Two-pion low-energy contribution to the muon $g-2$ with improved precision from analyticity and unitarity

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Outline

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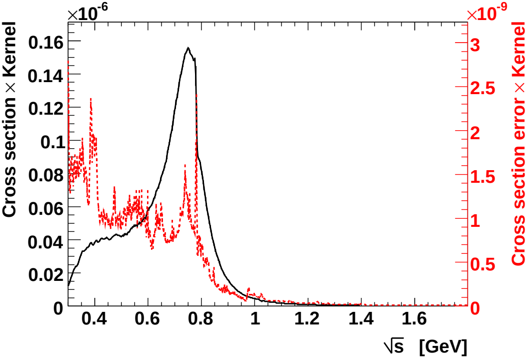
Motivation



- Black: the cross section for the combined e^+e^- data multiplied by the kernel function $K(s)$ in the integral for a_μ
- Red: the corresponding error contribution, with statistical and systematic errors added in quadrature

[M. Davier, A. Hoecker, B. Malaescu, C.Z. Yuan and Z. Zhang, Eur.Phys.J. C66 (2010) 1, arXiv:0908.4300]

Leading Order (LO) two-pion contribution to a_μ from the range $[t_l,\ t_u]$:

$$a_\mu^{\pi\pi,\,\mathrm{LO}}[\sqrt{t_l}, \sqrt{t_u}] = \frac{\alpha^2 m_\mu^2}{12\pi^2} \int_{t_l}^{t_u} dt\, K(t)\, \beta_\pi^3(t)\, |F(t)|^2 \left(1 + \frac{\alpha}{\pi}\, \eta_\pi(t)\right)$$

Particular values [Davier et al. 2010]

- Threshold region, no data, ChPT fit:

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[2m_\pi,\ 0.30\,\mathrm{GeV}] = (0.55 \pm 0.01) \times 10^{-10}$$

- From 0.3 GeV to 0.63 GeV, from combined e^+e^- experiments:

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\ 0.63\,\mathrm{GeV}] = (132.6 \pm 1.3) \times 10^{-10}$$

Problem: is it possible to reduce the error by exploiting the properties of $F(t)$ and using information from other energies?

Basic idea:

- Use the phase $\arg[F(t)]$ in the elastic region of the unitarity cut, where it is known with precision from Fermi-Watson theorem and Roy equations for $\pi\pi$ scattering
- Use measurements of $|F(t)|$ at higher energies where the precision is better

Requirements:

- Make a parametrization-free analytic extrapolation
- Results independent of the unknown phase of $F(t)$ above the inelastic threshold
- Have good control of the errors

Achieved by using:

- Analyticity and unitarity of the form factor
- Adequate mathematical methods: extremal problem for analytic functions

Extremal problem

Find optimal upper and lower bounds on $|F(t)|$ on the elastic unitarity cut, $t_+ < t < t_{in}$, for $F(t) \in \mathcal{C}$, where $\mathcal{C}$ is the class of functions real analytic in the t-plane cut along the real axis for $t \geq t_+$, which satisfy the following conditions:

1. Fermi-Watson theorem:

$$\mathrm{Arg}[F(t+i\epsilon)] = \delta_1^1(t), \qquad t_+ \leq t \leq t_{in}$$

2. An integral condition on the modulus squared above the inelastic threshold:

$$\frac{1}{\pi}\int_{t_{in}}^{\infty} dt \rho(t)|F(t)|^2 \leq I$$

3. Known first two Taylor coefficients at $t = 0$:

$$F(0) = 1, \qquad \left[\frac{dF(t)}{dt}\right]_{t=0} = \frac{1}{6}\langle r_\pi^2\rangle$$

4. Given value at one spacelike energy:

$$F(t_s) = F_s \pm \epsilon_s, \qquad t_s < 0$$

5. Given value of the modulus at one energy in the elastic region of the timelike axis:

$$|F(t_t)| = F_t \pm \epsilon_t, \qquad t_+ < t_t < t_{in}$$

Solution of the problem

- Define the Omnès function (for $t > t_{in}$, $\delta(t)$ arbitrary smooth function):

$$\mathcal{O}(t) = \exp\left(\frac{t}{\pi}\int_{t_+}^{\infty} dt \frac{\delta(t')}{t'(t'-t)}\right)$$

