Dispersive approach to hadronic light-by-light

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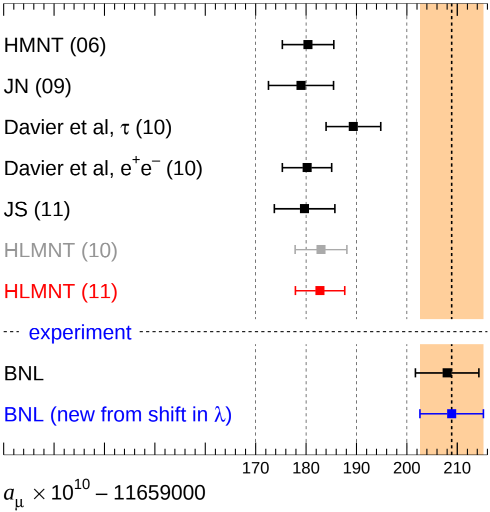
Outline

arXiv:1402.7081

in collaboration with M. Hoferichter, M. Procura and P. Stoffer

Status of $(g-2)_\mu$, experiment vs SM



Hagiwara et al. 2012

Status of $(g-2)_\mu$, experiment vs SM

Different contributions to the total SM result

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.95	0.04
electroweak, total	153.6	1.0
HVP (LO) [Hagiwara et al. 2011]	6 949.	43.
HVP (HO) [Hagiwara et al. 2011]	$-98.$	1.
HLbL [Jegerlehner-Nyffeler 2009]	116.	40.
theory	116 591 839.	59.

Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
  (but going much below 1% is hard – dealing with radiative corrections poses serious problems)
- Hadronic light-by-light (HLbL) is more problematic:
  - "it *cannot* be expressed in terms of measurable quantities"
  - reliability of uncertainty estimate based more on consensus than on a systematic method
  - only first-principle method in sight: lattice QCD
    (when will it become competitive?) $\rightarrow$ we'll know more after this workshop

Different evaluations of HLbL

Jegerlehner Nyffeler 2009

Table 13
Summary of the most recent results for the various contributions to $a_\mu^{\mathrm{LbL;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3 \pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (Ks are subdominant)
- heavier single-particle poles decreasingly important (unless one models them to resum the high-energy tail)

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light

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Conclusions

Some notation

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0|T\{j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\}|0\rangle$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x)\gamma^\mu q_i(x)$, $i = u, d, s$

$$k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

Helicity amplitudes

$$\begin{aligned} H_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s,t,u) &\equiv \mathcal{M}(\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2) \to \gamma^*(-q_3,\lambda_3)\gamma(k,\lambda_4)) \\ &= \epsilon_\mu(\lambda_1,q_1)\epsilon_\nu(\lambda_2,q_2)\epsilon^*_\lambda(\lambda_3,-q_3)\epsilon^*_\sigma(\lambda_4,k)\Pi^{\mu\nu\lambda\sigma} \end{aligned}$$

with Mandelstam variables

$$s = (q_1+q_2)^2 = (k-q_3)^2 \quad t = (q_1+q_3)^2 = (k-q_2)^2 \quad u = (q_2+q_3)^2 = (k-q_1)^2$$

and s-channel scattering angle

$$z_s = \cos\theta_s = \frac{s}{(s-q_3^2)\sqrt{\lambda_{12}}}\left(t - u + \frac{(q_1^2-q_2^2)q_3^2}{s}\right) \qquad \lambda_{12} = \lambda(s,q_1^2,q_2^2)$$

Contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2) = -k^\rho \frac{\partial}{\partial k^\sigma}\Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2).$$

