0 \Longrightarrow [N, N]$ Pade; $a_0 = 0 \Longrightarrow [N-1, N]$ Pade.

For instance:

- $\frac{a_1}{Q^2+b_1}$ is a [0,1] Pade $\Longrightarrow \Pi(Q^2) = \Pi(0) + Q^2 \left(\frac{a_1}{Q^2+b_1} \right)$
- $a_0 + \frac{a_1}{Q^2+b_1}$ is a [1,1] Pade $\Longrightarrow \Pi(Q^2) = \Pi(0) + Q^2 \left(a_0 + \frac{a_1}{Q^2+b_1} \right)$

etc...

Example: Stieltjes. No errors.

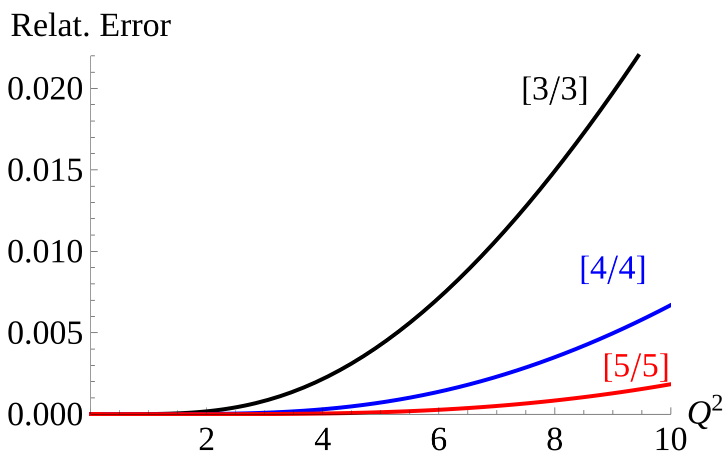
Toy Vac. Pol. function ($\mathrm{Im}\Pi \geq 0$), with a cut, $-\infty \leq Q^2 \leq -1$:

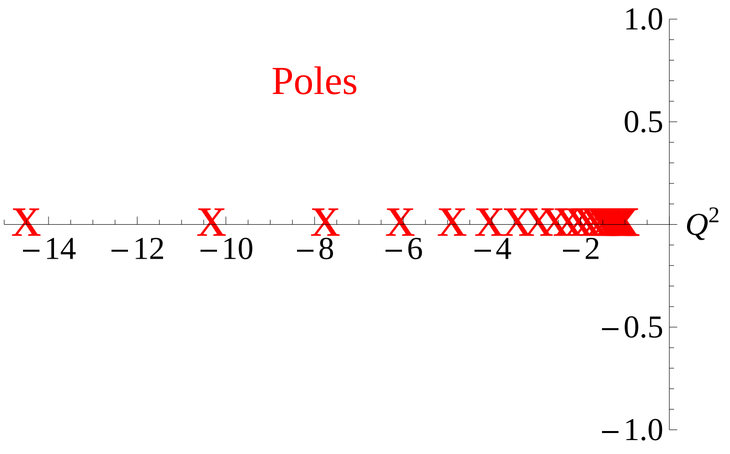
$$\frac{\Pi(0)-\Pi(Q^2)}{Q^2} = \int_1^{\infty} \frac{dt}{(t+Q^2)}\ \frac{1}{t} = \frac{1}{Q^2}\ \log\left(1+Q^2\right)$$

Theorem: As $N \to \infty$, with a_i, b_i determined from the function (and derivatives) at $Q^2 = 0$, or at multiple points,

$$\frac{a_0 + a_1 Q^2 + a_2 Q^4 + \dots + a_N Q^{2N}}{1 + b_1 Q^2 + \dots + b_N Q^{2N}} \longrightarrow \frac{\Pi(0)-\Pi(Q^2)}{Q^2}$$

everywhere in a compact region in complex Q^2, except on the cut.





