Leading hadronic contribution to $g_\mu - 2$ and lattice QCD

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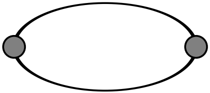
Special thanks to E. Gregory, R. Malak & C. McNeile

(work in progress)

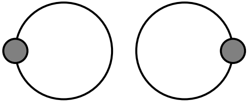
HVP LQCD challenges

Compute directly in Euclidean spacetime (Blum '02)

$$\Pi_{\mu\nu}(Q) = a^4 \sum_{x_\mu} e^{iQ\cdot x} \langle J_\mu^{\mathrm{EM}}(x) J_\nu^{\mathrm{EM}}(0) \rangle$$



- dominant and straightforward



- computationally demanding
- Zweig and doubly $SU(3)_f$ suppressed, but could be $\sim 10\%$ (Della Morte et al '10)

$\Rightarrow$ required for precision computation

To control all systematics, need:

- $N_f = 2+1+1$ simulations $\Rightarrow$ precision matching to real world HVP
- $m_{ud}^{\mathrm{lat}} \to m_{ud}^{\mathrm{phys}}$ and $L \to \infty \Rightarrow$ correctly describe dominant $\pi\pi$ contributions for a_μ
- $a \to 0 \Rightarrow$ reach perturbative Q^2 values and eliminate lattice artefacts

HVP LQCD challenges (cont'd)

Tradtional extraction

$$\Pi(Q^2) = \Pi_{\mu\nu}(Q)/\left(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2\right) = \Pi_{\mu\nu}(Q)/T_{\mu\nu}$$

$\Rightarrow$ $\Pi(0)$ not directly accessible and error blows up for $Q^2 \to 0$

$\Rightarrow$ Q_μ in units of $2\pi/L(,T) \gtrsim 200\,\text{MeV}$ for $T, L \lesssim 6\,\text{fm}$

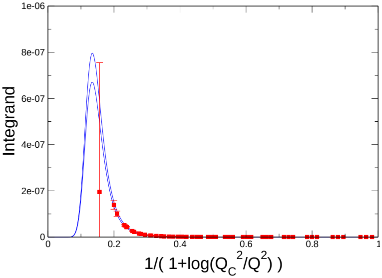
However (Lautrup et al '69, Blum '02)

$$a_\mu^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, w(Q^2/m_\mu^2)\hat{\Pi}(Q^2)$$

$$\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$$

[Plotted as function of $t = \left(1 + \ln(Q_C^2/Q^2)\right)^{-1}$ w/ $Q_C^2 = 11\,\text{GeV}^2$]

[Boyle et al '11, $a = 0.086\,\text{fm}$, $M_\pi \simeq 290\,\text{MeV}$, $L^3 \times T = (2.8^3 \times 5.6)\,\text{fm}^4$]



- extrapolate $Q^2 \to 0 \Rightarrow$ large uncertainties (Aubin et al '12)
- twisted BCs (Sachrajda et al '05) $\Rightarrow$ FV violation of unitarity and of QED WTI (Aubin et al '13)
- $\partial_Q^n \Pi_{\mu\nu}(Q)|_{Q=0}$ to get info. about $\Pi(Q^2)$ directly at $Q^2 = 0$ (de Divitiis et al '12, HPQCD '14)
- perform an analytic continuation (see Karl's talk) (Feng et al '13)

Zero-momentum moments and Padé's of $\hat{\Pi}(Q^2)$

HPQCD propose to compute on lattice (arXiv:1403.1778):

$$G_{2n} = a^4 \sum_{x_\mu} x_0^{2n} \langle J_i^{\rm EM}(x) J_i^{\rm EM}(0) \rangle$$

which give coefficients of Taylor expansion

$$\hat{\Pi}(Q^2) = \sum_{n=1}^{\infty} \Pi_n Q^{2n}, \qquad \Pi_n \equiv (-1)^{n+1} \frac{G_{2n+2}}{(2n+2)!}$$

Use to obtain Padé (see also Santi's talk & Aubin et al '12)

$$\hat{\Pi}_{[N/D]}(Q^2) \equiv \frac{\sum_{n=0}^{N} a_n Q^{2n}}{1 + \sum_{n=1}^{D} b_n Q^{2n}}$$

by matching term-by-term to Taylor expansion up to order $N + D$

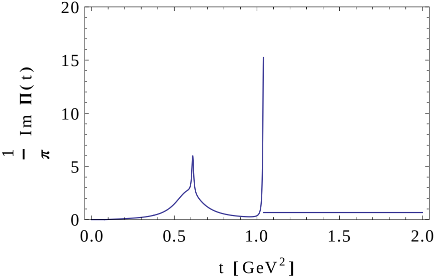
Perform calculation for connected s and c quark contributions