Contribution to a_μ:

$$a_\mu = \lim_{k\to 0} \mathrm{Tr}\left\{(\not{p}+m)\Lambda^\rho(p',p)(\not{p}'+m)\Gamma_\rho(p',p)\right\}$$

$$\Gamma_\rho = e^6\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2q_2^2q_3^2}\frac{\gamma^\mu(\not{p}'+\not{q}_1+m)\gamma^\lambda(\not{p}-\not{q}_2+m)\gamma^\nu}{((p'+q_1)^2-m^2)((p-q_2)^2-m^2)}k^\sigma\partial_{k^\rho}\Pi_{\mu\nu\lambda\sigma}$$

with the projector

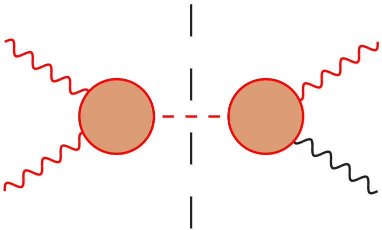
$$\Lambda^\rho(p',p) = \frac{m^2}{k^2(4m^2-k^2)}\left\{\gamma^\rho + \frac{k^2+2m^2}{m(k^2-4m^2)}(p+p')^\rho\right\}$$

m denotes the mass of the muon, p and $p' = p-k$ the momenta of the incoming and outgoing muon, respectively

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: known

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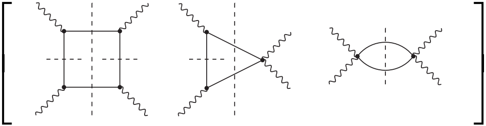
$F^V_\pi(q_1^2) F^V_\pi(q_2^2) F^V_\pi(q_3^2) \times$ 

Contribution with two simultaneous cuts

– analytic properties like the box diagram in sQED

– triangle and bulb diagram required by gauge invariance

– multiplication with F^V_π gives the correct q^2 dependence

it is not an approximation!

Setting up the dispersive calculation

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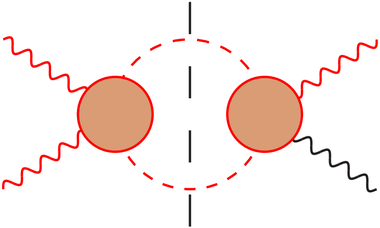

The “rest” with 2π intermediate states has cuts only in one channel and is what will be calculated dispersively

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will be neglected

Master formula

$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{I^{\pi\pi}}{q_1^2 q_2^2 s \left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)},$$

$$I^{\pi\pi} = \sum_{i\in\{1,2,3,6,14\}} \left(T_{i,s} I_{i,s} + 2 T_{i,u} I_{i,u} \right) + 2 T_{9,s} I_{9,s} + 2 T_{9,u} I_{9,u} + 2 T_{12,u} I_{12,u}$$

with $I_{i,(s,u)}$ dispersive integrals and $T_{i,(s,u)}$ integration kernels

$$I_{1,s} = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'-s} \left(\frac{1}{s'-s} - \frac{s' - q_1^2 - q_2^2}{\lambda(s', q_1^2, q_2^2)} \right) \mathrm{Im}\bar{h}^0_{++,++}(s'; q_1^2, q_2^2; s, 0),$$

$$T_{1,s} = \frac{16}{3} s \left\{ m^2 + \frac{8 P_{21}\, p \cdot q_1}{\lambda_{12}} \right\}, \qquad T_{1,u} = \frac{16}{3} \left\{ \frac{4 P_{12}^2}{\lambda_{12}} - P_{12} - Z_u \right\},$$

Master formula

$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{I^{\pi\pi}}{q_1^2 q_2^2 s\left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)},$$

$$I^{\pi\pi} = \sum_{i \in \{1,2,3,6,14\}} \left(T_{i,s} I_{i,s} + 2 T_{i,u} I_{i,u}\right) + 2 T_{9,s} I_{9,s} + 2 T_{9,u} I_{9,u} + 2 T_{12,u} I_{12,u}$$

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$$I_{6,s} = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{\left(s' - q_1^2 - q_2^2\right)\left(s' - s\right)^2} \mathrm{Im}\bar{h}^{2}_{+-,+-}(s'; q_1^2, q_2^2; s, 0) \left(\frac{75}{8}\right)$$

Helicity amplitudes contribute up to $J = 2$ (S and D waves)

Master formula

The bars on the helicity amplitudes mean that we must subtract the FsQED contribution.