Poles$\neq$ Physics

Model Fits

Golterman, Maltman, SP '13

"Exact result": $(g-2)_\mu^{HVP}|_{Q^2\leq 1\ \mathrm{GeV}^2} = 1.204\times 10^{-7}$.

Fit interval $0 < Q^2 \leq 1$ GeV^2 , (49 points).

Pull = (exact - fit) / error

	$(g-2)_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD			2189/47 ×	
VMD+			67.4/46	
$[0,1]$			285/46	
$[1,1]$			61.4/45	
$[1,2]$			55.0/44	
$[2,2]$			54.6/43	

VMD "flavors" :

- VMD: [0,1] Pade with $b_1 = M_\rho^2$.
- VMD+: [1,1] Pade with $b_1 = M_\rho^2$ (i.e. VMD + linear polynomial)

Model Fits

Golterman, Maltman, SP '13

"Exact result": $(g-2)_\mu^{HVP}|_{Q^2 \leq 1 \text{ GeV}^2} = 1.2059 \times 10^{-7}$.

Fit interval $0 < Q^2 \leq 1$ GeV^2 , (49 points).

Pull = (exact - fit) / error

	$(g-2)_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD	1.3201	0.0052	2189/47	-
VMD+			67.4/46	
$[0,1]$			285/46	
$[1,1]$			61.4/45	
$[1,2]$			55.0/44	
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VMD "flavors" :

- VMD has a bad χ^2 and $(g-2)$.

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	$(g-2)_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD	1.3201	0.0052	2189/47	-
VMD+	1.0658	0.0076	67.4/46	18
$[0,1]$			285/46	
$[1,1]$			61.4/45	
$[1,2]$			55.0/44	
$[2,2]$			54.6/43	

- VMD has a bad χ^2 and $(g-2)$.
- VMD+ also gets it wrong although the χ^2 is good $\Longrightarrow$ DANGER !

Model Fits

Golterman, Maltman, SP '13

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	$(g-2)_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD	1.3201	0.0052	2189/47	-
VMD+	1.0658	0.0076	67.4/46	18
$[0,1]$	0.8703	0.0095	285/46	-
$[1,1]$	1.116	0.022	61.4/45	4
$[1,2]$	1.182	0.043	55.0/44	0.5
$[2,2]$	1.177	0.058	54.6/43	0.5

