Total transition rates from lattice QCD

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Cluster of Excellence

Based on 1704.08993 (PRD), with M. Hansen and D. Robaina

What is this talk about?

Lattice QCD is formulated in Euclidean time, with states living on an $L \times L \times L$ torus.

- Generically, calculating the rate at which hadrons are produced in a reaction is not directly possible from Euclidean correlators (an analytic continuation is involved).
- When only one or two channels are open, the rate can be related to *stationary* observables (energy levels and matrix elements of states on a torus; Lüscher & Lüscher-Lellouch relations for $\langle\pi\pi|H_W|K\rangle$ or $\langle\pi\pi|J|0\rangle$)
- When several channels are open, the stationary states on the torus are linear superpositions of all possible channels. Extracting an exclusive transition rate requires undoing the quantum entanglement, which becomes untractable as the energy increases.
- Can we at least say something about the total transition rate?

Simplest example: $\langle j_\mu^{\rm e.m.}(x) j_\nu^{\rm e.m.}(0)\rangle_{\rm lattice}$ has a spectral representation in terms of $\sigma(e^+e^- \to \text{hadrons}) \Rightarrow$ consistency check. What if there is a hadron in the initial state?

Fermi's Golden Rule description of particle decay

- Let $|\Psi\rangle$ be a zero-momentum, one-particle state on a 3d torus of volume L^3, described by a Hamiltonian H_0;
- Let now $H = H_0 + V$ with a small perturbation V that allows is to decay in the $L \to \infty$ limit:

$$\Gamma = 2\pi \lim_{t\to\infty} \lim_{L\to\infty} \sum_k |V_k(L)|^2 \, \delta_{1/t}(E_k(L) - M), \qquad V_k(L) \equiv \langle k, L|V|\Psi\rangle.$$

- the states are unit-norm states and $\delta_{1/t}(\omega) = \frac{2}{\pi} \frac{\sin^2(\omega t/2)}{\omega^2 t}$ is a regularized delta-function.
- the k$^{\rm th}$ term is equal to the probability that in a measurement done at time t the system is observed in the unperturbed state k, divided by t.
- rewrite the width as

$$\Gamma = \frac{1}{2M} \lim_{\Delta\to 0} \lim_{L\to\infty} \int_0^\infty d\omega \, \underbrace{\left[4\pi M \sum_{k=0}^{\infty} |V_k(L)|^2 \, \delta(\omega - E_k(L))\right]}_{=\rho(\omega,L)} \, \widehat{\delta}_\Delta(M,\omega)$$

- $\widehat{\delta}_\Delta(M,\omega)$ is any regularized delta-fcnt going rapidly to zero for $\omega \to \infty$.

QFT-textbook expression of particle width

- In the infinite-volume limit, expect

$$\lim_{L\to\infty}\int_0^\infty d\omega\ \widehat{\delta}_\Delta(M,\omega)\ \rho(\omega,L) = \int_0^\infty d\omega\ \widehat{\delta}_\Delta(M,\omega)\ \rho(\omega), \qquad (\star)$$

$$\rho(\omega) = \sum_\alpha \frac{1}{S_\alpha}\int d\Phi_\alpha(k_1,\dots,k_{N_\alpha})\,\underbrace{|\langle P=(\omega,\mathbf{0}),\alpha;\mathrm{out}\,|\mathcal{V}(0)|\Psi\rangle|^2}_{\equiv|\mathcal{M}(\Psi\to\alpha)|^2}$$

  where $d\Phi_\alpha$ is the Lorentz-invariant phase-space measure for an N_α-particle state of energy ω, and $V=\int d^3x\ \mathcal{V}(x)$.
- Now take the limit $\Delta\to 0$ of expression $(\star)$.
- Hence the textbook QFT expression for the particle width,

$$\Gamma = \frac{1}{2M}\rho(\omega=M).$$

NB. $d\Phi_\alpha(k_1,\cdots,k_{N_\alpha}) \equiv \frac{d^3\mathbf{k}_1}{(2\pi)^3 2\omega_{\mathbf{k}_1}}\cdots\frac{d^3\mathbf{k}_{N_\alpha}}{(2\pi)^3 2\omega_{\mathbf{k}_{N_\alpha}}}(2\pi)^4\delta^4(P-\sum_{i=1}^{N_\alpha}k_i)\,.$

$$\langle \boldsymbol{k}|\boldsymbol{k}'\rangle = (2\pi)^3\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,2E_{\boldsymbol{k}}.$$