- Define the function $h(t)$ defined by $F(t) = \mathcal{O}(t)h(t)$. It has the properties:
  - $h(t)$ is real below t_{in}, *i.e.* is analytic in the t-plane cut only for $t > t_{in}$
  - $\frac{1}{\pi}\int\limits_{t_{in}}^{\infty} \rho(t)|\mathcal{O}(t)|^2|h(t)|^2 dt \leq I$

- Conformal map $z \equiv \tilde{z}(t)$ of the t-plane cut for $t > t_{in}$ onto the unit disc $|z| < 1$:

$$\tilde{z}(t) = \frac{\sqrt{t_{in}} - \sqrt{t_{in} - t}}{\sqrt{t_{in}} + \sqrt{t_{in} - t}}, \qquad \tilde{z}(0) = 0, \qquad t \equiv \tilde{t}(z)$$

- Define two outer functions (analytic without zeros) with modulus on the boundary equal to $\rho(t)|d\tilde{t}(z)/dz|$ and $|\mathcal{O}(t)|$, respectively:

$$w(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \, \frac{e^{i\theta}+z}{e^{i\theta}-z} \, \ln[\rho(\tilde{t}(e^{i\theta}))|\frac{d\tilde{t}}{dz}|]\right]$$

$$\omega(z) = \exp\left(\frac{\sqrt{t_{in} - \tilde{t}(z)}}{\pi}\int_{t_{in}}^{\infty} \mathrm{d}t' \frac{\ln|\mathcal{O}(t')|}{\sqrt{t' - t_{in}}(t' - \tilde{t}(z))}\right)$$

Then the function $g(z)$ defined by:

$$g(z) \equiv F(\tilde{t}(z))\,[\mathcal{O}(\tilde{t}(z))]^{-1}\,w(z)\,\omega(z)$$

is analytic in $|z| < 1$ and satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2 d\theta \equiv ||g||_{L^2} \leq I$$

$\Rightarrow$ a standard problem (Meiman problem): if $g(z)$ is analytic in $|z| < 1$ and satisfies the norm condition $||g||_{L^2} \leq I$, find rigorous correlations among the values $g(z_j)$ and the derivatives $g^{(k)}(z_m)$ at points inside the holomorphy domain [Duren 1970]

Solution of Meiman problem

Let $g(z)$ analytic in $|z| < 1$ and satisfies $\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2 d\theta \leq I$. Then, if

$$\left[\frac{1}{k!}\frac{d^k g(z)}{dz^k}\right]_{z=0} = g_k, \quad 0 \leq k \leq K-1, \quad g(z_n) = \xi_n, \quad z_n = z_n^*, \quad 1 \leq n \leq N$$

$$\bar{I} = I - \sum_{k=0}^{K-1} g_k^2, \qquad \bar{\xi}_n = \xi_n - \sum_{k=0}^{K-1} g_k z_n^k$$

$$\mathcal{D} = \begin{vmatrix} \bar{I} & \bar{\xi}_1 & \bar{\xi}_2 & \cdots & \bar{\xi}_N \\ \bar{\xi}_1 & \frac{z_1^{2K}}{1-z_1^2} & \frac{(z_1 z_2)^K}{1-z_1 z_2} & \cdots & \frac{(z_1 z_N)^K}{1-z_1 z_N} \\ \bar{\xi}_2 & \frac{(z_1 z_2)^K}{1-z_1 z_2} & \frac{(z_2)^{2K}}{1-z_2^2} & \cdots & \frac{(z_2 z_N)^K}{1-z_2 z_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\xi}_N & \frac{(z_1 z_N)^K}{1-z_1 z_N} & \frac{(z_2 z_N)^K}{1-z_2 z_N} & \cdots & \frac{z_N^{2K}}{1-z_N^2} \end{vmatrix}$$

$\Rightarrow$ **the determinant $\mathcal{D}$ and its minors are nonnegative**

Application

- $K = 2$: g_0 and g_1 depend on the charge radius $\langle r_\pi^2 \rangle$
- $N = 3$:
  - $N_{input} = 2$ points used as input, one spacelike and other timelike
  - t_{out} the point where we want to calculate bounds on $|F(t_{out})|$

- The condition $\mathcal{D} \geq 0$ gives a quadratic inequality with known coefficients for the unknown modulus $|F(t_{out})|$, from which upper and lower bounds on it are obtained
- The positivity of the minors provide consistency constraints on the quantities that enter as input, which ensures that the quadratic equations for the bounds have real solutions