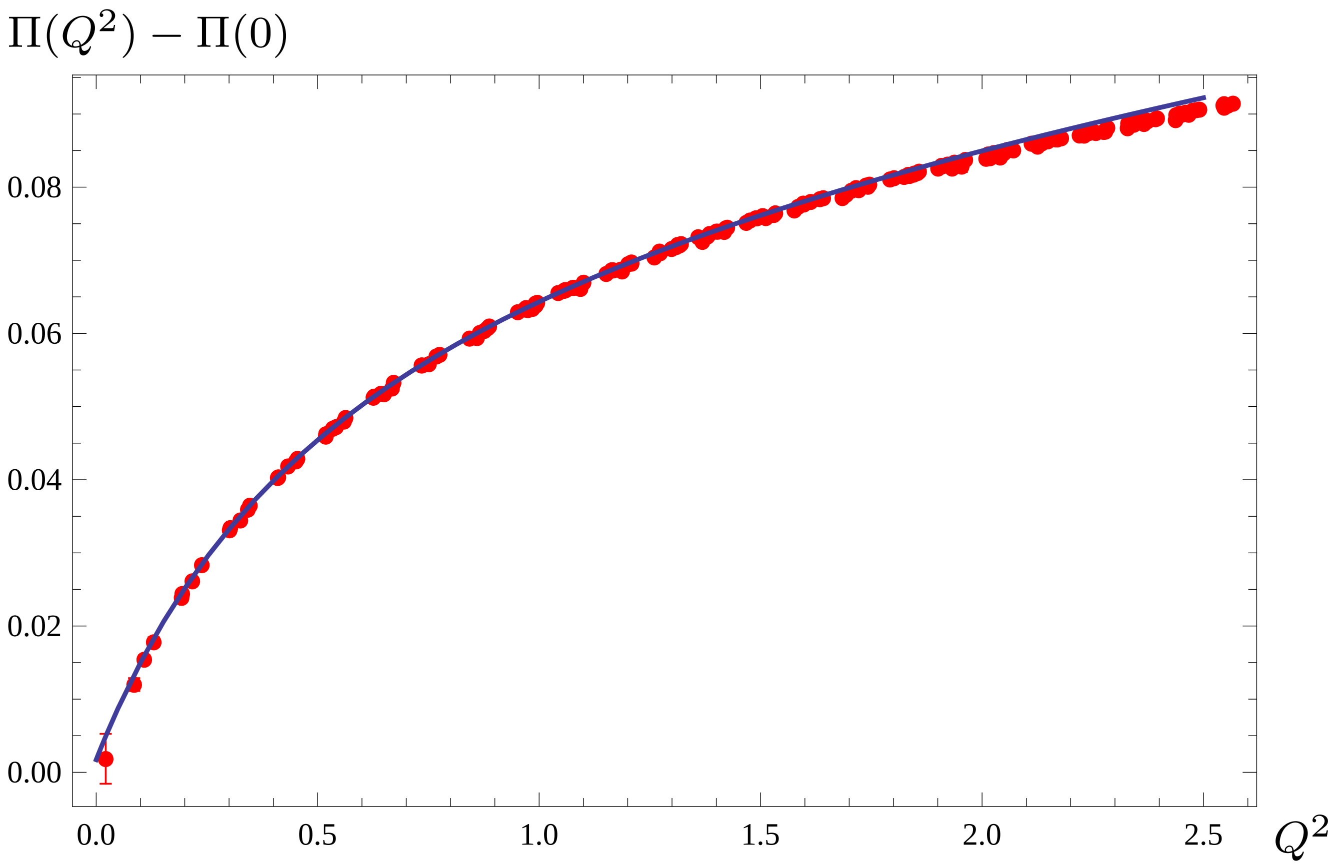
- VMD has a bad χ^2 and $(g-2)$.
- VMD+ also gets it wrong although the χ^2 is good $\Longrightarrow$ DANGER !
- Pades [1,2] and [2,2] get it right, but the error is $\sim 4\%$.

Moral

Take, e.g., the VMD+ case

You may think this is a good fit for an accurate $(g-2)_\mu$:

solid line=VMD+ Fit

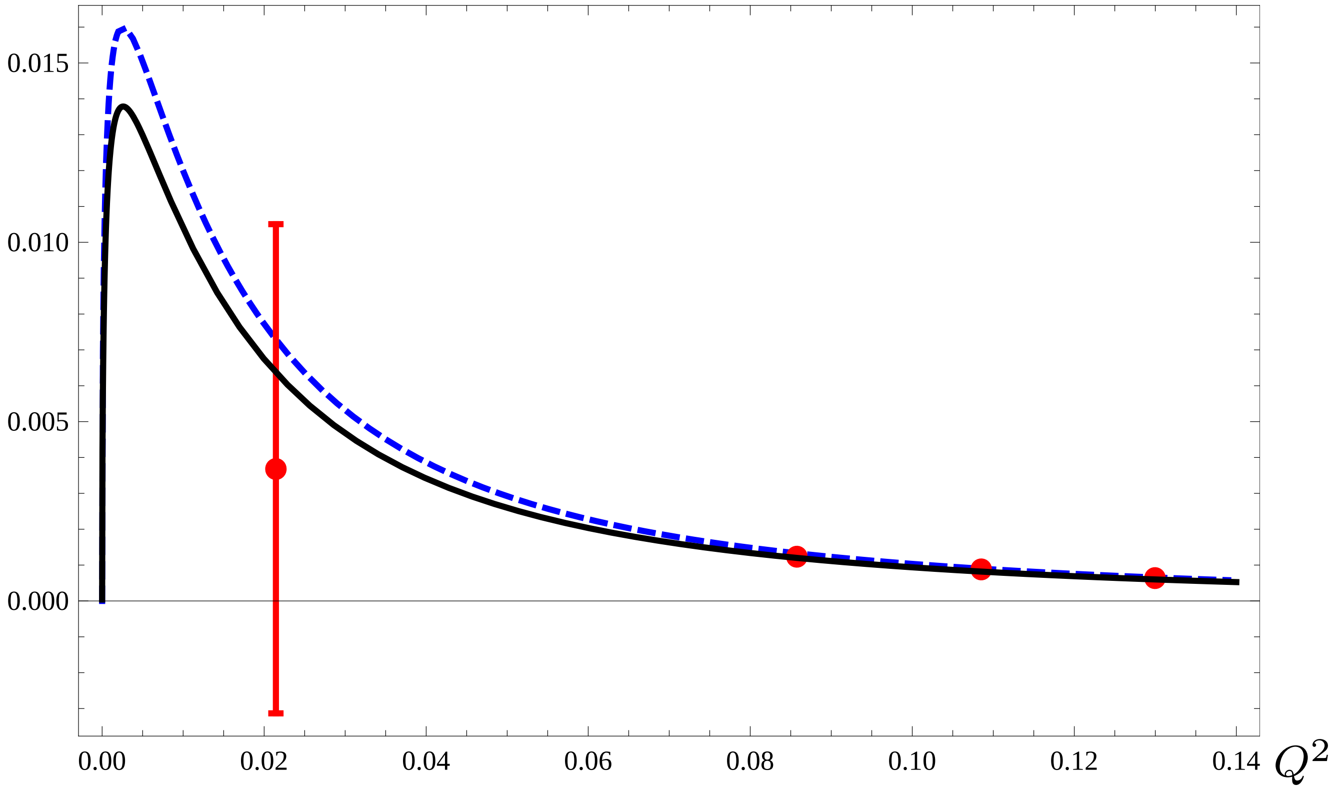


Moral

while, in fact, this is what you should be looking at:

dashed blue=Truth
solid line=VMD+ Fit

(g-2) Integrand

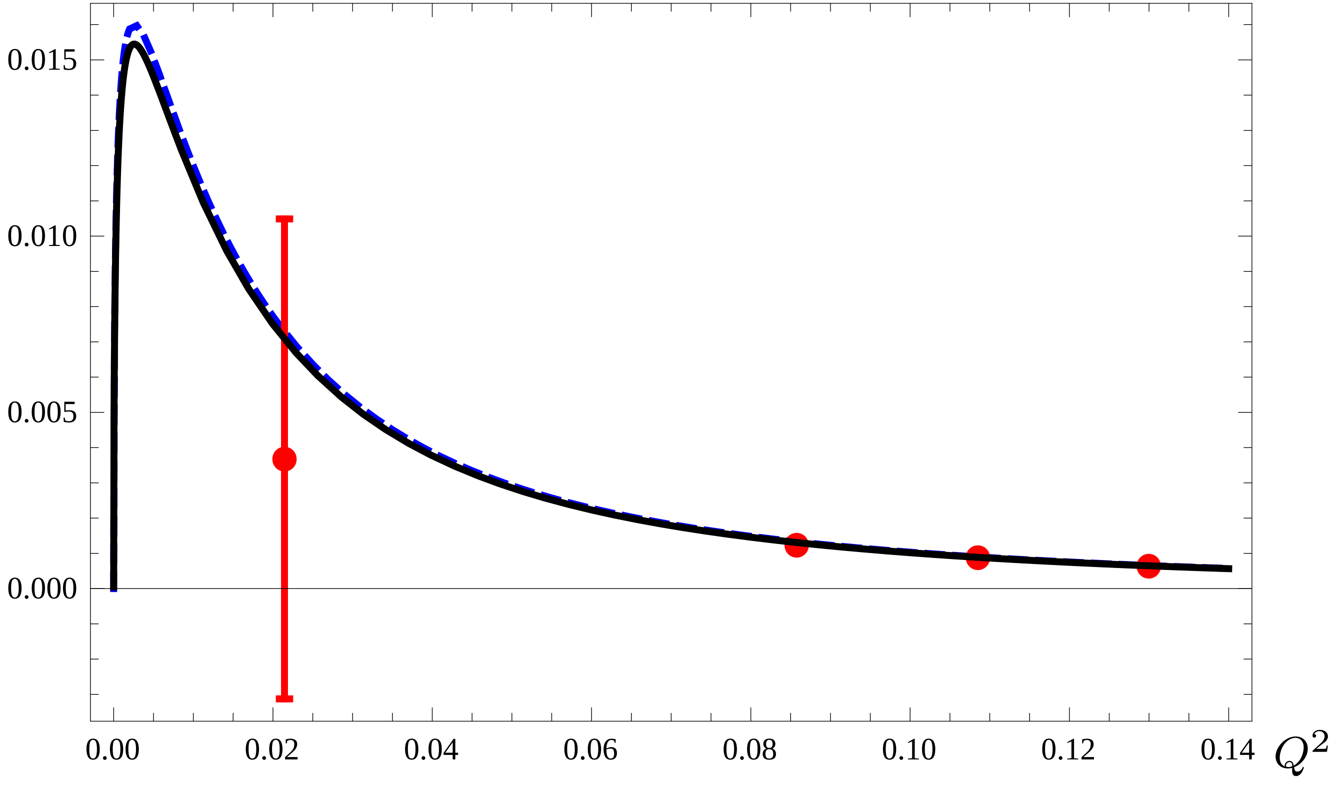


Moral

while, in fact, this is what you should be looking at:

dashed blue=Truth
solid line=[2,2] Pade Fit

(g-2) Integrand



Other Pade strategies are possible...

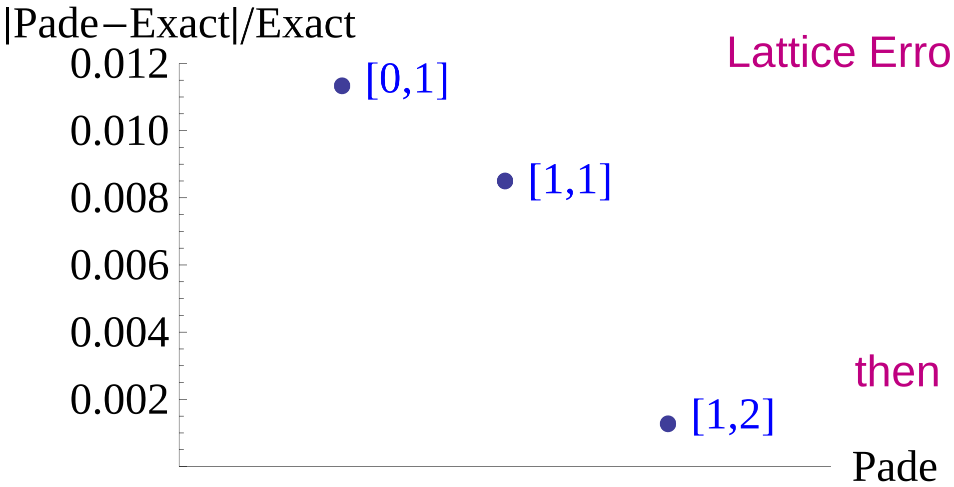
Chakraborty et al. arXiv:1403.1778
Aubin et al. (unpublished)

Instead of fitting Pades to lattice data in a Q^2 interval, determine Taylor coeff's around $Q^2 = 0$ and then build Pades to compute $a_\mu^{vac.pol.}$.

$$\Pi(Q^2) - \Pi(0) = Q^2 \left(\Pi_1 + \Pi_2 Q^2 + \Pi_3 Q^4 + \Pi_4 Q^6 + \ldots\right)$$

with $\Pi_j = \frac{(-1)^{j+1}}{(2j+2)!} \int d^4x \, t^{2j+2} \langle J_1(x) J_1(0) \rangle$.