The unitarity relation for the barred imaginary parts read

$$\mathrm{Im}_s \bar{h}_{J,ij}(s) = \\ = h^{c}_{J,i}(s;q_1^2,q_2^2)\left(h^{c}_{J,j}(s;q_3^2,0)\right)^* - N_{J,i}(s;q_1^2,q_2^2)N_{J,j}(s;q_3^2,0) \\ + \frac{1}{2}h^{n}_{J,i}(s;q_1^2,q_2^2)\left(h^{n}_{J,j}(s;q_3^2,0)\right)^*$$

where:

$h^{c,n}_{J,i}$ = helicity amplitudes for $\gamma^*\gamma^* \to \pi^+\pi^-$ and $\pi^0\pi^0$ resp.

$N_{J,i}$ = partial-wave projection of the $\gamma^*\gamma^* \to \pi^+\pi^-$ Born term

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A convenient basis

$\Pi^{\mu\nu\rho\sigma}$: gauge $+$ Lorentz inv. $+$ $(k^2=0)$ $\Rightarrow$ 29 scalar functions
But: in such a minimal basis crossing symmetry is hidden

A convenient (redundant) basis:

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \Big(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s,t,u) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t,s,u) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u,t,s) \Big)$$

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where (just one example):

$$A_{1,s}^{\mu\nu\lambda\sigma} = \frac{8}{(s-q_3^2)\lambda_{12}} \left(k^\lambda q_3^\sigma - k\cdot q_3\, g^{\lambda\sigma} \right) \left(q_{12}^{\mu\nu} + \frac{\lambda_{12}}{4} g^{\mu\nu} \right)$$

$A_{i,t}^{\mu\nu\lambda\sigma}$ from $(q_2,\nu) \leftrightarrow (q_3,\lambda)$ $A_{i,u}^{\mu\nu\lambda\sigma}$ from $(q_1,\mu) \leftrightarrow (q_3,\lambda)$

$\Rightarrow$ crossing symmetry is explicit

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Crucial property of this basis: the helicity amplitudes in each channel are "diagonal" :

$$\bar{H}_{++,++}(s,t,u) = \Pi_1(s,t,u) + \hat{H}_{++,++}(s,t,u)$$

$$\bar{H}_{00,++}(s,t,u) = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2} \Pi_2(s,t,u) + \hat{H}_{00,++}(s,t,u)$$

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Crucial property of this basis: the helicity amplitudes in each channel are "diagonal" and unitarity relations "simple":

$$\mathrm{Im}_s \bar{H}_{++,++}(s,t,u) = \mathrm{Im}_s \Pi_1(s,t,u)$$
$$\mathrm{Im}_s \bar{H}_{00,++}(s,t,u) = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2} \mathrm{Im}_s \Pi_2(s,t,u)$$

The cut in the s-channel of each s-channel helicity amplitude is only due to one single $\Pi_i(s,t,u)$ function (which only has a cut in s)

Unitarity relations for helicity amplitudes

Helicity amplitudes admit a partial wave expansion

$$H_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s,t,u) = \sum_J D^J(z_s) h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s)$$

where $D^J(z_s)$ is the appropriate Wigner function.

Each partial wave satisfies a simple unitarity relation (for $s > 0$)

$$\mathrm{Im} h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) = \frac{\sigma_s}{16\pi}\theta(s-4m_\pi^2)\, h_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2) h^*_{J,\lambda_3\lambda_4}(s;q_3^2,0)$$

where $h_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2)$ are partial-wave helicity amplitudes of the subprocess $\gamma^*\gamma^* \to \pi\pi$.

Dispersion relations for the $\Pi_i(s,t,u)$

- The $\Pi_i(s,t,u)$ only have a cut in s and for $s \geq 4m_\pi^2$
- Their imaginary part coincides with that of the related helicity amplitude
- The latter can be expanded in partial waves and for each of them unitarity fixes the imaginary part in terms of partial-wave helicity amplitudes of the subprocess $\gamma^*\gamma^* \to \pi\pi$
- a dispersive integral over the right-hand cut of each partial wave would in principle allow one to reconstruct the whole $\Pi_i(s,t,u)$, up to a polynomial

Simplified dispersion relations for the $\Pi_i(s,t,u)$

We will carry out the program outlined in the previous slide with one simplification (but see later!):

for each $\Pi_i(s,t,u)$ we only keep the discontinuity due to the lowest partial wave (*i.e.* *S* or *D*)

all $\Pi_i(s,t,u)$ become single-variable functions $\Rightarrow \Pi_i(s)$

This is analogous to the representation for the $\pi\pi$ scattering, $\eta \to 3\pi$ etc. amplitudes based on the "reconstruction theorem"
Amplitude $= \sum$ single-variable functs. (right-hand cut only)