Rigorous properties of the bounds

- Are optimal for a given input
- Are independent of the phase $\delta(t)$ of the Omnès function for $t > t_{in}$
- For a fixed weight $\rho(t)$, they depend in a monotonous way on I, becoming stronger/weaker when this value is decreased/increased

Input and optimization procedure

Phase

- Below $\sqrt{t_{in}} = 0.917$ GeV the phase $\delta_1^1(t)$ determined from ChPT and Roy equations [Ananthanarayan, Colangelo, Gasser, Leutwyler 2001, Garcia-Martin, Kaminski, Pelaez, Ruiz de Elvira, Yndurain 2011]
    - We used two phases denoted as Bern and Madrid phases
- Above t_{in}, $\delta(t)$ taken as a continuous smooth function that approaches asymptotically π
    - The dependence on the arbitrary phase for $t > t_{in}$ of the Omnès function $\mathcal{O}(t)$ and of the auxiliary outer function $\omega(z)$ exactly compensate each other: rigorous result proven for Lipschitz continuous functions and checked numerically [Abbas, Ananthanarayan, Caprini, Imsong, Ramanan 2010]

High-energy integral

- Input data on $|F(t)|$:
  - From t_{in} to $\sqrt{t} = 3$ GeV Babar data [Aubert et al. 2009]
  - For 3 GeV$\leq \sqrt{t} \leq 20$ GeV a constant modulus smoothly connected with a perturbative QCD inspired $1/t$ decrease above 20 GeV

    $\Rightarrow$ conservative choice (input modulus much higher than perturbative QCD)
- Weights of the form: $\rho(t) = \frac{t^b}{(t+Q^2)^c}$, $\qquad Q^2 \geq 0,\ b \leq c \leq b+2$

$$w(z) = (2\sqrt{t_{in}})^{1+b-c} \frac{(1-z)^{1/2}}{(1+z)^{3/2-c+b}} \frac{(1+\tilde{z}(-Q^2))^c}{(1-z\tilde{z}(-Q^2))^c}$$

- The weights with a rapid decrease allow a precise calculation of the integral, but lead to a weaker constraint
- The weights with a slower decrease have a bigger constraining power, but do not suppress the unknown high energy part
  - Suitable choice: $\rho(t) = \frac{1}{t}$ (the range above 3 GeV contributes with $\sim 1\%$)
  - $I = 0.578 + 0.022$ (0.022 the error set by BaBar data)

Input from the holomorphy domain

- Normalization condition $F(0) = 1$
- Input range for charge radius [Anantanarayan, Caprini, Das, Imsong 2013, 2014]

$$\langle r_\pi^2 \rangle \in (0.41,\, 0.45)\, \mathrm{fm}^2$$

  - Range of radius already restricted by the other constraints (mainly by the input modulus)
  - This condition has a weak constraining power

- Recent measurements on the spacelike axis [Horn et al. 2006, Huber et al. 2008]

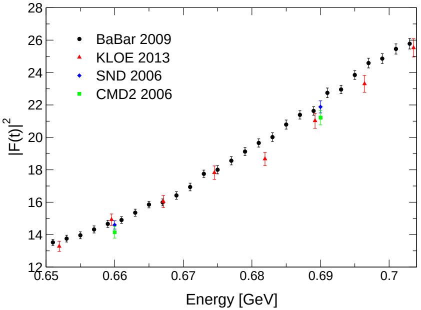
$$F(-1.60\,\mathrm{GeV}^2) = 0.243 \pm 0.012^{+0.019}_{-0.008}$$
$$F(-2.45\,\mathrm{GeV}^2) = 0.167 \pm 0.010^{+0.013}_{-0.007}$$

  - The spacelike input has a rather weak constraining power

Input from the elastic region of the cut

Modulus from e^+e^- experiments in the range 0.65-0.70 GeV (range considered in previous studies of the charge radius)

- SND: 2 points [M.N. Achasov et al. (SND Collaboration), 2006]
- CMD2: 2 points [R.R. Akhmetshin et al. (CMD-2 Collaboration), 2006]
- BaBar: 26 points [B. Aubert et al. (BABAR Collaboration), 2009]
- KLOE: 8 points [D. Babusci et al. (KLOE and KLOE-2 Collaborations), 2013]