Convergence and residual errors of Padé's

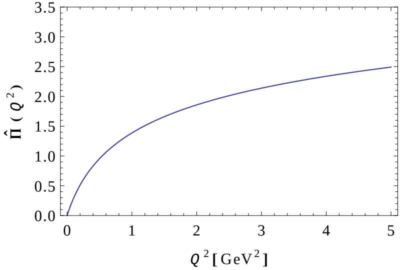
Test idea on phenomenological model of HVP:

$$\hat{\Pi}(Q^2) = \int_{4M_\pi^2}^{\infty} dt \frac{Q^2}{t(t+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(t)$$

w/ spectral function adapted from Bernecker et al '11 "fit" of $e^+e^- \to \mathrm{hadrons}$

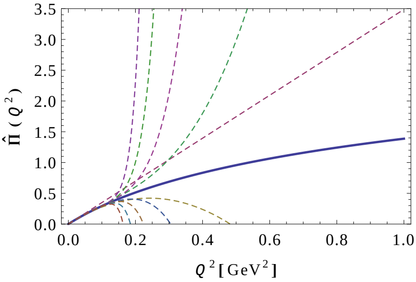


$\longrightarrow$



$$\longrightarrow a_\mu^{\mathrm{HVP,LO}} = 694. \times 10^{-10}$$

Convergence and residual errors of Padé's (cont'd)



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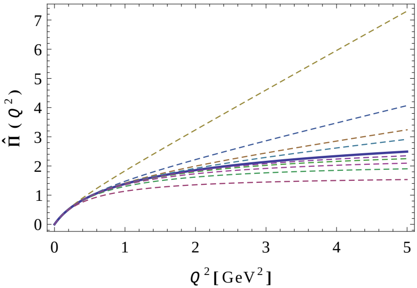
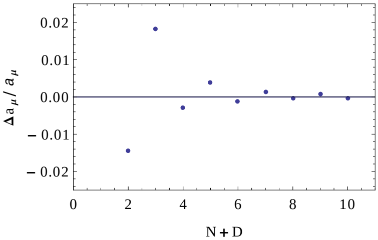


Taylor expansions for $N = 1, \cdots, 10$

$[N, N-1]$ & $[N, N]$ Padé's: $[1, 1] \to [5, 5]$



- $\Delta a_\mu / a_\mu \lesssim 0.003$ from $\hat{\Pi}_{[2,2]}(Q^2)$
- Requires G_{10}, i.e. 10th moment of $\Pi_{\mu\nu}$!

⇒ statistics and systematics?

⇒ unphysical poles, . . . ?

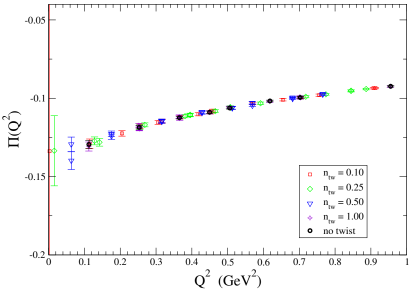
HVP simulations used in preliminary BMWc study

$am_{ud}^{\rm bare}$	$am_s^{\rm bare}$	volume	# cfgs	M_π [GeV]	$\theta^{\rm tw}/2\pi$
		2-HEX ($N_f = 2+1$)			
		$\beta = 3.31$, $a^{-1} = 1.697$ GeV			
-0.09933	-0.0400	$48^3 \times 48$	928	0.136(2)	
-0.09300	-0.0400	$24^3 \times 48$	210	0.255(2)	
		$\beta = 3.5$, $a^{-1} = 2.131$ GeV			
-0.05294	-0.0060	$64^3 \times 64$	83	0.130(2)	
-0.04900	-0.0120	$32^3 \times 64$	216	0.250(2)	
-0.04900	-0.0060	$32^3 \times 64$	110	0.258(2)	
-0.04630	-0.0120	$32^3 \times 64$	212	0.308(2)	
		$\beta = 3.61$, $a^{-1} = 2.561$ GeV			
-0.03000	-0.0042	$32^3 \times 48$	188	0.332(4)	0.5, 0.25, 0.1
		$\beta = 3.7$, $a^{-1} = 3.026$ GeV			
-0.02700	0.0000	$64^3 \times 64$	208	0.182(2)	

$am_u^{\rm bare}$	$am_d^{\rm bare}$	$am_s^{\rm bare}$	$am_c^{\rm bare}$	volume	# cfgs	M_π (GeV)
			3-HEX ($N_f = 4$)			
			$\beta = 3.2$, $a^{-1} = 1.897$ GeV			
-0.0806	-0.0794	-0.033	0.71	$32^3 \times 64$	240	0.250