In our model $\Pi_1 = -0.178565$ etc... and construct $[0,1], [1,1], [1,2], ..$ to compute $a_\mu^{vac.pol.}(0 \leq Q^2 \leq 1 \text{ GeV}^2)$.



Lattice Errors: Chakraborty et al. use s and c quarks

and find $\frac{\delta \Pi_j}{\Pi_j} \sim j \, \epsilon$

then $\frac{\delta a_\mu}{a_\mu} \sim \epsilon$ (both w/o corr's or w 100% corr's)

and u, d quarks ??

Conclusions

- VMD-type fits turn out to be unreliable for an accuracy in $(g-2)_\mu$ of few per cent.

- Benchmark your fitting method: Use a (reasonable) model to generate fake data with your lattice Q^2 values and cov. matrix to get a good check on your systematic error. (If you are interested we could try to help).

- Do not only rely on the χ^2 of your fit for assessing the accuracy in $(g-2)_\mu$.

- Blow up the region of the integrand around $Q^2 \sim m_\mu^2$. You need good data there. (Twisted BC's (Della Morte et al. '12, Aubin et al. '13), analytic continuation (Feng et al. '13), ...)

  Showing plots of $\Pi(Q^2)$ for large Q^2 is very misleading.

- Model independence is necessary: Pades are a useful example (convergent and efficient at low orders).

BACK-UP SLIDES

“IN GOD WE TRUST,

ALL THE OTHERS MUST BRING DATA.”

(W. Edwards Deming, American Statistician, 1900-1993.)

Science Fiction

(Recall exact value: $(g-2)_\mu^{HVP}|_{Q^2\leq 1\ \mathrm{GeV}^2} = 1.2059 \times 10^{-7}$.)

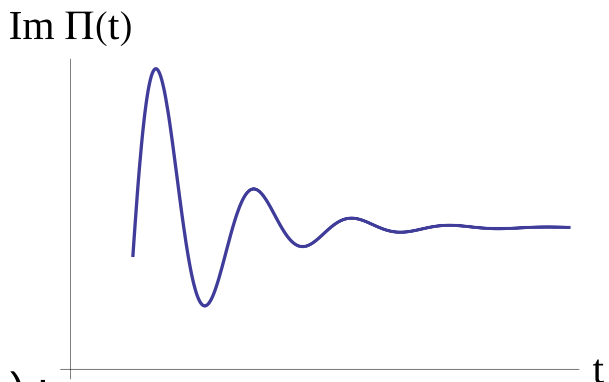
Reduce the previous covariance matrix by 10^4:

	$(g-2)_\mu \times 10^7$	Error$\times 10^7$	χ^2/dof	Pull
VMD	1.3210	0.00005	$2 \times 10^7/47$	-
VMD+	1.0732	0.00008	$7 \times 10^4/46$	-
$[0,1]$	0.89774	0.00010	$2 \times 10^7/46$	-
$[1,1]$	1.1011	0.0002	$5 \times 10^4/45$	-
$[1,2]$	1.1644	0.0004	$1340/44$	-
$[2,2]$	1.1884	0.0015	$76.4/43$ (?)	12
$[2,3]$	1.1987	0.0028	$42.0/42$	2.6

- VMD-type fits are very bad.
- Pade fits are eventually better, but need good data around the peak for χ^2 errors to be reliable.
- $[2,3]$ reaches error comparable to present e^+e^- and τ-data based determination.
- To reduce "Pull", need twisting or larger volumes to have very good data in the region of curvature *of the integrand.* See also the strategies in de Divitiis et al. '13 and Feng et al. '13.