Isospin breaking

- The formalism can be applied only in the isospin limit
- We corrected the experimental input for $\rho - \omega$ mixing, described by the factor [Leutwyler 2002, Hanhart 2012]

$$F_{\rho-\omega}(t) = \left(1 + \epsilon \frac{t}{t_\omega - t}\right), \quad t_\omega = (m_\omega - \frac{i}{2}\Gamma_\omega)^2$$

- At the end the factor $|F_{\rho-\omega}(t)|$ was reincluded in the bounds on the modulus used for the calculation of a_μ

Vacuum Polarization

- The experimental collaborations (CMD2, SND, BABAR, KLOE) include the vacuum polarization into the definition of the pion form factor
- The $|F(t)|$ used as input was obtained by removing the vacuum polarization from the experimental form factor
- Alternatively, we obtained $|F(t)|$ from the bare cross section:

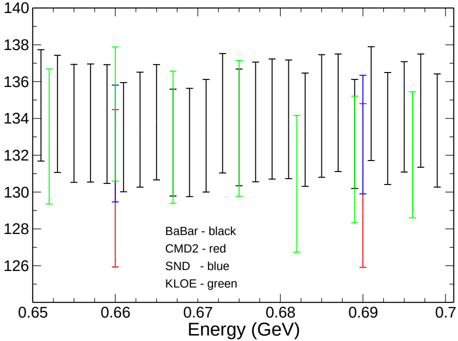
$$|F(t)|^2 = \frac{3t}{\alpha^2 \pi \beta_\pi(t)^3} \frac{\sigma^0_{\pi\pi(\gamma)}(t)}{1 + \frac{\alpha}{\pi}\,\eta_\pi(t)}$$

Optimization procedure

- For central values of the input quantities we obtained very narrow allowed intervals for the output modulus $|F(t)|$ in the range $[t_l, t_u]$ of interest
- To account for the uncertainties, we generated a large sample of data by varying the input quantities (phase, input modulus, spacelike value, charge radius) within their error intervals
- For each point in the sample we computed upper and lower bounds on $|F(t)|$
- We have taken the most conservative bounds, *i.e.* the largest upper bound and the smallest lower bound on $|F(t)|$ from the values obtained with the sample of generated data $\Rightarrow$ a larger allowed interval
- We finally varied the input spacelike and timelike points. Since the analyticity constraints provided by the values at different t must be valid simultaneously, we have taken the "intersection" of the individual allowed ranges, *i.e.* the smallest upper bound and the largest lower bound
- By inserting the upper and lower bounds on $|F(t)|$ into the integral we derived allowed intervals for $a_\mu^{\pi\pi,\,\mathrm{LO}}\,[\sqrt{t_l},\,\sqrt{t_u}]$

Results

Upper and lower bounds on a_μ



Alowed interval for $a_\mu^{\pi\pi,\,\mathrm{LO}}[0.30\,\mathrm{GeV},\,0.63\,\mathrm{GeV}] \times 10^{10}$ using as input the Bern phase and the modulus in the region 0.65-0.70 GeV from the e^+e^- experiments

- SND and CMD2: 2 points each
- BaBar: 26 points
- KLOE: 8 points

- For SND and CMD2 the intervals are rather large and consistent between them
- For BABAR the intervals are narrower and exhibit a moderate variation from point to point
- For KLOE the intervals exhibit a more pronounced variation with the input point

- Taking the intersection is equivalent with combining the central values and errors with a large correlation
- This prescription requires good, consistent input data, which produce narrow allowed intervals with a relatively large common part
- A too small overlap may signal inconsistencies among the input data

- In this analysis we kept all the points, assuming they are all reliable

	Bern phase	Madrid phase
CMD2 06	0.5528 ± 0.0089	0.5527 ± 0.0092
SND 06	0.5532 ± 0.0083	0.5530 ± 0.0086
BABAR 09	0.5534 ± 0.0080	0.5533 ± 0.0083
KLOE 13	0.5531 ± 0.0080	0.5530 ± 0.0084

Central values and errors for $a_\mu^{\pi\pi,\,\mathrm{LO}}\,[2m_\pi,\,0.30\,\mathrm{GeV}] \times 10^{10}$ given by the intersection of the alowed intervals obtained with timelike input from the region 0.65-0.70 GeV for each experiment