- One simulation with twisted BCs: $\psi(x + aL_\mu) = e^{i\theta_\mu^{\rm tw}}\psi(x)$
  $\Rightarrow Q_\mu = \frac{\theta_\mu^{\rm tw} + 2\pi n_\mu}{L_\mu}$ w/ n_μ integer in $[-L_\mu/2a, L_\mu/2a[$
- Point current at source, conserved at sink

Twisted vs non-twisted results and statistical errors



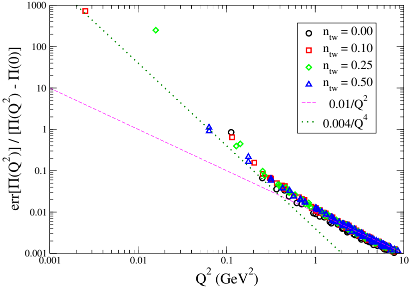
Twisted momenta $\rightarrow$
$Q_\mu^{\min} \simeq \{0.1, 0.25, 0.5\} \times 500\,\mathrm{MeV}$

[$\beta = 3.61$ ($a \simeq 0.077\,\mathrm{fm}$) and $M_\pi \simeq 350\,\mathrm{MeV}$]

Stat. err. grows as $1/Q^4$ for $Q^2 \rightarrow 0$

Due to division by $(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)$ and subtraction $\Pi(Q^2) - \Pi(0)$

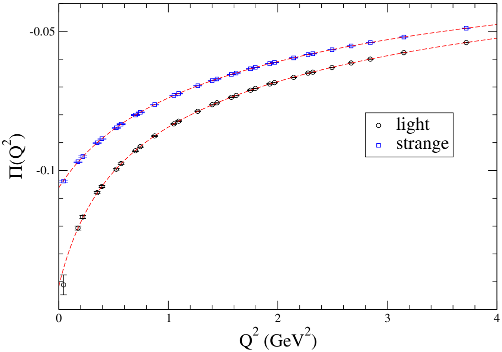
$\Rightarrow$ usefulness of twisting limited at fixed statistics



Sample fit of $\Pi(Q^2)$ vs Q^2

Fit $\Pi(Q^2)$ vs Q^2 to multiple pole with $N = 1, 2$ ($\leftrightarrow$ $[N, N]$ Padé):

$$\Pi(Q^2) = C + \sum_{i=1}^{N} \frac{f_V^2}{Q^2 + M_V^2}$$

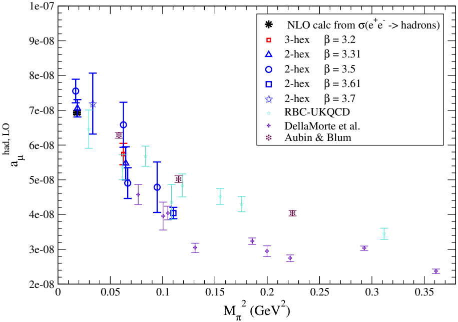


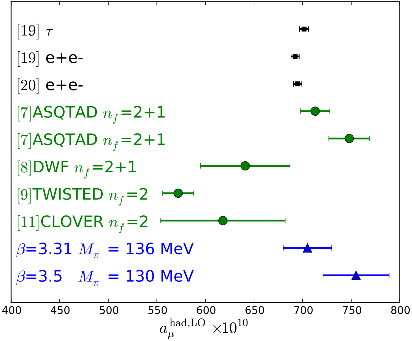
[$\beta = 3.5$ ($a \sim 0.092\,\mathrm{fm}$) and $M_\pi \simeq 250\,\mathrm{MeV}$]

Preliminary results for individual simulations

Warnings:

- Neglect contribution from $Q^2 \gtrsim 2\,\mathrm{GeV}^2$ for the moment
- No quark-disconnected contributions
- No estimate of systematic errors yet





- $\sim 3\%$ stat. err. per point at physical M_π
- Can easily be improved with more sources

Conclusion

- "Straightforward" to obtain quark-connected HVP and corresponding a_μ directly at physical M_π with percent level stat. errors

  $\Rightarrow$ may give impression that we are already there!
- Systematic errors still poorly understood
- Reaching low Q^2 w/ twisted BCs has limited interest due to $1/Q^4$ rise in relative statistical errors
- Statistics and systematics of zero-momentum spacetime moments/momentum derivatives of $\Pi_{\mu\nu}$ must be understood
- No reliable estimate of quark-disconnected HVP
- In next few years should have $a_\mu^{\mathrm{HVP,LO}}$ with fully controlled stat. and syst. uncertainties at few percent level

  $\Rightarrow$ non-trivial check of experiment-based, dispersive determinations of $a_\mu^{\mathrm{HVP,LO}}$ ...
  $\Rightarrow$ ... but what experimental numbers?
- In longer run, LQCD competitive with dispersive determinations of a_μ^{HVP}