Stern, Sazdjian, Fuchs (93)

Fixing subtraction constants: soft-photon zeros

Gauge-invariance implies the presence of so-called soft-photon zeros
Low (58), Moussallam (13)

$$H_{\lambda_1\lambda_2,\lambda_3\lambda_4} \stackrel{k\to 0}{\to} \propto (s - q_3^2)$$

and analogously

$$H_{\lambda_1\lambda_2,\lambda_3\lambda_4} \stackrel{q_{1,2}\to 0}{\to} \propto (s - q_{2,1}^2)$$

In a dispersive representation such a property must emerge from the kernels of the dispersive integrals

and constrains the subtraction polynomial

Soft-photon zeros in $\gamma^*\gamma^* \to \pi\pi$

These soft-photon zeros can be studied also in subprocess $\gamma^*\gamma^* \to \pi\pi$ where a dispersive representation for the helicity amplitudes reads

$$h_{J,i}(s) = \frac{1}{\pi} \sum_{J' \text{ even}} \sum_{j=1}^{5} \int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{ij}(s,s') \, \mathrm{Im} h_{J',j}(s') + \cdots, \quad i,j \in \{\lambda_1\lambda_2\}$$

the ellipsis stands for integrals of crossed-channel partial waves

The diagonal kernel functions

$$K_{00}^{++,++}(t,t') = K_{00}^{00,00}(t,t') = \frac{1}{t'-t} - \frac{t'-q_1^2-q_2^2}{\lambda(t',q_1^2,q_2^2)}$$

$$K_{22}^{++,++}(t,t') = K_{22}^{00,00}(t,t') = \frac{p_t^2 q_t^2}{p_t'^2 q_t'^2}\left(\frac{1}{t'-t} - \frac{t'-q_1^2-q_2^2}{\lambda(t',q_1^2,q_2^2)}\right)$$

display the desired soft-photon behaviour

Soft-photon zeros in $\gamma^*\gamma^* \to \pi\pi$

Soft-photon zeros of the $\gamma^*\gamma^* \to \pi\pi$ sub-amplitudes manifest themselves as a modification of the Cauchy kernel by a factor:

$$K_{12}(s,s') = \frac{f_{12}(s,s')}{s'-s}, \qquad K_{34}(s,s') = \frac{f_{34}(s,s')}{s'-s},$$

for the initial- and final-state photon pair, respectively.

A modified Cauchy kernel that gives the HLbL tensor the proper soft-photon zeros is obtained by factorization

$$K_{12,34}(s,s') = \frac{f_{12}(s,s')f_{34}(s,s')}{s'-s}.$$

Dispersion relations for the $\Pi_i(s)$

Imposing the same form of the soft-photon zeros as in the subamplitudes $\gamma^*\gamma^* \to \pi\pi$ we obtain the following dispersion relations:

$$\Pi_1^s = \bar{h}^0_{++,++}(s) = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left(\frac{1}{s'-s}-\frac{s'-q_1^2-q_2^2}{\lambda'_{12}}\right)\mathrm{Im}\bar{h}^0_{++,++}(s')$$

$$y\Pi_2^s = \bar{h}^0_{00,++}(s) = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left(\frac{1}{s'-s}-\frac{s'-q_1^2-q_2^2}{\lambda'_{12}}\right)\mathrm{Im}\bar{h}^0_{00,++}(s')$$

with $y = -\frac{q_1^2 q_2^2}{\xi_1\xi_2}$ [and similarly for the others]

Dispersion relations for the $\Pi_i(s)$

Soft-photon zeros for the Π_i^s or for the helicity amplitudes?