Extracting $\int_0^\infty d\omega\, \widehat{\delta}_\Delta(M,\omega)\, \rho(\omega,L)$ from Euclidean correlators (I)

Let $\Psi^\dagger(\tau)$ be an interpolating operator for the Ψ particle at rest:

$$\begin{aligned} G(\tau,L) &\equiv 2ML^3 e^{-M\tau} \lim_{\tau_f\to\infty} \lim_{\tau_i\to-\infty} \frac{\langle \Psi(\tau_f)\ \int d^3x\, \mathcal{V}(\tau,\boldsymbol{x})\mathcal{V}(0)\ \Psi^\dagger(\tau_i)\rangle}{\langle \Psi(\tau_f)\ \Psi^\dagger(\tau_i)\rangle} \\ &\overset{\tau\geq 0}{=} 2ML^3 \Big\langle \Psi \Big| \int d^3x\, \mathcal{V}(\tau,\boldsymbol{x})\mathcal{V}(0) \Big| \Psi \Big\rangle_L . \\ &= 2ML^6 \sum_k e^{-E_k\,\tau}\ |{}_L\langle E_k|\mathcal{V}(0)|\Psi\rangle|^2 = \int_0^\infty \frac{d\omega}{2\pi}\ \rho(\omega,L)\ e^{-\omega\tau}. \end{aligned}$$

$\rightsquigarrow$ In lattice QCD, we can compute the Laplace transform of the finite-volume spectral function $\rho(\omega,L)$.

Possible applications: $\Gamma(D\to X_s)$, $\Gamma(B\to X_c)$; $\Gamma(\Xi_{cc}^+)$, $\Gamma(\Xi_{cc}^{++})$.
Inclusive purely hadronic decay rates, where ‘inclusive’ means all final states with a given set of good quantum numbers in QCD.

Extracting $\int_0^\infty d\omega\, \widehat{\delta}_\Delta(M,\omega)\, \rho(\omega,L)$ from Euclidean correlators (II)

- There are many methods to regularize and solve the ill-posed problem of numerically inverting the Laplace transform (Maximum entropy/other Bayesian methods), but it is usually hard to estimate a systematic error.
- By linearity: (for determining Γ, choose $\bar{\omega} = M$)

$$G(\tau_j) = \int_0^\infty \frac{d\omega}{2\pi}\, \rho(\omega,L)\, e^{-\omega\tau_j}$$

$$\sum_j C_j(\bar{\omega}) G(\tau_j) = \int_0^\infty \frac{d\omega}{2\pi}\, \rho(\omega,L)\, \underbrace{\sum_j C_j(\bar{\omega}) e^{-\omega\tau_j}}_{=\widehat{\delta}_\Delta(\bar{\omega},\omega)}$$

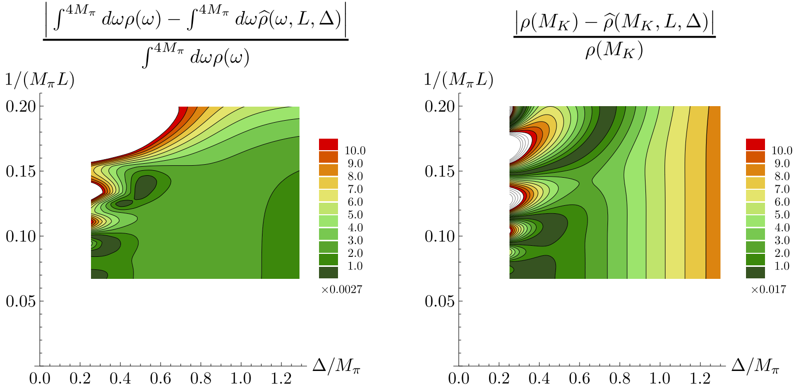
- The Backus-Gilbert method provides a recipe to determine coefficients C_j such that the energy resolution

$$\Delta = \int_0^\infty d\omega\, (\omega-\bar{\omega})^2\, \widehat{\delta}_\Delta(\omega,\bar{\omega})^2,$$

  is minimized under the normalization constraint $\int_0^\infty d\omega\, \widehat{\delta}_\Delta(\omega,\bar{\omega}) = 1$.
- Perform the calculation for several $(1/L, \Delta)$ and extrapolate to $(0,0)$.