	Bern phase	Madrid phase
CMD2 06	130.531 ± 3.955	129.739 ± 4.545
SND 06	132.775 ± 2.862	132.313 ± 2.759
BABAR 09	133.732 ± 1.761	133.484 ± 1.461
KLOE 13	132.380 ± 1.721	132.086 ± 1.451

Central values and errors for $a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\,0.63\,\mathrm{GeV}] \times 10^{10}$ given by the intersection of the alowed intervals obtained with timelike input from the region 0.65-0.70 GeV for each experiment

Combining the results

- For each experiment, we combine first the values obtained with the two phases (Bern and Madrid). For instance, assuming a correlation of 50% between them:

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\,0.63\,\mathrm{GeV}] = (130.243 \pm 3.637) \times 10^{-10} \qquad \text{CMD2}$$

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\,0.63\,\mathrm{GeV}] = (132.527 \pm 2.432) \times 10^{-10} \qquad \text{SND}$$

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\,0.63\,\mathrm{GeV}] = (133.563 \pm 1.365) \times 10^{-10} \qquad \text{BaBar}$$

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\,0.63\,\mathrm{GeV}] = (132.184 \pm 1.349) \times 10^{-10} \qquad \text{KLOE}$$

- The results obtained with input from the 4 experiments are not completely independent, since they include some common input. However, the experimental modulus has the dominant effect. Assuming no correlations, we obtain

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[2m_\pi,\,0.30\,\mathrm{GeV}] = (0.553 \pm 0.004) \times 10^{-10}$$

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\,0.63\,\mathrm{GeV}] = (132.673 \pm 0.866) \times 10^{-10}$$

Comparison

The direct determinations from low energy data [Davier et al 2010] vs. the present determinations based on data from higher energies:

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[2m_\pi,\ 0.30\,\mathrm{GeV}] = (0.55 \pm 0.01) \times 10^{-10}$$

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[2m_\pi,\ 0.30\,\mathrm{GeV}] = (0.553 \pm 0.004) \times 10^{-10}$$

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\ 0.63\,\mathrm{GeV}] = (132.6 \pm 1.3) \times 10^{-10}$$

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\ 0.63\,\mathrm{GeV}] = (132.673 \pm 0.866) \times 10^{-10}$$

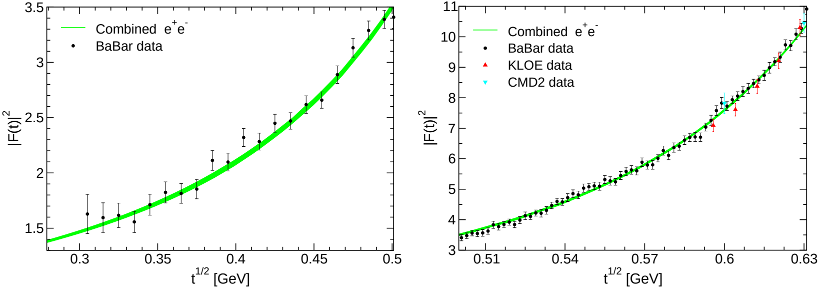
- Remarkable consistency of the central values and a slight improvement of the precision
- We can further combine the two predictions, the direct and the indirect one. If we assume that they are independent:

$$a_\mu^{\pi\pi,\,\mathrm{LO}}\,[0.30\,\mathrm{GeV},\ 0.63\,\mathrm{GeV}] = (132.651 \pm 0.721) \times 10^{-10}$$

Summary and outlook

- We proposed a formalism which exploits in an optimal way the knowledge of the phase in the elastic region with some information on the modulus
- The method is parametrization free; the price to be paid is that we can only derive upper and lower bounds on $|F(t)|$. But with the precise input available, the bounds are quite stringent
- The predicted contributions to $a_\mu^{\pi\pi,\,\mathrm{LO}}$ from below 0.63 GeV have central values almost identical to the direct determinations from ChPT fits and combined e^+e^- experiments, but slightly smaller errors
- By combining the result based on the technique of bounds with the direct determination from combined e^+e^- experiments, we obtained a reduction of the error $\delta_\mu^{\pi\pi}$ by $\sim 6 \times 10^{-11}$
- Perspectives:
  - Application to data from τ decays (under consideration)
  - Use more precise input at intermediate energies when will be available

Upper and lower bounds on $|F(t)|^2$



Allowed band for $|F(t)|^2$ in the low energy region, obtained by combining the bounds with input from e^+e^- experiments in the region 0.65-0.70 GeV, compared with the available experimental data.