Remember

$$\bar{H}_{++,++}(s,t,u) = \Pi_1^s + \hat{H}_{++,++}(s,t,u)$$

with

$$\hat{H}_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s,t,u) = \sum_{i=1}^{15} \left(f^i_{\lambda_1\lambda_2,\lambda_3\lambda_4} \Pi_i^t + \tilde{f}^i_{\lambda_1\lambda_2,\lambda_3\lambda_4} \Pi_i^u \right)$$

Dispersion relations for the $\Pi_i(s)$

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with

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By sheer kinematics the soft-photon zeros imposed on the Π_i^s imply the correct soft-photon zeros to the full helicity amplitudes

Our dispersive representation of the HLbL tensor

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u) \right)$$

- the $\Pi_i(s)$ are single-variable functions having only a right-hand cut
- for the discontinuity we keep only the lowest partial wave
- the dispersive integral that gives the $\Pi_i(s)$ in terms of its discontinuity has the required soft-photon zeros
- soft-photon zeros constrain the subtraction polynomial to vanish (unless one wanted to subtract more, which is unnecessary)

Contribution of $\bar{\Pi}_{\mu\nu\lambda\sigma}$ to a_μ

$$a_\mu = \lim_{k\to 0} \mathrm{Tr}\Big\{ (\not{p}+m)\Lambda^\rho(p',p)(\not{p}'+m)\Gamma_\rho(p',p) \Big\}$$

$$\Gamma_\rho = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 q_3^2} \frac{\gamma^\mu(\not{p}'+\not{q}_1+m)\gamma^\lambda(\not{p}-\not{q}_2+m)\gamma^\nu}{((p'+q_1)^2-m^2)((p-q_2)^2-m^2)} k^\sigma \partial_{k^\rho} \Pi_{\mu\nu\lambda\sigma}$$

Technical caveat

A disadvantage of our basis: the helicity amplitudes have kinematical singularities – the full HLbL tensor, however, doesn't.

$\Rightarrow$ In order to make sense of the limit $k_\mu \to 0$ for $\bar{\Pi}_{\mu\nu\lambda\sigma}$ we must average over the direction of k_μ first

Contribution of $\bar{\Pi}_{\mu\nu\lambda\sigma}$ to a_μ

$$a_\mu = \frac{1}{16m} \mathrm{Tr}\left\{ \left(\not{p} + m\right) \left[\gamma^\rho, \gamma^\tau\right] \left(\not{p} + m\right) \tilde{\Gamma}_{\rho\tau} \right\}$$

$$\tilde{\Gamma}_{\rho\tau} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s} \frac{\gamma^\mu (\not{p} + \not{q}_1 + m)\gamma^\lambda (\not{p} - \not{q}_2 + m)\gamma^\nu}{\left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)}$$
$$\times \left[\int \frac{d\Omega(p,k)}{4\pi} \frac{k_\tau k^\sigma}{k^2} \frac{\partial}{\partial k^\rho} \bar{\Pi}_{\mu\nu\lambda\sigma} \right]_{k=0}$$

Contribution of $\bar{\Pi}_{\mu\nu\lambda\sigma}$ to a_μ

$$a_\mu = \frac{1}{16m} \mathrm{Tr}\left\{ (\not{p} + m) \left[\gamma^\rho, \gamma^\tau\right] (\not{p} + m) \tilde{\Gamma}_{\rho\tau} \right\}$$

$$\tilde{\Gamma}_{\rho\tau} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s} \frac{\gamma^\mu (\not{p} + \not{q}_1 + m) \gamma^\lambda (\not{p} - \not{q}_2 + m) \gamma^\nu}{\left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)}$$
$$\times \left[\int \frac{d\Omega(p,k)}{4\pi} \frac{k_\tau k^\sigma}{k^2} \frac{\partial}{\partial k^\rho} \bar{\Pi}_{\mu\nu\lambda\sigma} \right]_{k=0}$$

- all $A^i_{\mu\nu\rho\sigma}$ tensors scale like $\mathcal{O}(k^0)$
- any term of $\mathcal{O}(k^2)$ in the $\Pi_i(s)$ does not contribute to a_μ
- higher partial waves in $\Pi_i(s)$ are suppressed by angular momentum factors:

$$q_{34}^2 = (s - q_3^2)^2/(4s) = \mathcal{O}(k^2)$$

- $\Rightarrow$ keeping only the lowest partial wave in the discontinuity of the $\Pi_i(s)$ is not an approximation for the calculation of a_μ