Duality Violations(I)

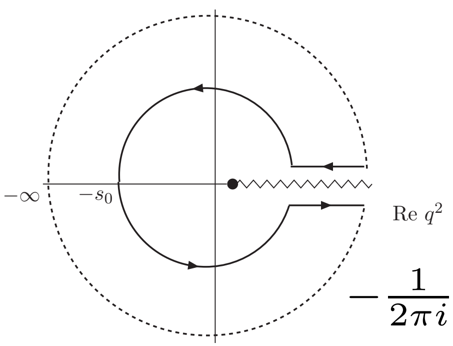
- OPE valid in euclidean, but not in minkowski. We know that spectrum $\neq$ OPE



- We expect (@ large t) :

$$\mathrm{Im}\Pi_{DV} \sim \mathrm{Im}(\Pi - \Pi_{OPE}) \sim \kappa \underbrace{e^{-\gamma t}}_{\text{OPE asympt.}} \underbrace{\sin(\alpha + \beta t)}_{\text{Regge}}$$

- $\Pi_{DV}(s) \to 0$ as $|s| \to \infty$. Then:



(Cata-Golterman-S.P. '05)

$$-\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\, \Pi_{DV}(z) = - \underbrace{\int_{s_0}^{\infty} ds}_{\text{extrapolation!}} \; w(s)\, \frac{1}{\pi}\mathrm{Im}\Pi_{DV}(s)$$

D. Violations(II)

Blok-Shifman-Zhang '98; Cata-Golterman-SP '05'08; Jamin '11

Explicit realization only models, no theory. Take $\Lambda_{QCD} = 1$; $F \sim 0.1$, decay constant.

- 1 resonance ($M \to M + i\Gamma/2$):

$$\frac{F^2}{q^2 - n} \longrightarrow \frac{F^2}{q^2 - n - i\sqrt{n}\,\Gamma}$$

- Regge-like tower: $n = 1, 2, 3, \ldots$

$$\Pi(q^2) \sim \sum_n^{\infty} \frac{F^2}{z+n} \quad , \quad z = \underbrace{(-q^2)^{\zeta}}_{\text{cut},\ q^2>0} \quad , \quad \zeta \simeq 1 - \mathcal{O}(\frac{1}{N_c})$$

$$\sim \quad \psi(z) = \frac{d \log \Gamma(z)}{dz}$$

- For $q^2 < 0 \longrightarrow \Pi(q^2) \sim \log z + \sum \frac{c_n}{z^n}$

- For $q^2 > 0 \longrightarrow \psi(z) = \psi(-z) - \frac{1}{z} - \pi \cot(\pi z)$,

$$\mathrm{Im}\Pi(q^2) \sim \mathrm{Im}(\log z) + \mathrm{Im} \sum \frac{c_n}{z^n} + \quad \underline{\underline{F^2 \ \mathrm{e}^{\frac{-q^2}{N_c}} \ \sin(\alpha + \beta\, q^2)}} \qquad F \sim 0.1 \quad ; \quad \alpha, \beta \sim 1$$

Uncorrelated Fits

(Recall exact value: $(g-2)_\mu^{HVP}|_{Q^2\leq 1\ \mathrm{GeV}^2} = 1.2059 \times 10^{-7}$.)

	$(g-2)_\mu \times 10^7$	Error$\times 10^7$	$\mathcal{Q}^2/\mathrm{dof}$	Pull
VMD	1.2145	0.0082	$75.2/47$	-1.1
VMD+	1.085	0.017	$15.0/46$	7
$[0,1]$	0.999	0.023	20.1/46	9
$[1,1]$	1.17	0.074	13.8/45	0.4
$[1,2]$	1.30	0.32	13.6/44	-0.3

- “Error" column is the result of linear propagation