G. Backus and F. Gilbert, Geophys. J.R. Astron. Soc. 16, 169 (1968); Phil. Trans. R. Soc. A 266, 123 (1970); Brandt et al. 1506.05732 (PRD).

Toy-model study of the $(1/L, \Delta) \to (0, 0)$ limit



- Case of a constant $K \to \pi\pi$ transition amplitude: spectral function only contains the two-body phase space.
- Intriguing: there are trajectories that lead quite rapidly to an accurate estimate of the infinite-volume spectral function.

Fig. by M. Hansen; see also HM, 1104.3708 (EPJA).

Generalization: transition with transfer of momentum to the final hadronic state

(think: semileptonic decay)

- transition spectral function in infinite volume:

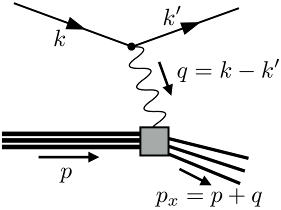
$$\rho_{\mathbf{P}}(E,\mathbf{p}) \equiv \frac{1}{n_\lambda}\sum_{\lambda,\alpha}\frac{1}{S_\alpha}\int d\Phi_\alpha(k_1,\cdots,k_{N_\alpha})|\langle E,\mathbf{p},\alpha;\mathrm{out}|\mathcal{J}(0)|\Psi,\mathbf{P},\lambda\rangle|^2$$

- Again, the Laplace transform of the finite-volume spectral function can be determined:

$$\begin{aligned} G_{\boldsymbol{P}}(\tau,\boldsymbol{p},L) &\equiv 2E_\Psi L^3 e^{-E_\Psi\tau}\lim_{\tau_f\to\infty}\lim_{\tau_i\to-\infty} \\ &\quad \frac{\langle\Psi(\tau_f,\boldsymbol{P})\ \int d^3x\ e^{i(\boldsymbol{P}-\boldsymbol{p})\cdot\boldsymbol{x}}\mathcal{J}(\tau,\boldsymbol{x})\mathcal{J}(0)\ \Psi^\dagger(\tau_i,\boldsymbol{P})\rangle}{\langle\Psi(\tau_f,\boldsymbol{P})\ \Psi^\dagger(\tau_i,\boldsymbol{P})\rangle} \\ &= 2E_\Psi L^6\sum_k e^{-E_k\tau}\ |{}_L\langle E_k,\boldsymbol{p}|\mathcal{J}(0)|\Psi,\boldsymbol{P}\rangle|^2 \\ &= \int_0^\infty \frac{d\omega}{2\pi}\ \rho_{\boldsymbol{P}}(\omega,\boldsymbol{p},L)\ e^{-\omega\tau}. \end{aligned}$$

Application: inelastic scattering on the nucleon

$$d\sigma \propto \underbrace{\ell^{\mu\nu}}_{\text{leptonic}} W_{\mu\nu}$$



The spin-averaged hadronic tensor:

$$W_{\mu\nu}(p,q) = \frac{1}{4\pi n_\lambda} \sum_\lambda \sum_\alpha \frac{1}{S_\alpha} \int d\Phi_\alpha(k_1, .., k_{N_\alpha}) \langle N\, \boldsymbol{p}\, \lambda | j_\mu(0) | p_x, \alpha\rangle \langle p_x, \alpha\, | j_\nu(0) | N, \boldsymbol{p}\, \lambda\rangle$$

where $j_\mu = \sum_f Q_f \overline{\psi}_f \gamma_\mu \psi_f$ is the electromagn. current. The task is to invert

$$G_{\mu\nu,\mathbf{p}}(\tau, \mathbf{p}_x, L) = \frac{1}{2\pi} \int_0^\infty dp_x^0\; e^{-p_x^0 \tau} \; \underbrace{\rho_{\mu\nu,\mathbf{p}}(p_x, L)}_{\text{“}\rightarrow\text{” } 4\pi W_{\mu\nu}(p,q)} \;\; ,$$

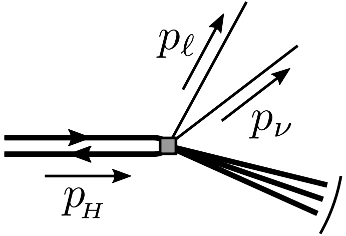
$$G_{\mu\nu,\mathbf{p}}(\tau, \mathbf{p}_x, L) \equiv 2E_{\mathbf{p}} L^3 e^{-E_{\mathbf{p}}\tau} \int d^3\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \lim_{\tau_f\to\infty} \lim_{\tau_i \to -\infty} \frac{\sum_\lambda \langle \Psi_\lambda(\tau_f, \mathbf{p}) j_\mu(\tau, \mathbf{x}) j_\nu(0) \Psi_\lambda^\dagger(\tau_i, \mathbf{p})\rangle_{\text{conn}}}{\sum_\lambda \langle \Psi_\lambda(\tau_f, \mathbf{p}) \Psi_\lambda^\dagger(\tau_i, \mathbf{p})\rangle}$$

The dispersion variable is the final-state energy. Here q is spacelike.

Application for q timelike: semileptonic decays

$$d\Gamma \propto \underbrace{\ell^{\mu\nu}}_{\text{leptonic}} W_{\mu\nu}$$

$$q = p_\ell + p_\nu$$



Here $\mathcal{J}_\mu = \overline{q}\gamma_\mu(1-\gamma_5)Q$. The task is to invert

$$G^{H_Q\to X}_{\mu\nu,\mathbf{p}}(\tau, \mathbf{p}_x, L) = \frac{1}{2\pi}\int_0^\infty d\omega\, e^{-\omega\tau}\, \underbrace{\rho^{H_Q\to X}_{\mu\nu,\mathbf{p}}(\omega, \mathbf{p}_x, L)}_{\text{“}\to\text{”}\, 2M W_{\mu\nu}(p/M, q)},$$

$$G^{H_Q\to X}_{\mu\nu,\mathbf{p}}(\tau, \mathbf{p}_x, L) \equiv 2E_{\mathbf{p}} L^3 e^{-E_{\mathbf{p}}\tau} \int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} \lim_{\tau_f\to\infty} \lim_{\tau_i\to-\infty} \frac{\langle \Psi_Q(\tau_f, \mathbf{p}) \mathcal{J}^\dagger_\mu(\tau, \mathbf{x}) \mathcal{J}_\nu(0) \Psi^\dagger_Q(\tau_i, \mathbf{p})\rangle_{\rm conn}}{\langle \Psi_Q(\tau_f, \mathbf{p}) \Psi^\dagger_Q(\tau_i, \mathbf{p})\rangle},$$

See also S. Hashimoto, Prog. Theor. Exp. Phys. 2017, 053B03 (2017);

U. Aglietti et al., Phys. Lett. B 432, 411 (1998) (this paper involves the shape function).

The inclusive semileptonic decay rate

$$W_{\mu\nu}(v,q) = -w_1 g_{\mu\nu} + w_2 v_\mu v_\nu - i w_3 \epsilon_{\mu\nu\alpha\beta} v^\alpha q^\beta + w_4 q_\mu q_\nu + w_5 (q_\mu v_\nu + v_\mu q_\nu)$$

with $w_i = w_i(v\cdot q, q^2)$. Then differential rate given by

$$\frac{d^3\Gamma}{dE_e dq^2 dq^0} = |V_{qQ}|^2 \frac{G_F^2}{32\pi^2} \Big[2q^2 w_1 + [4E_e(q^0 - E_e) - q^2] w_2 + 2q^2 (2E_e - q^0) w_3 \Big] \,,$$

See e.g. B. Blok, L. Koyrakh, M. Shifman, and A.I. Vainshtein, PRD 50, 3572 (1994).

Related work: Compton amplitude & hadronic tensor ($q^2 < 0$)

My subjective reading of the following papers:

- [W. Wilcox, NPB (Proc. Suppl.) 30 491 (1993); K.-F. Liu and S.-J. Dong, PRL72 1790 (1994)] the hadronic tensor $W_{\mu\nu}$ can be obtained by performing an inverse Laplace transform of a lattice four-point correlation function

- [X.-d. Ji and C.-w. Jung, PRL86 208 (2001)] the forward Compton amplitude

$$T_{\mu\nu}(p,q)_{\lambda,\lambda'} = i \int d^4x \; e^{iq\cdot x} \langle p, \lambda' | T\{j_\mu(x) j_\nu(0)\} | p, \lambda \rangle$$

  can be obtained from the lattice below threshold for particle production ($(p+q)^2 < M_N^2$, i.e. $\omega \equiv \frac{2p\cdot q}{-q^2} < 1$).

- QCDSF collaboration, PRL 118, 242001 (2017); [Gross & Treiman 1971]

$$\underbrace{\mathcal{P}^{\mu\nu}}_{\text{projector}} T_{\mu\nu}(p,q) = \int_0^1 \frac{dx}{1-\omega^2 x^2} F_2(x,q^2),$$

  $F_2(x,q^2)$ = unpolarized structure function, $x = 1/\omega$; and similar for $F_1(x,q^2)$. Various methods to solve the integral equations for F_1 and F_2.

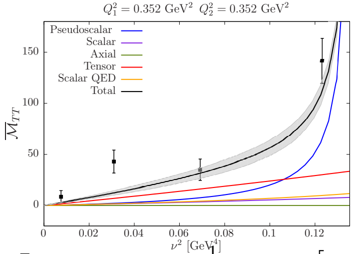
Forward $\gamma^*\gamma^* \to \gamma^*\gamma^*$ amplitude vs. $\sigma(\gamma^*\gamma^* \to \mathrm{hadrons})$

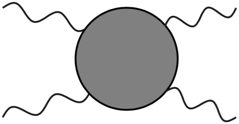
$$\Pi^E_{\mu_1\mu_2\mu_3\mu_4}(P_4; P_1, P_2) \equiv \int_{X_1,X_2,X_4} e^{-i\sum_a P_a\cdot X_a} \Big\langle J_{\mu_1}(X_1)J_{\mu_2}(X_2)J_{\mu_3}(0)J_{\mu_4}(X_4)\Big\rangle_{\mathrm{E}}$$

$$\mathcal{M}_{\mathrm{TT}}(-Q_1^2, -Q_2^2, -Q_1\cdot Q_2) = \frac{e^4}{4}\,\underbrace{R^E_{\mu_1\mu_3}R^E_{\mu_2\mu_4}}_{\text{projector}}\,\Pi^E_{\mu_1\mu_3\mu_4\mu_2}(-Q_2; -Q_1, Q_1),$$

Dispersive sum rule: $(\nu = \frac{1}{2}(s + Q_1^2 + Q_2^2))$

$$\mathcal{M}_{\mathrm{TT}}(q_1^2, q_2^2, \nu) - \mathcal{M}_{\mathrm{TT}}(q_1^2, q_2^2, 0) = \frac{2\nu^2}{\pi}\int_{\nu_0}^{\infty} d\nu'\,\frac{\sqrt{\nu'^2 - q_1^2 q_2^2}}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}(\sigma_0 + \sigma_2)(\nu'),$$





$m_\pi = 193$ MeV, $64^3 \times 128$, $a = 0.063$fm

$$R^E_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{1}{(Q_1\cdot Q_2)^2 - Q_1^2 Q_2^2}\cdot\Big[(Q_1\cdot Q_2)(Q_{1\mu}Q_{2\nu} + Q_{1\nu}Q_{2\mu}) - Q_1^2 Q_{2\mu}Q_{2\nu} - Q_2^2 Q_{1\mu}Q_{1\nu}\Big].$$

Green, Gryniuk, von Hippel, HM, Pascalutsa, PRL115 222003 (2015); Gérardin et al, 1712.00421

Going above the threshold for hadron production?

With H_W the $\Delta S = 1$ effective Hamiltonian:

$$\Delta M_K = 2 \underbrace{\mathcal{P}}_{\text{princ. value}} \sum_{\alpha} \frac{\langle \overline{K}^0 | H_W | \alpha \rangle \langle \alpha | H_W | K^0 \rangle}{M_K - E_\alpha}$$

Accessed on the lattice from the four-point function

$$G(t_f, t_A, t_B, t_i) = \frac{1}{2} \int_{t_A}^{t_B} dt_1 \int_{t_A}^{t_B} dt_2 \langle \overline{K}^0(t_f) H_W(t_2) H_W(t_1) \overline{K}^0(t_i) \rangle$$

Analogue of computing the Compton amplitude $T_{\mu\nu}(p,q)$ on the lattice above particle production threshold, $\omega > 1$.
Here it appears feasible to subtract the contributions of intermediate states with mass below M_K.

N. Christ et al., PRD 88, 014508 (2013); Z. Bai et al, PRL 113, 112003 (2014)

Conclusion

- Four-point functions computable on the lattice are sensitive to transition rates encoded in the hadronic tensor.
- Allows for confronting experimental information on the hadronic tensor data with lattice data $\Rightarrow$ consistency check.
- How far can we get in actually inverting the integral transform numerically?

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