Master formula

$$a_\mu^{\pi\pi} = e^6 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{I^{\pi\pi}}{q_1^2 q_2^2 s\left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)},$$

$$I^{\pi\pi} = \sum_{i\in\{1,2,3,6,14\}} \left(T_{i,s} I_{i,s} + 2T_{i,u} I_{i,u}\right) + 2T_{9,s} I_{9,s} + 2T_{9,u} I_{9,u} + 2T_{12,u} I_{12,u}$$

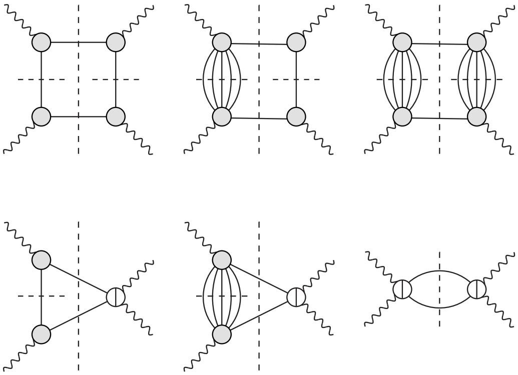
with $I_{i,(s,u)}$ dispersive integrals and $T_{i,(s,u)}$ integration kernels

Conclusions and outlook

- I have presented a dispersive framework for the calculation of the HLbL contribution to a_μ
- which takes into account only single- and double-pion intermediate states
  the extension to other single-particle intermediate states (η, η', etc.) is trivial
- we have derived a master formula which expresses the contribution of 2π intermediate states to a_μ in terms of (integrals over) $\gamma^*\gamma^* \to \pi\pi$ partial waves
- a numerical evaluation of the master formula is in progress
- we believe that this is a step towards a model-independent calculation of the HLbL contribution to a_μ

Backup Slides

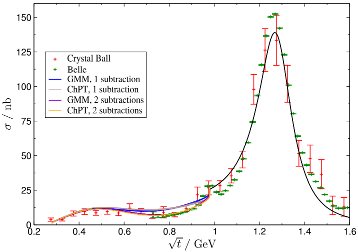
Master formula: what contributions are included? How?



Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$

Roy-Steiner eqs. $=$ Dispersion relations $+$ partial-wave expansion
$+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- On-shell $\gamma\gamma \to \pi\pi$: prominent *D*-wave reson. $f_2(1270)$ Moussallam (10) Hoferichter, Phillips, Schat (11)
- $\gamma^*\gamma \to \pi\pi$ Moussallam (13)
- $\gamma^*\gamma^* \to \pi\pi$, new feature: anomalous thresholds Hoferichter, GC, Procura, Stoffer (13)



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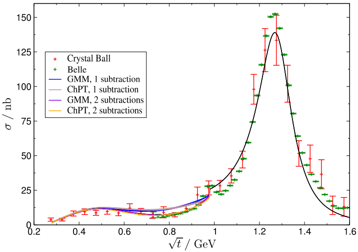
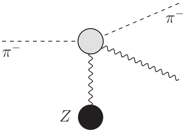
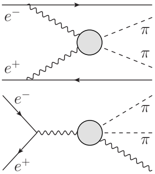


- Constraints
  - Low energy: pion polar., ChPT
  - Primakoff: $\gamma\pi \to \gamma\pi$ at COMPASS, JLAB
  - Scattering: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - Decays: $\omega, \phi \to \pi\pi\gamma$





Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$

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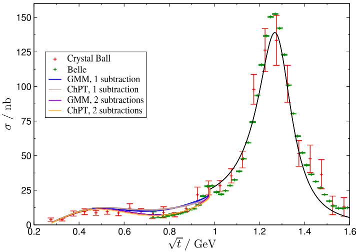
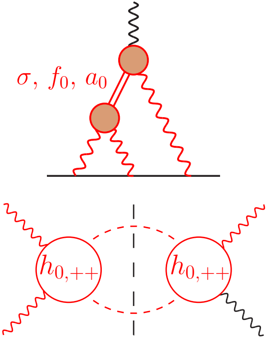


Analysis of the Roy-Steiner equations for $\gamma^*\gamma^* \to \pi\pi$ is in progress: any experimental input most welcome

$\to$ talks by Pennington, Hanhart and Kupsc

Physics of $\gamma^*\gamma^* \to \pi\pi$

- $\pi\pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- S-wave provides model-independent implementation of the σ

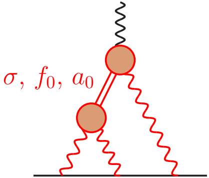


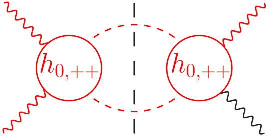
Physics of $\gamma^*\gamma^* \to \pi\pi$

- $\pi\pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- S-wave provides model-independent implementation of the σ
- Analytic continuation with dispersion theory: resonance properties
  - Precise determination of σ-pole from $\pi\pi$ scattering Caprini, GC, Leutwyler 2006

$$M_\sigma = 441^{+16}_{-8}\,\mathrm{MeV} \qquad \Gamma_\sigma = 544^{+18}_{-25}\,\mathrm{MeV}$$

  - Coupling $\sigma \to \gamma\gamma$ from $\gamma\gamma \to \pi\pi$ Hoferichter et al. 2011, Dai, Pennington 2014





$f_0(500)$ PARTIAL WIDTHS

$\Gamma(\gamma\gamma)$ **Γ_2**

VALUE (keV)	DOCUMENT ID		TECN	COMMENT
• • • We do not use the following data for averages, fits, limits, etc. • • •				
1.7 ±0.4	[54] HOFERICHTER	11	RVUE	Compilation
3.08±0.82	[55] MENNESSIER	11	RVUE	Compilation
2.08±0.2 $^{+0.07}_{-0.04}$	[56] MOUSSALLAM	11	RVUE	Compilation
2.08	[57] MAO	09	RVUE	Compilation
1.2 ±0.4	[58] BERNABEU	08	RVUE	
3.9 ±0.6	[55] MENNESSIER	08	RVUE	$\gamma\gamma \to \pi^+\pi^-$, $\pi^0\pi^0$
1.8 ±0.4	[59] OLLER	08	RVUE	Compilation

$f_0(500)$ or σ
was $f_0(600)$

$I^G(J^{PC}) = 0^+(0^{++})$

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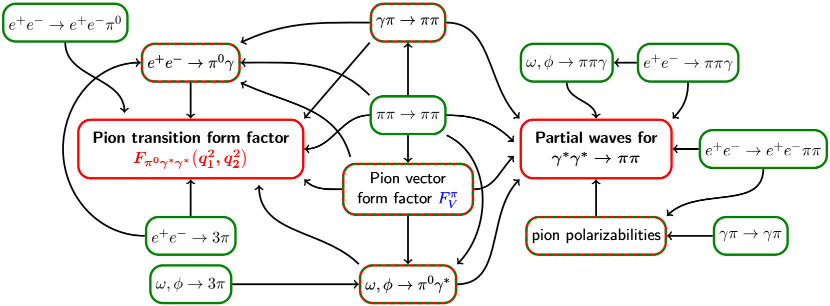
$f_0(500)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\ \mathrm{Im}(\sqrt{s_{\text{pole}}})$.

VALUE (MeV)	DOCUMENT ID		TECN	COMMENT
(400–550)−i(200–350) OUR ESTIMATE				
• • • We do not use the following data for averages, fits, limits, etc. • • •				
$(445 \pm 25)-i(278^{+22}_{-18})$	[1,2] GARCIA-MAR..	11	RVUE	Compilation
$(457^{+14}_{-13})-i(279^{+11}_{-7})$	[1,3] GARCIA-MAR..	11	RVUE	Compilation
$(442^{+5}_{-8})-i(274^{+6}_{-5})$	[4] MOUSSALLAM	11	RVUE	Compilation
$(452 \pm 13)-i(259 \pm 16)$	[5] MENNESSIER	10	RVUE	Compilation
$(448 \pm 43)-i(266 \pm 43)$	[6] MENNESSIER	10	RVUE	Compilation
(455 ± 6 +31 [illegible])−i(278 ± 6 +34 [illegible])	[7] CAPRINI	08	RVUE	Compilation

Hadronic light-by-light: a roadmap



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists